Chapter 7

Classical Lie Groups

A d-dimensional `Lie group` is a continuous group that can be specified locally by d differentiable parameters. An important Lie group is the group of all non-singular $n \times n$ matrices. It is called the `general linear group`, usually denoted as $\mathsf{GL(n)}$. It and all its subgroups are known as the `classical groups`. We will start by studying some of the more important subgroups.

The subgroup of $GL(n)$ consisting of matrices with unit determinant is call the `special linear group`, and is usually denoted as $\mathsf{SL(n)}$. Other important subgroups will be discussed below.

7.1 Orthogonal groups

$\mathsf{O(n)}$, $\mathsf{SO(n)}$

1. Let $\vec{x}$ be an n-dimensional vector with components x_i. A transformation $x_i \rightarrow x_i' := A_{ij}x_j$ (repeated indices are automatically summed) preserving $\vec{x}\cdot\vec{x} = \delta_{ij}x_ix_j$ (i.e., $\vec{x}\cdot\vec{x} = \vec{x}'\cdot\vec{x}'$) defines an orthogonal matrix A satisfying $A_{ij}A_{kl}\delta_{ik} = \delta_{jl}$. Since the product of two orthogonal matrices is an orthogonal matrix, and the inverse of A is A^T, the set of all $n \times n$ orthogonal matrices form a continuous group known as the `orthogonal group`, denoted as $O(n)$.

2. Since $A^TA = \mathbf{1} \Rightarrow \det(A)^2 = 1 \Rightarrow \det(A) = \pm 1$, $O(n)$ is divided into two pieces, those with $\det(A) = +1$, and those with $\det(A) = -1$. Those with $\det(A) = +1$ forms a connected subgroup of $O(n)$ known as the `special orthogonal group`, denoted as $SO(n)$. One usually

assumes the $SO(n)$ matrices to be real, so that it is the symmetry group of an n-dimensional sphere.

3. $SO(3)$ and $SO(2)$ are particularly important in physics. $A \in SO(2)$ can be parameterized by the rotation angle ξ as

$$A_\xi = \begin{pmatrix} \cos \xi & -\sin \xi \\ \sin \xi & \cos \xi \end{pmatrix}.$$

It is an abelian group because $A_{\xi_1} A_{\xi_2} = A_{\xi_1 + \xi_2} = A_{\xi_2} A_{\xi_1}$. It rotates a circle into a circle.

$SO(3)$ rotates a sphere into a sphere. Such rotations can be thought of as rotations about some axis, hence they are parameterized by three angles: the polar and the azimuth angles θ, ϕ to parameterize the direction of the axis, and the amount of rotation ψ about this axis. These are essentially the Euler angles. The detailed parameterization is a bit complicated so I will not show it here. Moreover, there are many parameterizations that can be used, but no matter what, one needs three parameters. In other words, the group $SO(3)$ is three dimensional.

4. More generally, the **dimension** of $SO(n)$ is $n(n-1)/2$ and it leaves an n-dimensional sphere invariant. Like in $SO(3)$, one can fix an axis in the n-dimensional space and perform a $(n-1)$th dimensional rotation about this axis. To make an $(n-1)$-dimensional rotation, one fixes an axis in the $(n-1)$-dimensional space, and make an $(n-2)$-dimensional rotation, so on down the line. An axis in n-dimensions needs $n-1$ parameters to fix, hence the total number of parameters needed to specify an $SO(n)$ rotation is $(n-1)+(n-2)+\cdots+2+1 = n(n-1)/2$.

5. There is another way to calculate the dimension of $SO(n)$. Write $A \in SO(n)$ as $A = e^{iH}$. Being an orthogonal matrix, $A^{-1} = e^{-iH} = A^T = e^{iH^T}$, hence $H^T = -H$ so H is antisymmetric. The diagonal elements of an antisymmetric matrix is zero, and the lower half is determined by the upper half. The upper half has $(n^2 - n)/2$ matrix elements. If A is real, then H must be imaginary, so A is determined by $n(n-1)/2$ real parameters.

6. This last argument has the advantage of also telling us what the **infinitesimal generators** are (see §1.3). Let $J_{ij} = -J_{ji}$ $(i \neq j)$ be an

antisymmetric matrix whose only non-zero entries are the (ij) matrix element, with a value $-i$, and the (ji) matrix element, with a value i. In other words,

$$(J_{ij})_{pq} = -i(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}).$$

Then

$$[J_{ij}, J_{kl}] = i\left(\delta_{ik}J_{jl} - \delta_{jk}J_{il} + \delta_{jl}J_{ik} - \delta_{il}J_{jk}\right). \tag{7.1}$$

