Chapter 1

Group and Symmetry

1.1 Introduction

1. A `group` (G) is a collection of elements that can 'MULTIPLY' and 'DIVIDE'. The 'multiplication' $*$ is a binary operation that is associative but not necessarily commutative. Formally, the defining properties are:

   (a) if $g_1, g_2 \in G$, then $g_1 * g_2 \in G$;

   (b) there is an identity $e \in G$ so that $g * e = e * g = g$ for every $g \in G$. The identity e is sometimes written as 1 or $\mathbf{1}$;

   (c) there is a unique inverse g^{-1} for every $g \in G$ so that $g * g^{-1} = g^{-1} * g = e$.

   The multiplication rules of a group can be listed in a `multiplication table`, in which every group element occurs once and only once in every row and every column (*prove this !*) .

   For example, the following is the multiplication table of a group with four elements named Z_4.

	e	g_1	g_2	g_3
e	e	g_1	g_2	g_3
g_1	g_1	g_2	g_3	e
g_2	g_2	g_3	e	g_1
g_3	g_3	e	g_1	g_2

Table 2.1 Multiplication table of Z_4

2. Two groups with identical multiplication tables are usually considered to be the same. We also say that these two groups are `isomorphic`.

   Group elements could be FAMILIAR MATHEMATICAL OBJECTS, such as numbers, matrices, and differential operators. In that case group multiplication is usually the ordinary multiplication, but it could also be ordinary addition. In the latter case the inverse of g is simply $-g$, the identity e is simply 0, and the group multiplication is commutative.

   The same `abstract group` can often be `represented` in several different ways. For example, the elements of the Z_4 group above could be the complex numbers $e = 1, g_a = e^{a\pi i/2}$, for $a = 1, 2, 3$, with the group multiplication taken to be ordinary multiplication. Alternatively, they could be the integers $e = 0, g_a = a$, with the group multiplication being ordinary addition mod 4.

   Group elements could also be OPERATIONS on an underlying space of objects. For example, for the Z_4 group above, the elements could be represented by rotations of a square, through the angles $\theta(e) = 0, \theta(g_1) = \pi/2, \theta(g_2) = \pi, \theta(g_3) = 3\pi/2$.

   In what follows we shall write the 'multiplication' $g_1 * g_2$ simply as g_1g_2, and g times itself m times as g^m. We will also often write the unit element e to be 1 or **1**. **[5.26]**

3. A `symmetry` operation is an operation that leaves certain objects unchanged. For example, the group Z_4 above is the symmetry group of a square. The set of symmetry operations taken together often (though not always) forms a group. Most of the groups used in physics arise from symmetry operations of physical objects.

4. $\mathsf{Z_n}$ `group`. It describes a symmetry of a plane figure invariant after a rotation of $2\pi/n$ degrees. We have discussed a Z_4 symmetry in the last section. The geometrical pattern below has a larger Z_{12} symmetry, which is particular also has a Z_4 symmetry. As we shall see, the smaller symmetry Z_4 is called a `subgroup` of the larger symmetry Z_{12}.

   The simplest non-trivial group is a group called Z_2 consisting of two elements e, σ such that $e\sigma = \sigma e = \sigma, \sigma^2 = \sigma\sigma = e$.

   There is another four-element group $Z_2 \times Z_2$, the `direct product` of two Z_2's, in which each of its elements is given by a pair $g = (a, b)$,

Figure 1.1: Z_{12} symmetry

with $a \in Z_2$ and b is in the other Z_2. Multiplication is carried out independently in each component, namely, $g_1 g_2 = (a_1 a_2, b_1 b_2)$. The operations of turning mattresses and flipping one's two hands illustrated below can both be considered as members of $Z_2 \times Z_2$.

