# Introduction to Group Theory
# Note 3 Continuous Group

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## Contents


**1 Infinite Group** **1**

**2 Group Integration** **2**
- 2.1 Group invariant measure . . . 2
- 2.2 $SO(2)$ group. . . . 3
- 2.3 $SU(2)$ . . . 3

**3 Rotation Group $O(3)$** **4**
- 3.1 Homomorphism to $SU(2)$ Group . . . 4
- 3.2 Rotation about z-axis . . . 5
- 3.3 Rotation about y-axis . . . 6
- 3.4 Euler Angles . . . 6
- 3.5 Properties of $D^j_{m'm}(\alpha,\beta,\gamma)$ . . . 8
- 3.6 Basis Functions . . . 10

**4 Product Representations** **11**
- 4.1 Addition of Angular Momentum . . . 11
- 4.2 Clebsch-Gordon Coefficients . . . 12
- 4.3 Rotation group and Quantum Mechanics . . . 13

**5 Wigner-Eckart Theorem** **14**
- 5.1 Tensor operators . . . 14
- 5.2 Theorem$(Wigner-Eckart)$ . . . 15


## 1 Infinite Group

**Infinite group** : group which contains infinite number of elements.

It is convenient to label the group elements by one or more real parameters (group parameters)

$$A(\alpha_1, \alpha_2, \cdots \alpha_n) \qquad \alpha_1, \alpha_2, \cdots \alpha_n \quad \text{group parameters}$$

**Continuous group** : group parameters take continuous values.

**Compact group**: group parameters vary over some compact domain.

For example, 3-dimensional rotational group is compact because the group parametes, the angles of rotation, varies over compact interval $[0,2\pi]$, while Lorentz group is not compact because it has a group parameter, $\beta = \dfrac{v}{c}$ which varies over the non-compact interval, $[0,1)$.

Example: $SO(2)$ , set of rotations about a fixed axis( or $2\times 2$ real orthogonal matrices with unit determinant) and is an one-parameter continuous group. We can choose the group parameter to be the angle of rotation $\phi$, $0 \leq \phi \leq 2\pi$. Thus it is a compact group. Denote the group elements by, $R(\phi)$. The group multiplication is then

$$R(\phi_1)\, R(\phi_2) = R(\phi_1 + \phi_2)$$

We can visualize the group elements as points on the unit circle and are labeled by the angle. This is called the group parameter space.

# 2 Group Integration

## 2.1 Group invariant measure

In the theory of finite group, the rearrangement theorem,

$$\sum_{j=1}^{n} f(A_j) = \sum_{j=1}^{n} f(A_j B) = \sum_{j=1}^{n} f(BA_j), \qquad B, A_j \in G$$

plays an essential role in proving important theorems on the representation theory. In the continuous group, the summation over group elements is replaced by the group integration which is defined on the group parameter space,

$$\int dA \equiv \int W(\alpha_1, \cdots \alpha_n) \, d\alpha_1 \cdots d\alpha_n$$

where $W(\alpha_1, \cdots \alpha_n)$ is a measure (or weight function). In order to carry over the useful results of the finite group we want to define the group integration (or choose a measure $W$) in such a way that that the rearrangement theorem is still true. This is called the **group invariant integration** (measure). This means that the measure $W(\alpha_1, \cdots \alpha_n)$ should be chosen such that for arbitrary group element $B$ and arbitrary continuous function $u(A) \equiv u(\alpha_1, \cdots \alpha_n)$ defined over the group parameter space, we require

$$\int u(A) \, dA = \int u(AB) \, dA = \int u(BA) \, dA \qquad (1)$$

where

$$\int u(A) \, dA \equiv \int u(\alpha_1, \cdots \alpha_n) W(\alpha_1, \cdots \alpha_n) \, d\alpha_1 \cdots d\alpha_n$$

We shall adopt the notation,

$$A = A(\alpha_1, \cdots \alpha_n) = A\left(\vec{\alpha}\right), \qquad B = A(\beta_1, \cdots \beta_n) = A\left(\vec{\beta}\right)$$