7. Suppose $H = \frac{1}{2}\omega_{ij}J_{ij}$ is parametrized by the antisymmetric parameters ω_{ij}. For $|\omega_{ij}| \ll 1$, an $SO(n)$ rotation of a vector v changes v_k into

$$v_l A_{lk} \simeq v_l(\mathbf{1} + iH)_{lk} + \cdots = v_k + i\frac{\omega_{ij}}{2}(i)(\delta_{ik}v_j - \delta_{jk}v_i) + \cdots.$$

8. If we use a hat to denote a Hilbert-space operator, then $\hat{v}_k$ is transformed into

$$\hat{A}\hat{v}_k\hat{A}^{-1} \simeq \hat{v}_k + i\frac{\omega_{ij}}{2}[\hat{J}_{ij}, \hat{v}_k] + \cdots \simeq \hat{v}_k + i\frac{\omega_{ij}}{2}(i)(\delta_{ik}\hat{v}_j - \delta_{jk}\hat{v}_i) + \cdots \tag{7.2}$$

hence

$$[\hat{J}_{ij}, \hat{v}_k] = i(\delta_{ik}\hat{v}_j - \delta_{jk}\hat{v}_i). \tag{7.3}$$

The second term in this realtion is necessary because $\hat{J}_{ij} = -\hat{J}_{ji}$.

The Hilbert-space commutator relation corresponding to (7.1) is

$$\left[\hat{J}_{ij}, \hat{J}_{kl}\right] = i\left(\delta_{ik}\hat{J}_{jl} - \delta_{jk}\hat{J}_{il} + \delta_{jl}\hat{J}_{ik} - \delta_{il}\hat{J}_{jk}\right). \tag{7.4}$$

In light of (7.3), this seemingly complicated commutation relation simply says that $\hat{J}_{kl}$ is an antisymmetric second-rank tensor operator.

9. In particular, for $SO(3)$, if we let $J_{12} = J_3$, $J_{23} = J_1$, $J_{31} = J_2$, then the commutation relation is the well-known angular-momentum commutation relation in quantum mechanics:

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{7.5}$$

10. A r-index object $T_{i_1i_2\cdots i_r}$ is called a rth rank `tensor` if it transforms like $x_{i_1}x_{i_2}\cdots x_{i_r}$. Namely, $T_{i_1i_2\cdots i_r} \to T'_{i_1i_2\cdots i_r} = A_{i_1j_1}A_{i_2j_2}\cdots A_{i_rj_r}T_{j_1j_2\cdots j_r}$.

11. A tensor is called an `invariant tensor` if $T' = T$ for every A. For $SO(n)$, δ_{ij} is a second rank invariant tensor because of the orthogonal nature of every $A \in SO(n)$. The nth rank totally antisymmetric tensor $\epsilon_{i_1 i_2 \cdots i_n}$ with $\epsilon_{12\cdots n} := +1$ is also an invariant tensor for $SO(n)$ because $A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_n j_n} \epsilon_{j_1 j_2 \cdots j_n} = \det(A)\epsilon_{i_1 i_2 \cdots i_n} = \epsilon_{i_1 i_2 \cdots i_n}$.

12. **contraction of tensor indices.** A tensor or a product of tensors remain a tensor after some of the repeated indices are summed over. Such a sum of repeated indices is usually referred to as an indix `contraction`. This is so because the transformation matrix A_{ij} is orthogonal. For example, if T_{ijk} is a third-rank tensor, then $U_i = T_{ijj}$ is a first-rank tensor because under an orthogonal transformation, because

$$U_i = T_{ijj} \rightarrow T'_{ijj} = A_{ii'} A_{jj'} A_{jk'} T_{i'j'k'} = A_{ii'} T_{i'j'j'} = A_{ii'} U_{i'}.$$

13. A tensor is called an `invariant tensor` if $T' = T$ for every A. For $SO(n)$, δ_{ij} is a second rank invariant tensor because of the orthogonal nature of every $A \in SO(n)$. The nth rank totally antisymmetric tensor $\epsilon_{i_1 i_2 \cdots i_n}$ with $\epsilon_{12\cdots n} := +1$ is also an invariant tensor for $SO(n)$ because $A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_n j_n} \epsilon_{j_1 j_2 \cdots j_n} = \det(A)\epsilon_{i_1 i_2 \cdots i_n} = \epsilon_{i_1 i_2 \cdots i_n}$.