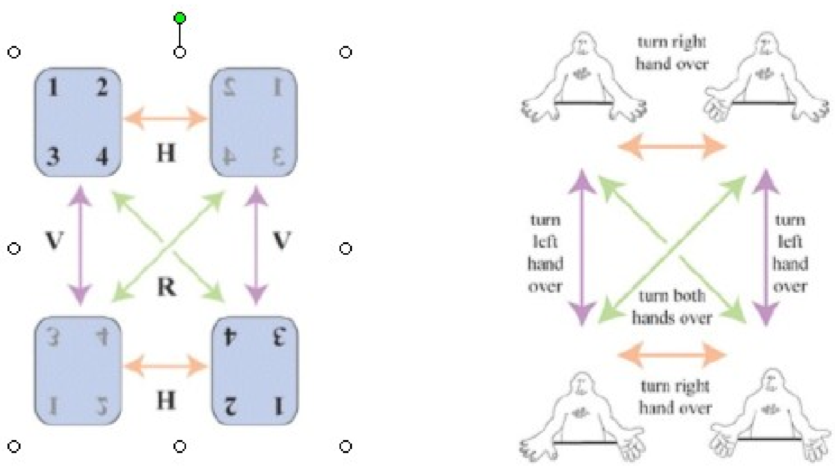

Figure 1.2: $Z_2 \times Z_2$

5. Most of us are somewhat familiar with how symmetry helps to simplify the analysis of `physical systems`. For example, if a potential is SPHERICALLY SYMMETRIC, then the force is radial, and angular mo-

mentum is conserved. If a two dimensional potential has the Z_4 symmetry discussed above, then the force along any direction is identical to the force along the orthogonal direction. If the potential is INDEPENDENT OF x, then there is no force along the x-direction and momentum along that direction is conserved.

Similar restrictions are also present on quantum mechanical systems. The general connection between quantum physics and symmetry is relatively simple, and is outlined at the end of this chapter.

1.2 Definitions

1. The number of elements in a group G is called the `order` of the group, and will be denoted by $|G|$.

   If $|G|$ is a finite number, then the group is a `finite group`. If $|G|$ is infinite and if its elements are labeled by n continuous real parameters, then G is a `continuous group` with `dimension` n. The only continuous groups we will discuss are the smooth ones, those that are also differentiable in their parameters, at least locally. Those are the `Lie groups`.

2. `Group summation`. For a finite group, $|G| = \sum_{g'\in G} 1$. In what follows, we will find other sums over elements of a finite group useful. If $f(g)$ is a function of g, then we can define another function $c(g)$ by a group sum, $c(g) := \sum_{g'\in G} f(g'g)$. This function turns out to be a constant, independent of g. To see that, let $g'' = g'g$. As g' ranges over G, g'' also ranges over G, hence $c(g) = \sum_{g''\in G} f(g'')$ is independent of g. This invariant property of a group sum will be used fairly frequently in group theory.

   For a continuous group, we must replace summation over g' by an integral over g': $\sum_{g'} \to \int dg'$. To retain the invariant group sum property of a finite group, the integration measure dg should have the property that $dg' = d(g'g)$. A measure satisfying this property is known as a `Haar measure` or an `invariant measure`. If the group is compact, then the volume $\int dg$ of G is finite, and this can be used in place of $|G|$ of a finite group. With this replacement, many of the properties of a finite group which we find in subsequent chapters remain valid for compace Lie groups.

For a concrete way to compute the Haar measure, see §2 of Ref. [3] and §8.12 of Ref. [6].

3. A group G is said to be `isomorphic` to another group G', in symbols, $G \cong G'$, if there is a one-one correspondence between the elements of the two groups that preserves multiplication and inverses. This means, the one-one correspondence $g_i \leftrightarrow g'_i$ for every $g_i \in G$ and every $g'_i \in G'$ is such that $g_1g_2 \leftrightarrow g'_1g'_2$ and $g_i^{-1} \leftrightarrow g_i'^{-1}$.

   Two groups isomorphic to each other are essentially the same except for names. For that reason, most of the time we shall not make a distinction between isomorphic groups, so we also write $G = G'$ to mean $G \cong G'$.

   G is said to be `homomorphic` to G' if the mapping goes one way only. In that case, G is usually larger than G' and the correspondence $g_i \rightarrow g'_i$ implies $g_1g_2 \rightarrow g'_1g'_2$ and $g_i^{-1} \rightarrow g_I'^{-1}$ is many to one. We can think of G' as a projection of G. If there are exactly n g_i's, $g_i^1, g_i^2, \cdots, g_i^n$, that corresponds to the same g'_i, then the group G is said to `cover` G' n times. G' is like a planar view of an n story building. If $n = 2$, the cover is a `double cover`; if $n = 3$, it is a `triple cover`, etc.