Then

$$BA = A\left(\vec{\beta}\right) A\left(\vec{\alpha}\right) = A\left(\vec{\gamma}\right)$$

where

$$\vec{\gamma} = \vec{\gamma}\left(\vec{\beta}, \vec{\alpha}\right)$$

are some functions of group parameters, $\vec{\alpha}$ , $\vec{\beta}$ . These functions are determined by the group multiplication and contains all the information about the structure of the group. We write the integrations as,

$$dA = W\left(\vec{\alpha}\right) d^n\alpha = W(\alpha_1, \cdots \alpha_n) \, d\alpha_1 \cdots d\alpha_n$$

$$d(BA) = W\left(\vec{\gamma}\right) d^n\gamma = W(\gamma_1, \cdots \gamma_n) \, d\gamma_1 \cdots d\gamma_n$$

Note that left (or right) multiplication by a fixed group element, say $B$, is a 1-1 mapping of $G$ onto itself. Thus giving a set of group elements in some region $V$ of the parameter space, under the left (or right) multiplication by $B$, these elements will move to other region $V'$. Since the total number of element in $V$ is the same as those in $V'$, we get the relation,

$$\rho V = \rho' V',$$

where $\rho(\rho')$ is the density of elements at $V(V')$. This means that if we take the measure $W(\alpha_1, \cdots \alpha_n)$ to be the density of the group elements at $\vec{\alpha}$ we will have

$$W\left(\vec{\alpha}\right) d^n\alpha = W\left(\vec{\gamma}\right) d^n\gamma, \qquad \text{where } \vec{\gamma} = \vec{\gamma}\left(\vec{\beta}, \vec{\alpha}\right)$$

i.e. $W\left(\vec{\alpha}\right)$ is a group invariant measure.

To get the density of elements $W\left(\vec{\alpha}\right)$ we consider an infinitesmal volume element $V_0 = d\alpha_1 \cdots d\alpha_n$ in the neighborhood of the origin(identity) i.e. $I = A(0)$. Under left multiplication by $B$, they move to $V_1$,

$$W\left(\vec{\alpha}\right) V_0 = W\left(\vec{\gamma}\right) V_1$$

Or

$$\frac{W\left(\vec{\alpha}\right)}{W\left(\vec{\gamma}\right)} = \frac{V_1}{V_0}$$

Thus the ratio of the weight functions are determined by the ratio of the volume elements. We will normalize the density such that $W\left(\vec{0}\right) = 1$, i.e. density is 1 at origin. Setting $\vec{\alpha} = \vec{0}$, we get

$$W\left(\vec{\alpha}\right) = W\left(\vec{0}\right) = 1, \qquad \text{and} \quad W\left(\vec{\gamma}\right) = W\left(\vec{\beta}\right)$$

Then

$$W\left(\vec{\beta}\right) = \frac{V_0}{V_1}$$

Recall that the change in the volume elements under the transformation induced by $B = A\left(\vec{\beta}\right)$ is given by the Jacobian of the change of the variables $\vec{\alpha} \to \vec{\gamma}\left(\vec{\beta}, \vec{\alpha}\right)$. Thus we get

$$V_0 = d\alpha_1 \cdots d\alpha_n = \left.\frac{\partial\left(\alpha_1, \cdots \alpha_n\right)}{\partial\left(\gamma_1, \cdots \gamma_n\right)}\right|_{\vec{\alpha}=\vec{0}} d\gamma_1 \cdots d\gamma_n = \left.\frac{\partial\left(\alpha_1, \cdots \alpha_n\right)}{\partial\left(\gamma_1, \cdots \gamma_n\right)}\right|_{\vec{\alpha}=\vec{0}} V_1$$

Thus the invariant group measure is given by

$$W\left(\vec{\beta}\right) = \left[\left.\frac{\partial\left(\gamma_1, \cdots \gamma_n\right)}{\partial\left(\alpha_1, \cdots \alpha_n\right)}\right|_{\vec{\alpha}=\vec{0}}\right]^{-1} \tag{2}$$

## 2.2 $SO(2)$ group.

The group multiplication is given by

$$R(\alpha) R(\alpha) = R(\gamma), \qquad \text{with } \gamma = \alpha + \beta$$

The group invariant measure is

$$W(\beta) = \left[\frac{\partial\gamma}{\partial\alpha}\right]^{-1} = 1$$

Note that the group elements of $SO(2)$ are uniformly populated along the unit circle. This means that the points per unit length is same everywhere. This explains the feature $W(\beta) = 1$ in this case. The the group integration is then

$$\int_0^{2\pi} d\alpha$$

Since this is an Abelian group, all irreps are one-dimensional and of the form

$$e^{\pm im\alpha}, \qquad m = \text{integer}\, s$$

The great orthogonality theorem is of the form,

$$\int_0^{2\pi} d\alpha\, e^{im\alpha} \left(e^{im'\alpha}\right)^* = \delta_{mm'} 2\pi$$

If we had chosen any other measure $f(\alpha) \neq$ constant, then we would not have the great orthogonality theorem.