14. The orthogonal matrices A defines what is usually known as the `fundamental representation`, or the `defining representation`. This representation is clearly irreducible because the group consists of *all* orthogonal matrices with unit determinant. The vector space V for this IR is simply the original n-dimensional space upon which the matrices are defined. From this we can form the tensors $T_{i_1 i_2 \cdots i_k} = x_{i_1} x_{i_2} \cdots x_{i_k}$ which are symmetric in all the indices. A contraction with ϵ_{cdots} would vanish because ϵ is totally antisymmetric but T is totally symmetric in their indices. However, a contraction with $\delta_{..}$ would produce a non-zero tensor two ranks lower, so on. Hence the traceless expressions in (4.1) are really tensors, and they remain traceless upon any orthogonal transformation. Hence the kth rank traceless symmetric tensors span a subspace of all kth rank symmetric tensors, invariant under rotation, hence they form a basis to reduce the tensor represenatations of $SO(n)$. It turns out that they are irreducible, a fact that will be shown in later chapters.

15. If the tensor $T_{i_1 i_2 \cdots i_k}$ is not symmetric in all its indices, then its reduction

into irreducible spaces is more complicated. That will be discussed later.

7.2 Unitary groups

$\mathsf{U(n)}$ and $\mathsf{SU(n)}$

1. A transformation $x_i \to x_i' := A_i^j x_j$ that preserves the complex scalar product for all $\vec{x}$ ($x'^i x_i' = x^i x_i$, with $x^i := (x_i)^*$) defines a unitary matrix A satisfying $A^\dagger A = \mathbf{1}$. Since the product of unitary matrices is a unitary matrix, and the inverse of A is $A^\dagger$, all the $n \times n$ unitary matrices form a group known as the `unitary group`, $U(n)$. The unitary matrices of unit determinant form a subgroup called the `special unitary group`, $SU(n)$. Taking the determinant on both sides of the unitarity condition, we get $|\det A|^2 = 1$ for any $A \in U(n)$. If we let $\det A = \alpha$, with $|\alpha| = 1$, then every A can be decomposed uniquely into the product $\alpha \bar{A}$, where $\alpha \in U(1)$ and $\bar{A} \in SU(n)$. Thus $U(n) = U(1) \times SU(n)$.

2. Unitary groups are important in quantum mechanics because a quantum mechanical evolution, or a simply change of basis, have to be unitary to conserve probabilities.

3. Note that we have made a distinction between superscript indices and subscript indices, and have written the matrix element A as $A_i^{\;j}$. The reason is twofold. First, by defining $x^i = (x_i)^*$ and $A^i_{\;j} = (A_i^{\;j})^*$, we replace the task of complex conjugation by raising and lowering indices. Secondly, if A is the `defining` or `fundamental representation`, then A^* is also a representation because $(A_1 A_2)^* = A_1^* A_2^*$, and generally, other than $SU(2)$, the representation A is not equivalent to A^*. To make that clear, we distinguish the vector spaces of these two by using lower and upper vector indices respectively. Correspondingly, tensors T may have upper as well as lower indices, which are transformed by A^* and A respectively. We are now allowed to sum over repeated indices only when one is upper and the other is lower.

4. The unitarity condition is $(A_j^{\;i})^* A_k^{\;j} = A^j_{\;i} A_j^{\;k} = \delta_i^k$, and the determinant condition of $SU(n)$ can be written either as $\epsilon^{i_1 \cdots i_n} \prod_{a=1}^n A_{i_a}^{\;j_a} = \epsilon^{j_1 \cdots j_n}$, or as $\epsilon_{j_1 \cdots j_n} \prod_{a=1}^n A_{i_a}^{\;j_a} = \epsilon_{i_1 \cdots i_n}$. Hence δ_i^k is an invariant tensor for $U(n)$, and $\epsilon_{i_1 \cdots i_n}$, $\epsilon^{i_1 \cdots i_n}$ are also invariant tensors for $SU(n)$.

However, δ^{ij} and δ_{ij} cannot be invariant tensors, for otherwise it would declare $y^i = \delta^{ij} y_j$ should transform like y_j^* but in reality this is just equal to y_i so that cannot be the case.

5. As in $SO(n)$, these invariant tensors can be used to reduce the dimension of the representations. For example, we may contract x_i and y^j using δ_j^i to get a scalar $x_j y^j$.

6. The **dimension** of the $SU(n)$ group is $n^2 - 1$. The easiest way to see that is to write an $SU(n)$ element in the form $A = e^{iH}$, or equivalently, let $iH = \ln(A)$. Unitarity of A implies $\mathbf{1} = AA^\dagger = e^{iH} e^{-iH^\dagger}$, hence $H = H^\dagger$. Using the formula $\det(A) = \exp(\mathrm{Tr}\ln(A))$ (*prove this !*) , we see that $\det(A) = 1$ implies $\mathrm{Tr}(H) = 0$.