1.3 Examples of symmetries and groups

1.3.1 Finite groups

1. `Polyhedral groups`. They describe the symmetry of a regular polyhedron (`tetrahedron, cube, octahedron, icosahedron, dodecahedron`), otherwise known as a `Platonic solid`.

	F	E	V
tetrahedron	4	6	4
cube	6	12	8
octahedron	8	12	6
dodecahedron	12	30	20
icosahedron	20	30	12

Table 1.1 The five Platonic solids. F, E, V are respectively the number of faces, edges, and vertices of the solid

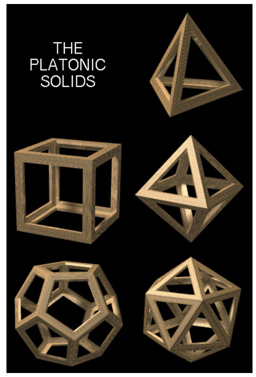

Figure 1.3: Platonic solids

(a) The number of faces (F), edges (E), and vertices (V) of these five solids are given in Table 1.1.

(b) The cube and the octahedron (second row of Fig. 1.2) are DUAL to each other, and so are the icosahedron and the dodecahedron (third row of Fig. 1.2), in the sense that the number of vertices of one is equal to the number of faces of another, so that one can be fit snuggly into another by placing its vertices at the center of the faces of another. For that reason, the cube and the octahedron have the same symmetry O, and the icosahedron and the dodecahedron have the same symmetry I. The tetrahedron is self dual and its symmetry is denoted by T. Strictly speaking, these symbols refer to their rotational symmetries only; reflections and inversions are additional.

(c) These geometrical symmetries could be symmetry of MOLECULES. For example, the molecules CH_4 and CCl_4 have the tetrahedral symmetry (T), and the molecule UF_6 (U in the middle, and F occupying the six vertices of the octahedron) has the octahedral symmetry (O).

2. `Permutation symmetry`

(a) We shall use the CYCLIC NOTATION to denote a permutation of n identical objects. For example, (135)(7986) is used to denote a permutation of 9 objects, with the object in the top row below replaced by the corresponding object in the bottom row. Any

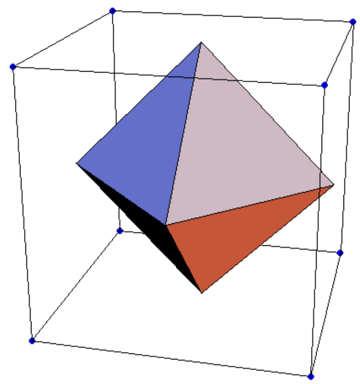

Figure 1.4: An octagon inside a cube

number (2 and 4 above) not appearing the cycles is unchanged during the permutation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 5 & 4 & 1 & 7 & 9 & 6 & 8 \end{pmatrix}$$

(135) is called a `3-cycle` because the three symbols are cyclically invariant: (135)= (351)= (513). (7986)=(9867)=(8679)=(6798) is called a `4-cycle`.

(b) The cycle notation is cyclic. For example, (1234)=(2341)=(3412) =(4123). **[5.26]**

(c) The product of permutations is obtained by successive replacements, starting from the rightmost cycle. For example, (243)(36)(64)= (63)(42)

(d) The inverse of a permutation is obtained by reversing all the cycles. For example, the inverse of (143)(25)(768) is (341)(52)(867).

(e) A cycle of odd/even length can be written as a product of an even/odd number of 2-cycles. For example, (123)= (12)(23), (1234)= (12)(23)(34).

(f) A useful rule (*prove this !*) : $t' = sts^{-1}$ is obtained from t by making the substitution specified by s. Example: $s = (1342), t = (3516)$, then $t' = (4536)$.

(g) A permutation is called `odd` (`even`) if it consists of an odd (even) number of 2-cycles. Hence an n-cycle is odd/even if n is even/odd.

The `signature` of a permutation σ, denoted δ_σ, is ± 1 for even/odd permutations.