## 2.3 $SU(2)$

This is the group consists of $2 \times 2$ unitary matrice with determinant=1,

$$UU^\dagger = U^\dagger U = 1, \qquad \det U = 1$$

Write the unitary matrix $U$ as

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then the conditions

$$U^{-1} = U^{\dagger}, \text{ and } \det U = 1$$

imply that

$$a^* = d, \qquad c^* = -b, \qquad |a|^2 + |b|^2 = 1$$

The most general form a $2 \times 2$ unitary matrix can have is

$$U = \begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix}, \qquad \text{with} \quad |a|^2 + |b|^2 = 1$$

If we parametrize $a$ and $b$ in terms of real variables,

$$a = u_1 + iu_2, \qquad b = u_3 + iu_4, \qquad u_i \text{ real}$$

then we have the constraint

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$$

Thus group elementss of $SU(2)$ can be represented by points on the surface of a sphere in 4-dimensional space. This suggests the choice of invariant measure as

$$\int dR = \int du_1 \cdots du_4 \delta\left(u_1^2 + u_2^2 + u_3^2 + u_4^2 - 1\right) \tag{3}$$

It is intuitively clear that multiplication by an $SU(2)$ matrix corresponds to a rotation on the surface of the sphere and this measure is invariant. To show this explicitly, we need to prove that

$$\int dR' f(R) = \int dR f(R),$$

where $R' = SR$ and $f$ is some arbitrary function. Using the parametrizations,

$$R' = \begin{pmatrix} u_1' + iu_2' & -u_3' - iu_4' \\ u_3' - iu_4' & u_1' - iu_2' \end{pmatrix}, \qquad S = \begin{pmatrix} s_1 + is_2 & -s_3 - is_4 \\ s_3 - is_4 & s_1 - is_2 \end{pmatrix}, \qquad R = \begin{pmatrix} u_1 + iu_2 & -u_3 - iu_4 \\ u_3 - iu_4 & u_1 - iu_2 \end{pmatrix},$$

we get from the relation $R' = SR$,

$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{pmatrix} = \begin{pmatrix} s_1 & -s_2 & -s_3 & -s_4 \\ s_2 & s_1 & -s_4 & s_3 \\ s_3 & s_4 & s_1 & -s_2 \\ s_4 & -s_3 & s_2 & s_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

Using $s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1$, we see that the matrix which transform $u_i$ to $u_i'$ is an orthogonal matrix and the Jacobian is unity

$$J = \frac{\partial(u_1', u_2', u_3', u_4')}{\partial(u_1, u_2, u_3, u_4)} = \begin{vmatrix} s_1 & -s_2 & -s_3 & -s_4 \\ s_2 & s_1 & -s_4 & s_3 \\ s_3 & s_4 & s_1 & s_2 \\ s_4 & s_3 & -s_2 & s_1 \end{vmatrix} = 1$$

Then

$$\begin{aligned} du_1' \cdots du_4' \delta\left(u_1'^2 + u_2'^2 + u_3'^2 + u_4'^2 - 1\right) &= \frac{\partial(u_1', u_2', u_3', u_4')}{\partial(u_1, u_2, u_3, u_4)} du_1 \cdots du_4 \delta\left(u_1^2 + u_2^2 + u_3^2 + u_4^2 - 1\right) \\ &= du_1 \cdots du_4 \delta\left(u_1^2 + u_2^2 + u_3^2 + u_4^2 - 1\right) \end{aligned}$$

Thus group measure given in Eq (3) is invariant under left multiplication.