   Being hermitian, $H_{ij} = H_{ji}^*$, its diagonal elements are determined by n real parameters. The off-diagonal elements may be complex, but those below the diagonal are the complex conjugates of those above the diagonal. The number of complex entries above the diagonal is $(n^2 - n)/2$, so the off-diagonal matrix elements of an hermitian H is specified by $n^2 - n$ real parameters. Thus the total number of real parameters needed to fix a hermitian matrix is n^2. If the matrix is to be traceless, one condition must be imposed on the diagonal matrix elements, so the number of free real parameters needed to specify a traceless Hermitian matrix H, or equivalently a unitary matrix A of unit determinant, is $n^2 - 1$. In particular, for $SU(2)$, this is 3, the same number as $SO(3)$.

7. Another way to calculate the dimension of $U(n)$ and $SU(n)$ is as follows. Each row of an $U(n)$ matrix has a unit norm and different rows are orthogonal to each other. The first row is determined by n complex, or $2n$ real parameters, but the requirement of having an unit norm reduces the free parameters to be $2n - 1$. The second row must be orthogonal to the first tow, and this orthogonality determines one of the n complex parameters, leaving $n - 1$ of them free. Unit norm condition fixes another real parameter, leaving behind only $2n - 3$ real free parameters for the second row. So on down the line, the total number of free real parameters for an $U(n)$ matrix is therefore

$$(2n - 1) + (2n - 3) + \cdots + 1 = n^2.$$

If $A \in U(n)$, then $|det A| = 1$, so if A has unit determinant, then the phase of $\det A$ must be zero. This costs another real parameter, hence the dimension of $SU(n)$ is $n^2 - 1$.

8. $U(1)$ is just the set of all complex numbers of modulus 1, so it can be parameterized by a single parameter ξ as $e^{i\xi}$. Clearly it is an abelian group, and there is a 1-1 correspondence between this ξ and the ξ in $SO(2)$, so $SO(2) \cong U(1)$.

9. The most important $SU(n)$ group is $SU(2)$, not only because it gives rise to spin and isotopic spin in physics, but also because it occupies an important position in determining the structures of many other Lie groups. See Chap. 9 for details. For that reason let us look at $SU(2)$ more closely.

10. $SU(2)$

   (a) The infinitesimal traceless Hermitian generators can be taken to be the three Pauli matrices

   $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   If we let $J_i = \sigma_i/2$, then the $SU(2)$ commutation relation is the well known angular-mometnum commutation relation in quantum mechanics:

   $$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{7.6}$$

   (b) For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be of determinant 1, we need $ad - bc = 1$. For it to be unitary, we need $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$ and $ac^* = -bd^*$. The solution can be written as

   $$A = \begin{pmatrix} e^{i(\alpha+\beta)/2}\cos\xi & e^{i(\alpha-\beta)/2}\sin\xi \\ -e^{i(\beta-\alpha)/2}\sin\xi & e^{-i(\alpha+\beta)/2}\cos\xi \end{pmatrix}. \tag{7.7}$$

   It takes three real parameters α, β, ξ to parameterize. This shows explicitly that the dimension of $SU(2)$ is three, the same as $SO(3)$. Moreover, the commutations (7.6) of $SU(2)$ and (7.5) of $SU(3)$ are identical, showing that the local structure of the two groups are

identical. One might therefore expect that the two groups might be identical, but it turns out that $SU(2)$ covers $SO(3)$. This homomorphism can be seen in the following way.

(c) Let the two complex coordinates for an $SU(2)$ vector to be x_1, x_2. The quadratic combination $x_1^* x_1 + x_2^* x_2$ is invariant under an $SU(2)$ transformation, but the other 3 real bilinear combinations, $Z = x_1^* x_1 - x_2^* x_2 = x^\dagger \sigma_3 x$, $X = x_1^* x_2 + x_2^* x_1 = x^\dagger \sigma_1 x$, $Y = -i(x_1^* x_2 - x_2^* x_1) = x^\dagger \sigma_2 x$ undergoes a linear transformation upon an $SU(2)$ rotation $x_i \to x_i'$. However,

$$X^2 + Y^2 + Z^2 = (x_1^* x_1 + x_2^* x_2)^2 \qquad (7.8)$$

remains invariant, hence the linear transformation induced on (X, Y, Z) is an $SO(3)$ transformation.

(d) To see this more explicitly, note that A in (7.7) can be decomposed into

$$
\begin{aligned}
A &= s_1 s_2 s_3, \quad s_1 = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \\
s_2 &= \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix}, \quad s_3 = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix} \qquad (7.9)
\end{aligned}
$$

If we denote the induced $SO(3)$ transformation of s_i by S_i, then a straight forward computation shows that

$$
\begin{aligned}
S_1 &= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
S_2 &= \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \\
S_3 &= \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (7.10)
\end{aligned}
$$

so these are just the Euler rotations with the Euler angles ($\psi = 2\alpha, \theta = 2\xi, \phi = 2\beta$), in which ($\theta, \phi$) specifies the direction of the axis of rotation, and ψ is the amount of rotation about that axis.