(h) The product of two odd or two even cycles is an even cycle, because the product contains odd+odd, or even+even, number of 2-cycles. Similarly the product of an odd and an even cycle is an odd cycle. Consequently, the signature of a product is the product of the signatures. [**5.26**]

(i) The set of all permutations of n objects forms a group called the `symmetric group` (S_n). The order of this group is the number of ways permuting n objects, so it is $n!$. [**5.26**]

(j) The set of all even permutations of n objects forms a group called the `alternating group` (A_n). The order of this group is $n!/2$, because (12) times an odd cycle is an even cycle, and (12) times an even cycle is an odd cycle. [**5.26**]

1.3.2 Continuous groups

1. SO(2) `symmetry`. Symmetry of a circle; 2-dim rotational symmetry.

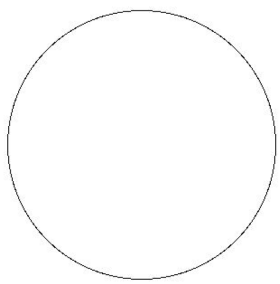

Figure 1.5: $SO(2)$ symmetry

   Z_n are finite subgroups of $SO(2)$.

2. SO(3) `symmetry`. Symmetry of a sphere; 3-dim rotational symmetry. See the picture on top of the next page.

   The polyhedral groups are finite subgroups of $SO(3)$.

3. `spacetime groups`. Other continuous group encountered in physics include

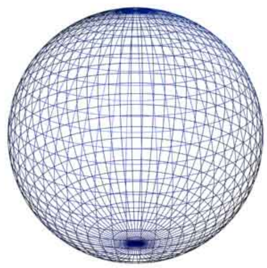

Figure 1.6: $SO(3)$ symmetry

(a) Lorentz group of Lorentz transformations.

(b) Spacetime translation group T_4.

(c) The Poincaré group generated by the Lorentz group and T_4. **[05.20]**

(d) The conformal group valid for some scale-invariant systems, which contains the Poincaré group.

1.3.3 Infinite discrete groups

There are groups which are neither finite, nor continuous. The translation group of a `crystal lattice` is a typical example. Molecules in a 3-dimensional crystal are located at positions $\vec{r}_n = n_1\vec{e}_1 + n_2\vec{e}_2 + n_3\vec{e}_3$, where n_i are integers and $\vec{e}_i$ are three linearly independent vectors which differ from crystal to crystal. The crystal is invariant under a group G consisting of all translations $\vec{r}_n$.

1.4 Generators and Relations

1. Group elements are related by multiplication and inverse, hence they are not all independent. The independent elements of a group, from which all other elements can be obtained through multiplication and inverse operations, are called the `generators` of the group. For example, the 2-dimensional rotation

$$g := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \cos\theta\mathbf{1} + i\sin\theta\ \sigma_2, \qquad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \tag{1.1}$$

through an angle $\theta = 2\pi/n$ generates the Z_n group because every group element can be expressed as g^m for some $0 \le m < n$.

2. Since we like to identify isomorphic groups, there are usually different ways to specify the generators, depending on how the group is expressed.

   Here are several ways that can be used to specify the Z_n generators g:

   $$g = \cos\theta \mathbf{1} + i \sin\theta\ \sigma_2 = e^{i\theta\sigma_2}; \qquad e^{i\theta}; \qquad 1 \text{ mod } n. \tag{1.2}$$

   In the first two cases, the group operation $*$ is ordinary multiplication. In the third case, it is ordinary addition mod n.

3. There is a way to specify Z_n and its generator g without going into specific representations in (1.2). Note that whatever the representation is, the generator satisfies $g^n = 1$. This way of specifying an abstract group element is called a `relation`. A finite group G can often be specified this way by the expression G={generators|relations}. In the case of Z_n, that would be $Z_n = \{g|g^n = e\}$, or simply $\{g|g^n\}$ for short. Any of the four ways to define Z_n in (1.2) all satisfy this abstract definition.

   This abstract way of defining a group, through generators and relations, is known as a `presentation` of the group.

   If you want to know more about that way of doing things, take a look at the book 'generators and relations for discrete groups' by Coxeter and Moser. See also §3.11.

4. For a Lie group of n dimension, an element g is specified by n parameters. In particular, the neighborhood of the identity e can be written in the form $g \simeq e + i\vec{\xi}\cdot\vec{t} + \cdots$, where $\vec{\xi} = (\xi_1, \cdots, \xi_n)$ are n infinitesimal parameters, $\vec{t} = (t_1, \cdots, t_n)$ are the corresponding `generators` (sometimes known as 'infinitesimal generators') along the n directions, and $\cdots$ means terms of order ξ^2. The t_i's are generators in the same sense as above because they not only specify the group elements near identity, but by multiplying the elements within this identity, we can gradually enlarge this neighborhood until it covers the whole group, in much the same way as integrating a differential equation to get the complete global solution, so they are indeed the generators of the whole

group. In fact, as we shall see in later chapters, in order for them to generate a group, they must be closed under commutation, and obey other relations to form a structure known as a `Lie algebra`.