# 3 Rotation Group $O(3)$

## 3.1 Homomorphism to $SU(2)$ Group

The group elements of rotation group $O(3)$ will be denoted by $P_{\vec{n}}(\theta)$, the operator which rotates the system by angle $\theta$ about the axis $\vec{n}$. If we write

$$P_{\vec{n}}(\theta) = \exp\left(-i\frac{\theta \vec{J} \cdot \vec{n}}{\hbar}\right)$$

then $\vec{J}$ will be called the generator of rotation which correspond to infinitesmal rotation,

$$P_{\vec{n}}(\theta) \simeq 1 - i\frac{\theta \vec{J} \cdot \vec{n}}{\hbar}, \qquad \text{for } \theta \ll 1$$

We can also parametrize the rotation operator in terms of the familiar Euler rotations,

$$R(\alpha, \beta, \gamma) = P_z(\alpha) P_y(\beta) P_z(\gamma) = \exp\left(-\frac{iJ_z}{\hbar}\alpha\right) \exp\left(-\frac{iJ_y}{\hbar}\beta\right) \exp\left(-\frac{iJ_z}{\hbar}\gamma\right),$$

where $0 \le \alpha, \gamma \le 2\pi$, $0 \le \beta \le \pi$. We will now show that $O(3)$ is homomorphic to $SU(2)$ group. For any vector $\vec{r} = (x, y, z)$ in 3-dimensional space, we define a $2 \times 2$ hermitian matrix by

$$h = \vec{\sigma} \cdot \vec{r} = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

where $\vec{\sigma}$ are the Pauli matrices. It is easy to see that $h$ has the properties

$$Tr(h) = 0, \qquad \det h = -\left(x^2 + y^2 + z^2\right) = -r^2$$

Let $U$ any $2 \times 2$ unitary matrix $U$ and define a new matrix $h'$ by

$$h' = UhU^\dagger \tag{4}$$

It is clear that $h'$ is also hermitan, traceless and has the same determinant as $h$,

$$h' = (h')^\dagger, \qquad Tr(h') = 0, \qquad \det h' = \det h$$

If we expand $h'$ in terms of Pauli matrices,

$$h' = \vec{\sigma} \cdot \vec{r}'$$

then the relation between $\vec{r}$ and $\vec{r}'$ is just a 3-dimensional rotation. This is due to the equality of the determinants,

$$\det h = -\left(x^2 + y^2 + z^2\right) = \det h' = -\left(x'^2 + y'^2 + z'^2\right)$$

In other words, the relation between $\vec{r}$ and $\vec{r}'$ is a linear transformation and can be written as

$$r'_i = R_{ij} r_j$$

and $R$ is an orthogonal matrix. This establishes the correspondence between $2 \times 2$ unitary matrix with unit determinant and 3-dimensional rotation.

Note that from Eq(4) we see that $U$ and $-U$, give the same $h'$ and hence the same rotation. So the correspondence

$$\pm U \to R$$

is a homomorphism rather than isomorphism.

## 3.2 Rotation about z-axis

Suppose $U$ is diagonal. Then the general form is given by,

$$U = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

and

$$h' = \begin{pmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{pmatrix} = UhU^\dagger = \begin{pmatrix} z & (x - iy) e^{i\alpha} \\ (x + iy) e^{-i\alpha} & -z \end{pmatrix}$$

This gives the relation

$$\begin{cases} x' = \cos\alpha x + \sin\alpha y \\ y' = -\sin\alpha x + \cos\alpha y \\ z' = z \end{cases}$$

which is clearly a rotation around $z - axis$. Thus a diagonal $U$ given above corresponds to rotation about $z$-axis by an angle $\alpha$.

## 3.3 Rotation about y-axis

For the case $U$ is real, we can write

$$U=\begin{pmatrix}\cos\frac{\beta}{2} & \sin\frac{\beta}{2}\\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}$$

Then we get

$$\begin{aligned}h' &= \begin{pmatrix}\cos\frac{\beta}{2} & \sin\frac{\beta}{2}\\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} z & x-iy\\ x+iy & -z\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\\ &= \begin{pmatrix} z\cos\beta+x\sin\beta & x\cos\beta-iy-z\sin\beta\\ iy+x\cos\beta-z\sin\beta & -z\cos\beta-x\sin\beta\end{pmatrix}\end{aligned}$$

which is clearly a rotation about $y-axis$ by an angle $\beta$.