(e) The mapping $(x_1, x_2, x_1^*, x_2^*) \mapsto (X, Y, Z) = (X_1, X_2, X_3)$ is quadratic, hence x_i and $-x_i$ map to the same (X, Y, Z), showing that the mapping x_a to X_i is 2 to 1. Moreover, the since (α, ξ, β) and $(\alpha + \pi, \xi + \pi, \beta + \pi)$ give rise to two distinct $SU(2)$ matrices but the same $SO(3)$ matrix, the mapping from $SU(2)$ to $SO(3)$ is 2 to 1. In other words, $SU(2)$ is a double cover of $SO(3)$.

In quantum mechanics, this $2 \to 1$ relation can be seen by considering a rotation about the z-axis by an angle θ. Remembering that such rotations are given by the operator $\exp(i\theta \hat{J}_z)$, whose eigenvalues are $\exp(im\theta)$ ($\hbar = 1$ is used thruoghout in these notes). $SU(2)$ rotations correspond to $m = 1/2$, and $SO(3)$ rotations correspond to $m = 1$. Since $\exp(im\theta)$ is periodic with a period 2π when m is an integer, and the period is 4π when m is a half integer, this again shows that $SU(2)$ covers $SO(3)$ twice.
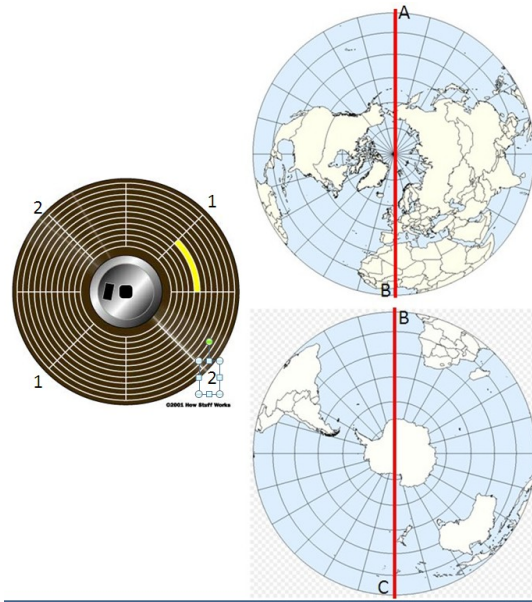

Figure 7.1: Topology of $SO(3)$ and $SU(2)$

(f) We can picture the manifold of $SO(3)$ to be a 3-dimensional ball B^3 with radius π. A rotation $g \in SO(3)$ can be specified by an axis, and the amount of rotation θ about that axis. Since a rotation by π is identical to a rotation about the opposite axis

by $-\pi$, we have identify these two. This element corresponds to a point on B^3, of distance θ from the center, and located in the direction of the axis. However, a point on the boundary S^2 of the sphere must be identified with the opposite point on S^2.

It is easier to picture things in one lower dimension. We have drawn on the left of Fig. 7.1 a disk B^2 to represent B^3, even so it is hard to picture what the resulting object looks like after we identify opposite points, 1 with 1, and 2 with 2, etc. Any curve from the top 1 to the bottom 1 is a closed curve, because the top 1 is identified with the bottom 1, but this curve can never be continuously shrunk to a single point. If we move the top 1 towards the bottom 1 to try to make them coincide to shrink the closed curve to a point, we can never succeed because the bottom 1 keeps on running away from the top 1. So, $SO(3)$ is not `simply connected`. However, if we wind around $SO(3)$ twice, we can shrink the doubly wound closed curve to a point, so $SO(3)$ is doubly connected.

The group manifold for $SU(2)$ is a sphere S^3, namely, it consists of points (y_1, y_2, y_3, y_4) is a real 4-dimensional phase satisfying $\sum_{i=1}^{4} y_i^2 = 1$. This is so because $SU(2)$ is the group that preserves the invariant form $|x_1|^2 + |x_2|^2$. If we let $x_1 = y_1 + iy_2, x_2 = y_3 + iy_4$, then this invariant form defines a sphere S^3. Starting from any point on S^3, we can reach every other point of S^3 by an $SU(2)$ rotation, hence the group manifold of $SU(2)$ is S^3. Again it is easier to picture things in one lower dimension, so we will think of it as S^2, an ordinary sphere. It is clear that any closed curve on S^2 can be continuously shrunk to a point, so $SU(2)$ is simply connected.