(a) For example, an element $g = g(\theta) \in SO(2)$ is given by (1.1). The identity e is the 2-dimensional unit matrix $\mathbf{1}$. For $|\theta| \ll 1$, we can write $g(\theta) = \mathbf{1} + i\theta t + O(\theta^2)$, with the generator t given by the Pauli matrix σ_2.

(b) You may also have learned from quantum mechanics that the generators of the 3-dimensional rotational group $SO(3)$, which as a group also has dimension 3, are the angular momentum operators: $\vec{t} = (J_x, J_y, J_z)$.

1.5 Physics Applications

Here are some illustrations of how symmetries and groups are useful in physics. They by no means exhaust all the possibilities how groups can be used.

1.5.1 Finite groups

Finite groups like Z_n and the polyhedral symmetry groups T, O, I are used to study symmetry of molecules, crystals (Secs. 3.13 to 3.15), as well as symmetry of the three fermionic generations in particle physics, known as `horizontal symmetry` (§3.12). PERMUTATION SYMMETRY is relevant for identical particles in physics, and useful in constructing tensor representations in mathematics. Moreover, as we shall see, every finite group can be considered as a subgroup of the symmetric group.

1.5.2 Lie groups

Our spacetime is parametrized by four continuous coordinates, hence symmetries in spacetime are given mostly by Lie groups. This includes the rotation group $SO(3)$, the Lorentz group, etc. A more detailed discussion of these groups is given in Chap. 7.

1.5.3 Quantum mechanical applications

1. In quantum mechanics, a group G of Hilbert-space operators is called a `symmetry group` (of the Hamiltonian $\mathcal{H}$) if $\mathcal{H} = g^{-1}\mathcal{H}g$ for all $g \in G$.

2. If G is a symmetry group and $\psi(\vec{x})$ is an energy eigenfunction with energy eigenvalue E, *i.e.,* $\mathcal{H}\psi(\vec{x}) = E\psi(\vec{x})$, then every $g\psi(\vec{x})$ is an energy eigenfunction with the same eigenvalue.

   Suppose the functions $G\psi$ form an n-dimensional vector space, then every function in that space has the same energy eigenvalue.

   A homomorphism or an isomorphism of G into the linear operators of an n-dimensional vector space V_n is called an n-dimensional `representation` of G. If Gv spans V_n for every $v \in V_n$, then the representation is called `irreducible`. In principle, given a group G, all its irreducible representation can be determined. See later chapters to see how it is done.

   This is a useful way to find out potential degenerate energy eigenstates of a system possessing a symmetry, without having to solve the Schroedinger equation first. All that we have to do is to find out the irreducible representations (see §4.2) of the group G. If the dimension of the representation is n, then we know that potentially the system possess a degenerate energy `multiplet` of `multiplicity` n. Although we do not know the energy of such multiplets until we solve the Schroedinger equation, we do know the relations between the states in the same multiplet. For example, in a system possessing an $SO(3)$ symmetry (rotationally invariant), $n = 2j + 1$, and one can obtain any state of such angular momentum multiplets from another in the same multiplet by the angular-momentum creation and annihilation operators $J_\pm$.

   For $n = 1, 2, 3, \cdots$, such multiplets are called `singlet, doublet, triplet, etc.,` respectively.

3. The elements of a Lie group can be written in the form $g = \exp(i\vec{\xi}\cdot\vec{T}) := e^{\eta Q}$. The invariant condition $g\mathcal{H}g^{-1} = \mathcal{H}$ gives rise to the relation $[Q, \mathcal{H}] = 0 = [\vec{T}, \mathcal{H}]$, which implies that Q and $\vec{T}$ are conserved operators.