## 3.4 Euler Angles

The $2\times 2$ unitary matrix corresponding to rotation characterized by Euler angles, is then of the form,

$$\begin{aligned}U(\alpha,\beta,\gamma) &= P_z(\alpha)P_y(\beta)P_z(\gamma)=\begin{pmatrix} e^{-i\alpha/2} & 0\\ 0 & e^{i\alpha/2}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & \sin\frac{\beta}{2}\\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma/2} & 0\\ 0 & e^{i\gamma/2}\end{pmatrix} \quad (5)\\ &= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2}\\ e^{i(\alpha+\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix} a & -b\\ b^* & a^*\end{pmatrix}, \qquad \begin{matrix} a=e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2}\\ b=-e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2}\end{matrix}\end{aligned}$$

If we write

$$a=u_1+iu_2=e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2}, \qquad b=u_3+iu_4=e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \quad (6)$$

the group invariant integration can be converted to the integration over Euler angles by computing the Jacobian as follows. First we write

$$\int dR=\int du_1\cdots du_4\delta\left(u_1^2+u_2^2+u_3^2+u_4^2-1\right)=\int dR=\int du_1du_2du_3\frac{1}{2\left|u_4\right|}$$

and

$$\int du_1du_2du_3=\int d\alpha d\beta d\gamma J$$

The Jacobian can be computed from relations in Eq(6) to give

$$\begin{aligned}J &= \frac{\partial(u_1,u_2,u_3)}{\partial(\alpha,\beta,\delta)}=\frac{1}{8}\begin{bmatrix} -\sin(\alpha+\gamma)/2\cos\beta/2 & -\cos(\alpha+\gamma)/2\sin\beta/2 & -\sin(\alpha+\gamma)/2\cos\beta/2\\ -\cos(\alpha+\gamma)/2\cos\beta/2 & \sin(\alpha+\gamma)/2\sin\beta/2 & -\cos(\alpha+\gamma)/2\cos\beta/2\\ -\sin(\alpha-\gamma)/2\sin\beta/2 & \cos(\alpha-\gamma)/2\cos\beta/2 & \sin(\alpha-\gamma)/2\sin\beta/2\end{bmatrix}\\ &= \frac{1}{8}2\sin\frac{(\alpha-\gamma)}{2}\left[\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\right]\end{aligned}$$

and

$$\frac{J}{2\left|u_4\right|}=\frac{1}{16}d\alpha\sin\beta d\beta d\gamma$$

The integration measure in terms of Euler angles is of the form,

$$\int dR=\int du_1\cdots du_4\delta\left(u_1^2+u_2^2+u_3^2+u_4^2-1\right)=\frac{1}{16}\int_0^{2\pi}d\alpha\int_0^{\pi}\sin\beta d\beta\int_0^{2\pi}d\gamma$$

The factor $\frac{1}{16}$ is an overall normalization factor and is usually neglected for conveience.

The $2\times 2$ matrice of $SU(2)$ can be viewed as a rotations in the complex 2−dimensional space $C_2$. We can then use the induced transformation on functions of these 2–dimensional coordinates to generate other representationss. Let $\xi$ and $\eta$ be the basis vector for the $2\times 2$ matrice of $SU(2)$. Then under the $SU(2)$ transformation, we have

$$P_U(\xi,\eta) = (\xi,\eta)\,U = (\xi,\eta)\begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix} = (a\xi + b^*\eta, -b\xi + a^*\eta)$$

i.e.

$$\xi' = P_U\xi = a\xi + b^*\eta$$
$$\eta' = P_U\eta = -b\xi + a^*\eta$$

We will work out a simple case before going to the more general case. Consider a set of 3 mononomials of the form,

$$\phi_1^1 = \frac{1}{\sqrt{2}}\xi^2, \qquad \phi_1^0 = \xi\eta, \qquad \phi_1^{-1} = \frac{1}{\sqrt{2}}\eta^2$$

Under the $SU(2)$ transformation we get

$$\phi_1^1 \longrightarrow \phi_1'^1 = \frac{1}{\sqrt{2}}\xi'^2 = \frac{1}{\sqrt{2}}(a\xi + b^*\eta)^2 = a^2\phi_1^1 + \sqrt{2}ab^*\phi_1^0 + (b^*)^2\phi_1^{-1}$$

$$\phi_1^0 \longrightarrow \phi_1'^0 = \xi'\eta' = (a\xi + b^*\eta)(-b\xi + a^*\eta) = -\sqrt{2}ab\phi_1^1 + (-bb^* + a^*a^*)\phi_1^0 + \sqrt{2}a^*b^*\phi_1^{-1}$$

$$\phi_1^{-1} \longrightarrow \phi_1'^{-1} = \frac{1}{\sqrt{2}}\eta'^2 = \frac{1}{\sqrt{2}}(-b\xi + a^*\eta)^2 = b^2\phi_1^1 - \sqrt{2}a^*b\phi_1^0 + (a^*)^2\phi_1^{-1}$$