To illustrate the connection between $SU(2)$ with $SO(3)$ in one lower dimension, consider Fig. 7.1. Instead of the disk on the left, we can equally well picture $SO(3)$ as the top disk on the right, or the bottom disk on the right, in which points A and B on top, and points B and C at the bottom are identified. Now take the upper disk as the northern hemisphere and the lower disk as the southern hemisphere and assemble the two together into a globe, or S^2. We see in this way that two copies of $SO(3)$ can be assembled into a $SU(2)$, so $SU(2)$ covers $SO(3)$ twice. The

red line connecting A to B is a closed curve in $SO(3)$ but not in $SU(2)$, and so is the red line connecting B with C. However, the line ABC, wraps around $SO(3)$ twice, is a closed curve in S^2 that can be continuously shrunk to a point. We see in this way that $SO(3)$ is doubly connected.

11. More generally, $SO(n)$ for $n > 2$ is always doubly connected, with a double cover group called Spin(n). Thus $Spin(3) = SU(2)$.

12. Here is a point that might lead to some confusion. $SU(2)$ leaves $|x_1|^2 + |x_2|^2$ invariant. If we write $x_1 = y_1 + iy_2$, $x_2 = y_3 + iy_4$, with y_i real, then $|x_1|^2 + |x_2|^2 = \sum_{i=1}^{4} y_i^2$, so one might think than $SU(2)$ would be the same as $SO(4)$, because they leave the same quadratic form invariant. However, as we have seen, $SU(2)$ is more like $SO(3)$ than $SO(4)$. The reason is that $SO(4)$ is described by 4×4 orthogonal matrices but $SU(2)$ is described by 2×2 complex matrices, so they are not the same. In fact, $SU(2)$ has 3 parameters, so does $SO(3)$, but $SO(4)$ has 6 parameters, so as far as number of parameters go, it is more like two $SU(2)$ put together. Actually, $SO(4)$ is locally isomorphic to $SU(2) \times SU(2)$. Not only they both have 6 parameters, but their commutation relations are the same as can be seen in the following way.

The $SO(4)$ commutation is given by (7.1) to be

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \ [J_i, K_j] = i\epsilon_{ijk}K_k, \ [K_i, K_j] = i\epsilon_{ijk}J_k, \qquad (7.11)$$

where $i, j, k = 1 \cdots 3$, $J_{ij} = \epsilon_{ijk}J_k$, $J_{i4} = K_i$. Letting $J_{\pm i} = \frac{1}{2}(J_i \pm K_i)$, we deduce from (7.11) that

$$[J_{\pm i}, J_{\pm j}] = i\epsilon_{ijk}J_{\pm k}, \quad [J_{\pm i}, J_{\mp j}] = 0, \qquad (7.12)$$

which is the commutation relation of $SU(2) \times SU(2)$ (see (7.6)).

7.3 Symplectic group

SP(2n)

1. Consider a $2n$-dimensional vector space labeled by the coordinates x_a, $1 \le a \le 2n$. Any transformation $x_a \to x'_a = A_{ab}x_b$ keeping the bilinear form $\langle x, y \rangle := \sum_{i=1}^{n} (x_i y_{i+n} - x_{i+n}y_i) = x^T \Omega y$ invariant $[\langle x', y' \rangle =$

$\langle x, y \rangle]$ is called a symplectic transformation, and the corresponding matrix A is called a `symplectic matrix`. In other words, a symplectic matrix is one that satisfies $A^T \Omega A = \Omega$, where $\Omega = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} = \Omega^{-1T}$. Note that $\det(A) = \pm 1$ and the matrix $A = \Omega$ is itself symplectic.

2. The product of two simplectic matrices is symplectic, and the inverse of a symplectic matrix is symplectic. The unit matrix $\mathbf{1}_{2n}$ is obviously symplectic. The group of all symplectic matrices with unit determinant is called the `special symplectic group`, denoted by the symbol $SP(2n)$.

3. Suppose u, v are two functions of $2n$ variables, $\xi_a = (q_1, \cdots, q_n, p_1, \cdots p_n)$, and suppose

$$\vec{x} = \nabla_\xi u = \left(\frac{\partial u}{\partial q_1}, \cdots, \frac{\partial u}{\partial q_n}, \frac{\partial u}{\partial p_1}, \frac{\partial u}{\partial q_1}, \cdots, \frac{\partial u}{\partial p_n} \right),$$

and $\vec{y} = \nabla_\xi v$. Then

$$\langle x, y \rangle = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$$

is nothing but the Poisson bracket $\{u, v\}$. A transformation of $(p_i, q_i) \rightarrow (P_i, Q_i)$ preserving the Poisson brackets is a `canonical transformation`. Hence a symplectic transformation is really a linear canonical transformation in the phase space. We know that in general canonical transformations always leave a phase-space volume invariant. In the present case it is obvious because $\det A = 1$.

4. $SP(2)$ consists of all matrices with a unit determinant. In particular, it includes $\mathbf{1}_2$ and Ω as it should.