`Proof` : The first statement follows from the formula

$$e^{\eta Q}\mathcal{H}e^{-\eta Q} = \sum_{n=0}^{\infty} \frac{\eta^n}{n!} ad^n(Q).\mathcal{H}, \qquad ad(Q).\mathcal{H} := [Q, \mathcal{H}],$$

which is obtained by Taylor-expanding the left. The second statement is a consequence of Heisenberg's equation of motion. ■

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4. The exponential form is really not required for the proof. Near the identity, $g \simeq 1 + i\vec{\xi}\vec{T} + \cdots, \quad (|\vec{\xi}| \ll 1)$, the invariant condition $g\mathcal{H}g^{-1} = \mathcal{H}$ therefore requires $[\vec{T}, \mathcal{H}] = 0$. **[5.28]**

   Note that a symmetry group of order $|G|$ does not lead to $|G|$ *independent* conserved quantum numbers; only as many as the NUMBER OF GENERATORS.

1.5.4 Harmonic analysis

We are familiar with using Fourier series to analyze periodic functions, and spherical harmonics to expand solutions of spherically symmetric differential equations. The exponential function in Fourier analysis, and the spherical harmonics, are sometimes called 'harmonic functions'. They are the basis functions of a symmetry group, the translation symmetry over a period in the former case, and rotation symmetry in the latter case. These 'harmonic analyses' can be generalized to other groups G. The corresponding harmonic functions are the basis of the 'irreducible representations', and their orthonormality is closely related to the orthonormality of 'unitary irreducible representations' to be explained in §4.2.

1.5.5 ☒ Quantum field theory

A quantum field theory is simply a quantum mechanical system with an infinite number of degrees of freedom, one for each spatial point $\vec{x}$. As such, any application of group theory to quantum mechanics is equally useful in quantum field theories, but not vice versa. Here are some examples how symmetries can be used in quantum field theories.

Noether currents

1. In this section, the conserved group element g discussed before shall be renamed Q to conform to usual notations. We shall refer to them as `conserved charges`. It is to be regarded as a Hilbert-space operator, and the symmetry group G will be assumed to be am n-dimensional Lie group with generators $\vec{T}$. Hence there aer n independent conserved charges which we will denote as $\vec{Q}$.

2. From the additional spatial information now available, one can derive a conserved current $\vec{j}^\mu(x) = \vec{j}^\mu(\vec{x}, t)$ $(\mu = 0, 1, 2, 3)$ such that $\partial_\mu \vec{j}^\mu = 0$ and $\vec{Q} = \int \vec{j}^0(\vec{x}, t) d^3x$. This current is known as the `Noether current`, and is given in terms of the quantum fields $\phi_i(x)$ by the formula

$$\vec{j}^\mu(x) = \sum_i \Pi_i^\mu(x) \vec{T} \phi_i(x),$$

   where $\Pi_i^\mu(x) := \delta\mathcal{L}/\delta(\partial_\mu \phi_i(x))$ is the canonical momenta for $\phi_i(x)$ and $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ is the Lagrangian density invariant under G so that $\mathcal{L}(\phi_i, \partial_\mu \phi_i) = \mathcal{L}(g\phi_i, \partial_\mu g\phi_i)$ for every $g \in G$. For $g \simeq e := \mathbf{1}$, we can write $g \simeq \mathbf{1} + i\vec{\xi} \cdot \vec{T}$.

3. $\partial_\mu \vec{j}^\mu(x) = 0$ because

   `Proof` : Let $\delta\phi_i = i\vec{\xi} \cdot \vec{T} \phi_i$, then the fact that $\delta\mathcal{L} = 0$ under this change implies

$$\delta\mathcal{L} = \sum_i \left\{ \frac{\delta\mathcal{L}}{\delta\phi_i(x)} \delta\phi_i(x) + \frac{\delta\mathcal{L}}{\delta(\partial_\mu \phi_i(x))} \delta\partial_\mu \phi_i(x) \right\} = 0.$$

   Using the Euler-Lagrange equation of motion,

$$\partial_\mu \Pi^\mu(x) = \frac{\delta\mathcal{L}}{\delta\phi_i(x)}, \quad \Pi_i^\mu(x) := \frac{\delta\mathcal{L}}{\delta(\partial_\mu \phi_i(x))},$$

   we obtain $\partial_\mu \sum_i \Pi_i^\mu(x) \delta\phi_i(x) = 0$ for every $\vec{\xi}$, hence $\partial_\mu \vec{j}^\mu(x) = 0$. ■
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4. It follows that the integrated quantity $\vec{Q} = \int d^3x \, \vec{j}^0(\vec{x}, t)$ is a conserved charge.

