

Group Theory for Physicists

Contents

1	Group and Symmetry	3
1.1	Introduction	3
1.2	Definitions	6
1.3	Examples of symmetries and groups	7
1.3.1	Finite groups	7
1.3.2	Continuous groups	10
1.3.3	Infinite discrete groups	11
1.4	Generators and Relations	11
1.5	Physics Applications	13
1.5.1	Finite groups	13
1.5.2	Lie groups	13
1.5.3	Quantum mechanical applications	13
1.5.4	Harmonic analysis	14
1.5.5	☒ Quantum field theory	14
2	Group Structure	19
2.1	Classes	19
2.2	Subgroups	21
3	Some Finite and Discrete Groups	27
3.1	cyclic group	28
3.2	☒ Sylow p -subgroups	30
3.2.1	Sylow's theorem	30
3.2.2	Sample applications	31
3.3	dihedral group	33
3.4	symmetric group	34
3.5	alternating group	35
3.6	Quaternion groups	36

3.7	Dicyclic groups	37
3.8	Classical groups over a finite field	38
3.9	Modular Group	40
3.10	A list of groups of order ≤ 16	41
3.11	\boxtimes A short list of group presentations	42
3.12	Horizontal Symmetry of Leptons	45
3.13	Crystal Point Groups and Space Groups	46
	3.13.1 Point groups	47
	3.13.2 Space groups and the Bravais lattices	50
3.14	Piezoelectricity, optical activity, and dipole moments of crystals	58
	3.14.1 Dipole moments	58
	3.14.2 Piezoelectricity	59
	3.14.3 Optical activity	59
3.15	Brillouin zone	59
4	Group Representation	63
4.1	Basic facts	63
4.2	Irreducible representations	68
	4.2.1 Basic theorems	68
	4.2.2 Basis vectors of IR	69
	4.2.3 Another view of the orthogonality relation	70
5	Characters	73
5.1	Character construction	75
	5.1.1 Product with a linear character	75
	5.1.2 Character from fixed points of a permutation group	75
	5.1.3 Character induced from a quotient group	75
	5.1.4 Character induced from any subgroup	76
5.2	Characters of some finite groups	78
5.3	Complex-conjugate representation	83
5.4	Clebsch-Gordan series	89
5.5	Clebsch-Gordan coefficients	90
5.6	Simply reducible groups	91
5.7	Wigner-Eckart theorem	94
6	Class Sums and Projection Operators	95
6.1	Class sums	95
6.2	Projection operators	98

6.3	Computing IR	100
7	Classical Lie Groups	103
7.1	Orthogonal groups	103
7.2	Unitary groups	107
7.3	Symplectic group	113
7.4	Lorentz and Poincaré groups	114
7.5	Conformal group	116
8	Symmetric Group	119
8.1	Introductory review	119
8.2	Frobenius' character formula	120
8.3	Using the character formula	124
8.4	Frobenius' degree formula	126
8.5	Graphical application	128
8.5.1	Examples of character computation	129
8.5.2	Dimension of IR.	131
8.5.3	Recursion formula:	135
8.6	Schur's character formula	136
8.7	Conjugate representation	137
8.8	Young Tableau and Young Operators	138
8.8.1	Examples:	140
8.8.2	A combinatorial relation	142
8.8.3	Young Operators	144
8.8.4	Young Projection Operators	146
8.8.5	Already 'orthogonal' Young projection operators	146
8.8.6	Examples of S_2 , S_3 , and S_4	147
8.8.7	☒ Projection operators	150
8.8.8	☒ Orthogonalization of Young operators	151
8.9	Reducing tensorial representations	154
8.10	Reducing tensorial representations	154
8.10.1	Basic reduction techniques	154
8.10.2	Second rank tensors	156
8.10.3	Tensors of higher ranks	160
8.10.4	Number of tensor components for $SU(n)$	161
8.10.5	Number of tensor components for $SO(n)$	166

9	Lie algebra	169
9.1	Lie group and Lie algebra	169
9.2	Adjoint representation	171
9.3	Killing form as a classification tool	171
9.4	Semi-simple and solvable algebras	173
9.5	$su(2)$	174
9.6	Roots of semi-simple algebras	175
9.7	Geometry of the Cartan algebra	177
9.8	Cartan matrix	183
9.9	Dynkin diagrams	186
9.10	Classical Lie algebras	187
	9.10.1 $su(l+1)$	188
	9.10.2 $so(2l)$	188
	9.10.3 $so(2l+1)$	190
	9.10.4 $sp(2l)$	190
9.11	Isomorphism of low-order algebras	191
	9.11.1 Remarks	191
9.12	Weyl and Coxeter groups	191
9.13	Extended Dynkin Diagrams	194
9.14	Weights	196
9.15	$su(3)$ representations and quark model	198
9.16	Weyl character formula	199
9.17	Proof of the Weyl character formula	203
9.18	An example in $su(3)$	204
9.19	Weyl denominator formula	206
9.20	Weyl dimensional formula	207
9.21	Kac-Moody algebra	208