In matrix form, we see

$$\begin{pmatrix} \phi_1'^1 & \phi_1'^0 & \phi_1'^{-1} \end{pmatrix} = \begin{pmatrix} \phi_1^1 & \phi_1^0 & \phi_1^{-1} \end{pmatrix}\begin{pmatrix} a^2 & -\sqrt{2}ab & b^2 \\ \sqrt{2}ab^* & (aa^* - bb^*) & -\sqrt{2}a^*b \\ (b^*)^2 & \sqrt{2}a^*b^* & (a^*)^2 \end{pmatrix} = \begin{pmatrix} \phi_1^1 & \phi_1^0 & \phi_1^{-1} \end{pmatrix} D^{(1)}$$

In terms of Euler angles we have

$$D^{(1)} = \begin{pmatrix} \frac{1}{2}(1+\cos\beta)e^{-i(\alpha+\gamma)} & -\frac{1}{\sqrt{2}}\sin\beta e^{-i\alpha} & \frac{1}{2}(1-\cos\beta)e^{-i(\alpha-\gamma)} \\ \frac{1}{\sqrt{2}}\sin\beta e^{-i\gamma} & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta e^{i\gamma} \\ \frac{1}{2}(1-\cos\beta)e^{i(\alpha-\gamma)} & \frac{1}{\sqrt{2}}\sin\beta e^{i\alpha} & \frac{1}{2}(1+\cos\beta)e^{i(\alpha+\gamma)} \end{pmatrix}$$

These matrices form a representation of the rotation group as we have discussed in Note 2. It is not hard to see that this can be written as

$$D^{(1)} = \begin{pmatrix} e^{-i\alpha} & & \\ & 1 & \\ & & e^{i\alpha} \end{pmatrix}\begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}\begin{pmatrix} e^{-i\gamma} & & \\ & 1 & \\ & & e^{i\gamma} \end{pmatrix}$$

We see that the rotation about z-axis is just multiplication of phases. Furthermore the rotation around y-axis can be written as

$$\begin{aligned} d^{(1)}(\beta) &= \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix} \\ & \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

This shows that it is related to the usual rotation of $(x,y,z)$ by a similarity transformation. This similarity transformation will take the Cartesian coordinates $(x,y,z)$ to spherical basis $\left(-\frac{x+iy}{\sqrt{2}}, z, \frac{x-iy}{\sqrt{2}}\right)$. In other words, the spherical basis give the irreducible representation of rotation group in the standard form. The rotation about the z-axis can also be transformed by similarity transformation into the more familiar form,

$$R_z(\alpha) = \begin{pmatrix} e^{-i\alpha} & & \\ & 1 & \\ & & e^{i\alpha} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

For more general case, consider the action of $P_U$ on the mononomial $\xi^\lambda \eta^\mu$, with $\lambda+\mu=2j$, where $j$ is an integer or half integer. It is more convenient to write it as

$$\phi_m^j = \frac{1}{n_{jm}} (\xi)^{j+m} (\eta)^{j-m}, \qquad m=-j,-j+1,\cdots,j$$

where $n_{jm}$ is a normalization factor to be determined later. Since the transformation from $(\xi,\eta)$ to $(\xi',\eta')$ is linear and homogeneous, the transform of $\phi_m^j$ will have the same value of $j$ but different values of $m$. These functions will be used to generate representation matrices. More explicitly,

$$\begin{aligned} P_U \phi_m^j &= \frac{1}{n_{jm}} (a\xi + b^*\eta)^{j+m} (-b\xi + a^*\eta)^{j-m} \\ &= \frac{1}{n_{jm}} \sum_{s=0}^{j-m} \sum_{r=0}^{j+m} \left[\frac{(j+m)!}{r!\,(j+m-r)!}\right] \left[\frac{(j-m)!}{s!\,(j-m-s)!}\right] (-b)^s (a^*)^{j-m-s} a^r (b^*)^{j+m-r} \xi^r \xi^s (\eta)^{j+m-r} (\eta)^{j-m-s} \end{aligned}$$