7.4   Lorentz, Translation, and Poincaré groups

1. The group of linear transformations in a $(p+q)$-dimensional space that keeps the quadratic form $\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2$ fixed is called $O(p, q)$, with $O(p, 0) = O(p)$. Let $g = \text{diag}(+1, \cdots, +1, -1, \cdots, -1)$, with p

$(+1)$'s and q (-1)'s, a $O(p, q)$ matrix A is one that satisfies the condition

$$A^T g A = g.$$

Taking the determinant on both sides, we see that $\det(A)^2 = 1$. The subgroup with $\det = +1$ is called $\mathsf{SO(p, q)}$.

2. The infinitesimal generators of $SO(p, q)$ satisfies the same commutation relations as (7.1), except with δ_{ij} replaced by g_{ij}:

$$[J_{ij}, J_{kl}] = -i \left(g_{jk} J_{il} - g_{ik} J_{jl} - g_{jl} J_{ik} + g_{il} J_{jk} \right). \qquad (7.13)$$

3. $SO(3, 1)$ is the `Lorentz group` describing relativistic invariance. $SO(n+1, 2)$ is the conformal group in n spatial and one time dimensions which will be discussed in the next section.

   In the same way as $SO(4)$, we can show that the generators of $SO(3, 1)$ can be combined to obtain the generators $J_{\pm a} := J_a \pm i K_a$ $(a = 1, 2, 3,\ K_a = J_{a4})$ of $SU(2) \times SU(2)$.

4. Recall from quantum mechanics that the irreducible states of $SO(3)$ (or more appropriately, $SU(2)$) are labelled by the quantum numbers j, m. Since $SO(3, 1)$ is locally the same as $SU(2) \times SU(2)$, relativistic states are labelled by the quantum numbers (j_+, m_+, j_-, m_-), and relativistic multiplets are labelled by (j_+, j_-). Since $J_{+i} + J_{-i} = J_i$ is the angular momentum operator, the spin (angular momentum) of the multiplet (j_+, j_-) rangers from $j_+ + j_-$ down to $|j_+ - j_-|$. Note that spatial reflection interchanges j_+ and j_-, such a multiplet does not have a definite parity unless $j_+ = j_-$.

   For example, the $(\frac{1}{2}, \frac{1}{2})$ multiplet has $2 \times 2 = 4$ components, and it consists of a spin 1 state and a spin 0 state. This is just the familiar relativistic four-vector A^μ, with the spatial vector A^i $(i = 1, 2, 3)$ being a spin-1 object and the spatial scalar A^4 a spin-0 object. Note that even though the multiplet $(\frac{1}{2}, \frac{1}{2})$ has a definite parity, A^i and A^0 have opposite parities because spin-1 calls for symmetric combination of two spin-$\frac{1}{2}$ states and spin-0 calls for an antisymmetric combination.

   Another example is to look at the relativistic second-rank tensor $T^{\mu\nu} = A^\mu B^\nu$. Note that if $A \neq B$, then this tensor is NOT symmetric in μ and ν. This tensor is the tensor product of two $(\frac{1}{2}, \frac{1}{2})$ states, so we will

end up with $4 \times 4 = 16$ states residing in four relativistic multiplets $(1,1),(1,0),(0,1),(0,0)$. The state $(0,0)$ corresponds to $T^{\mu\nu} g_{\mu\nu}$, the six states in $(1,0),(0,1)$ corresponds to the antisymmetric tensor $F^{\mu\nu} = T^{\mu\nu} - T^{\nu\mu}$, with $(1,0)$ corresponds to $F^{ij} + F^{k4}$ and $(0,1)$ to $F^{ij} - F^{k4}$, so that one becomes another under a parity transformation. Finally, the nine states $(1,1)$ corresponds to a symmetric traceless tensor $I^{\mu\nu} = T^{\mu\nu} + T^{\nu\mu} - g^{\mu\nu} g_{\alpha\beta} T^{\alpha\beta}/2$.

5. Translational group T. The translations changing $\hat{x}_\mu$ to $\hat{x}'_\mu = \hat{x}_\mu + a_\mu$, where $\hat{x}_\mu$ is the spacetime position operator and a_μ is a constant 4-vector, is an abelian group generated by $\exp(ia \cdot \hat{P})$:

$$e^{ia\hat{P}} \hat{x}_\mu e^{-ia\hat{P}} = \hat{x}_\mu + a_\mu.$$

This implies

$$[\hat{P}_\nu, \hat{x}_\mu] = -ig_{\mu\nu}.$$

In the representation where $\hat{x}_\mu = x_\mu$ is diagonal, $\hat{P}_\nu = -i\partial/\partial x^\nu$, so $\hat{P}_\nu$ is the energy-momentum operator.