5. A similar Noether current can be worked out for continuous spacetime symmetries. The formula and the derivation are more complicated because G varies x as well as ϕ_i. See any quantum field theory book for its derivation.

Spontaneous breaking of symmetry and Goldstone bosons

6. If a quantum field theory has a set of scalar fields $\phi_i(x)$, and if they interact with one another through a potential $V(\phi_i)$ whose minimum occurs at a value $\phi_i = v_i \neq 0$, then dynamically such a value of the scalar field is preferred in the ground state. In that case, the original symmetry G of the Lagrangian is reduced to a smaller symmetry H which transforms the scalar fields in such a way that it leaves the minimum unchanged: $Hv_i = v_i$. This is known as a `spontaneous breaking of symmetry`, and the v_i's are called the `condensates`.

7. Massless bosons known as `Goldstone bosons` will appear whenever spontaneous symmetry breaking of a Lie group occurs. This is known as the `Goldstone theorem`. There are as many Goldstone bosons as the broken generators, namely, generators in G but not in H.

`Proof` : Let $V(\phi_i)$ be the scalar potential. Then $(\partial V/\partial\phi_i)_0 = 0$ because the vacuum expectation value v_i lies at the bottom of the potential. The subscript 0 indicates the substitution of ϕ_i by v_i. Moreover, $(\partial^2 V/\partial\phi_i\partial\phi_j)_0 = m_{ij}$ is the mass matrix of the scalar particles, because small excitations about the vacuum takes on the form

$$V(\phi) \simeq V(v_i) + \frac{1}{2}\left(\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right)_0 (\phi_i - v_i)(\phi_j - v_j) + \cdots \tag{1.3}$$

Let $\Delta\phi_i$ be the infinitesimal variation of ϕ_i generated by a group element in G. Since V is invariant under G, $V(\phi_i) = V(\phi_i + \Delta\phi_i)$. Specializing $\phi_i - v_i$ in (1.3) to $\Delta\phi_i$, we see that the second term on its right (as well as all the subsequent terms) must vanish. If $\Delta\phi_i$ comes from the variation of H, then this is trivial because $\phi_i = v_i$ in that case. However, if $\Delta\phi_i$ is not generated by H, then it must be an eigenvector of m_{ij} with zero eigenvalues, hence a massless state. This is the Goldstone boson. ■

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8. A standard example of a Goldstone boson is the pion. In the limit of zero up- and down-quark masses, there is no coupling between the left-handed and the right-handed quarks, hence the `isospin symmetry` of (u, d) is extended to an $SU(2)_L \times SU(2)_R$ symmetry, called the `chiral symmetry`. The strong attraction between a quark and an antiquark causes a scalar `chiral condensate` to be formed, which breaks the chiral symmetry down to an isospin symmetry $SU(2)_{L+R}$. As a result, three massless Goldstone bosons emerge because there are three broken generators, and they correspond to the pions in this massless limit of the quarks.

Higgs mechanism

9. Gauge particles are massless spin-1 particles, carrying only transverse polarizations. When the gauge group G is spontaneously broken down to a gauge group H, the gauge bosons corresponding to the broken generators can incorporate the Goldstone bosons to become their longitudinal polarized degree of freedom, and thereby gaining a mass. This is known as the Higgs mechanism.

10. For example, in the Standard Model, the `electroweak symmetry` $SU(2) \times U(1)_Y$ is broken down spontaneously to $U(1)_Q$. There are three broken generators, and correspondingly the $W^{\pm}$ and Z^0 bosons gain a mass through the Higgs mechanism.

Symmetry and dynamics

11. A gauge theory has a symmetry at every spacetime point. This leads to an infinite symmetry group, with an infinite number of Noether currents. In the case of pure electrodynamics, for example, these infinite number of conservation laws are equivalent to the Maxwell equations $\partial_\nu F^{\mu\nu} = 0$. Thus for gauge theories the *symmetry is so large that it completely determines the dynamics of the gauge fields.* This is also the case for string theory.

Supersymmetry

12. Nowadays, supersymmetry (SUSY) is also popular. It this conjectured symmetry between fermions and bosons is present, it must be badly broken because we have found no trace of such symmetry among the known particles. The mechanism causing such a breaking is unclear. We will not discuss supersymmetry in these lectures.