Since $\hat{P}_\mu$ must transform like a Lorentz vector, it follows from (7.3) that

$$[\hat{J}_{\mu\nu}, \hat{P}_\alpha] = i(g_{\mu\alpha}\hat{P}_\nu - g_{\nu\alpha}\hat{P}_\mu), \tag{7.14}$$

and of course we also have

$$[\hat{P}_\alpha, \hat{P}_\beta] = 0 \tag{7.15}$$

because the translation group is abelian. Note that rotation (or more generally Lorentz transformation) and spacetime translation do not commute, because if we translate first, the translation vector must be rotated later, but that does not happen if we rotate first.

The group that includes the Lorentz group $SO(3,1)$ and the spacetime translational group T is called the `Poincaré group`. It is an important group because physics is invariant under Poincaré transformations.

7.5 Conformal group

1. Relativistic theories containing no dimensional parameters (such as masses) may be invariant under the `conformal group`. This is for example true for free Maxwell equations. The conformal group includes in it the Poincaré group, as well as the following transformations:

   (a) *scaling*: $x_\mu \to x'_\mu = \lambda x_\mu$, with an arbitrary constant λ.

   (b) *special conformal transformation*: $x_\mu \to x'_\mu = (x_\mu - a_\mu x^2)/D_a(x)$, where $D_a(x) := 1 - 2a{\cdot}x + a^2 x^2$, and $x^2 := -x_0^2 + \sum_{i=1}^{3} x_i^2$, $x_0 = ct$. Notice that this implies $x^2 \to x'^2 = x^2/D_a(x)$.

   It is clear that these two transformation cannot possibly leave the Hamiltonian or Lagrangian invariant if it contains dimensional parameters.

2. The special conformal transformation can be better understood by writing it in one of the two equivalent forms:

$$\frac{x_\mu}{x^2} \to \frac{x'_\mu}{x'^2} = \frac{x_\mu}{x^2} - a_\mu, \quad x'_\mu = \frac{x_\mu/x^2 - a_\mu}{(x/x^2 - a)^2},$$

   namely, an inversion followed by a translation and then an inversion. It is also easier to see in this form that the special conformal transformations form a one-parameter group.

3. In an n-dimensional spacetime, Lorentz transformation $SO(n-1,1)$ is specified by $\frac{1}{2}n(n-1)$ parameters, translations by n parameters, special conformal transformation by n parameters, and scaling by 1 parameter. Altogether, there are $\frac{1}{2}(n+2)(n+1)$ parameters, just the right number of parameters for rotations in an $(n+2)$-dimensional space. In fact, the conformal group can be represented by $SO(n,2)$.

4. Although the number of parameters is the same, $SO(n,2)$ is a *linear homogeneous* transformation while the Poincaré transformation is non-homogeneous, and the special conformal transformation is non-linear and non-homogeneous, so how can they be contained in $SO(n,2)$? The answer is, they are not contained in $SO(n,2)$, but some homogeneous linear transformations equivalent to them are.

5. To see that, consider the following mapping of the n-dimensional space-time with coordinates x_μ and signature $(n-1,1)$ into the light cone of an $(n+2)$-dimensional space with coordinates y_a and signature $(n,2)$:

$$x = (x_0, \vec{x}) \mapsto y = (x_0, u, \vec{x}, w),$$
$$u = (1 - x^2)/2, \ w = \tfrac{1}{2}(1 + x^2), \ x^2 := x_0^2 - \vec{x}^{\,2}.$$

The target is on the light-cone because $y^2 = x^2 - u^2 + w^2 = 0$ identically.

In the y-space, the Lorentz transformation remains linear and homogeneous, so we do not have to worry about it. The Poincaré transformation becomes $x_\mu \rightarrow x_\mu + a_\mu(u + w)$, which is also linear and homogeneous. The special conformal transformation $y \rightarrow y'$ becomes

$$
\begin{aligned}
x_\mu &\rightarrow (x_\mu - a_\mu x^2)/D_a(x) = [x_\mu - a_\mu(w - u)]/D_a(x), \\
u = (1 - x^2)/2 &\rightarrow (1 - x^2/D_a(x))/2 = (D_a(x) - x^2)/2D_a(x) \\
w = (1 + x^2)/2 &\rightarrow (D_a(x) + x^2)/2D_a(x).
\end{aligned}
$$

If we replace $D_a(x) \pm x^2$ by $(u + w) - 2a \cdot x + a^2(w - u)$ in the numerator, then all the numerators y_a'' are linear and homogeneous in the y-coordinates, and $y_a' = y_a''/D_a(x)$. Since $(y')^2 = 0$ implies $(y'')^2 = 0$, the linear homogeneous mapping $y_i \rightarrow y_i''$ preserves the scalar product, $y^2 = 0 = (y'')^2$, hence this mapping is a member of $SO(n, 2)$.