Define $m'$ by

$$s = j+m'-r, \qquad j \le m' \le j$$

Then

$$P_U \phi_m^j = \sum_{r,m'} \left(\frac{n_{jm'}}{n_{jm}}\right) \left[\frac{(j+m)!\,(j-m)!\,(-1)^{j+m'-r}}{r!\,(j+m-r)!\,(j+m'-r)!\,(r-m-m')!}\right] a^r (b^*)^{j+m-r} (b)^{j+m'-r} (a^*)^{r-m-m'} \phi_{m'}^j$$

In terms of Euler angles we have

$$a = e^{-i(\alpha+\gamma)/2} \cos\frac{\beta}{2}, \qquad b = e^{-i(\alpha-\gamma)/2} \sin\frac{\beta}{2}$$

and we can write

$$P_U \phi_m^j = \sum_{m'} \phi_{m'}^j D_{m'm}^j (\alpha,\beta,\gamma)$$

where

$$D_{m'm}^j (\alpha,\beta,\gamma) = e^{-im'\alpha} d_{m'm}^j (\beta)\, e^{-im\gamma} \tag{7}$$

with

$$\begin{aligned} d_{m'm}^j (\beta) \quad &= \quad \frac{(j+m)!\,(j-m)!\,n_{jm'}}{n_{jm}} \sum_k \frac{(-1)^{j+m'-k}}{k!\,(j+m-k)!\,(j+m'-k)!\,(k-m-m')!} \\ & \qquad \left(\cos\frac{\beta}{2}\right)^{2k-m-m'} \left(\sin\frac{\beta}{2}\right)^{2j-2k+m+m'} \end{aligned} \tag{8}$$

Note that the sum over $k$ covers all thos values for which the argument of the factorial functions are positive.

### 3.5 Properties of $D_{m'm}^j (\alpha,\beta,\gamma)$

1. $D_{m'm}^j (\alpha,\beta,\gamma)$ 's form $(2j+1)$ dimensional representation of $SU(2)$. This follows from the fact that the group induced transformation always generates a representation of the group as discussed in Note 2.
2. $D_{m'm}^j (0,0,0) = \delta_{mm'}$, the identiy matrix. To see this we set $\beta = 0$ in Eq(8) and the non-zero term is where $2k = 2j+m+m'$. This implies that

$$j+m-k = \frac{1}{2}(m-m'), \qquad j+m'-k = \frac{1}{2}(m'-m)$$

   and the positivity of the arguments of the factorial functions gives $m=m'$ and $d_{m'm}^j(0) = \delta_{mm'}$.
3. For the matrix $D_{m'm}^j (\alpha,\beta,\gamma)$ to be unitary, we require

$$D_{m'm}^j (\alpha,\beta,\gamma)^\dagger = D_{m'm}^j (-\gamma,-\beta,-\alpha) \qquad \Rightarrow \qquad d_{m'm}^j(-\beta) = d_{mm'}^j(\beta)$$

   It is straightforward to show that this fixes the constant $n_{jm}$ to be

$$n_{jm} = \sqrt{(j+m)!\,(j-m)!}$$

   and the $d-$finction is then

$$d_{m'm}^j(\beta) = \sum_k \frac{(-1)^{j+m'-k} \sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{k!\,(j+m-k)!\,(j+m'-k)!\,(k-m-m')!} \left(\cos\frac{\beta}{2}\right)^{2k-m-m'} \left(\sin\frac{\beta}{2}\right)^{2j-2k+m+m'} \tag{9}$$

4. For each value of $j$, the representation $D^j_{m'm}(\alpha,\beta,\gamma)$ is irreducible. This can be seen as follows. Suppose there exists a matrix $M$ such that
$$MD^j(\alpha,\beta,\gamma) = D^j(\alpha,\beta,\gamma)M \tag{10}$$
We consider following cases:

   (a) $\alpha \neq 0, \beta = \gamma = 0$
   From Eq(9), we see that as before only non-zero term in the summation is where
   $$2j - 2k + m + m' = 0$$
   which gives
   $$d^j_{m'm}(0) = \delta_{mm'}$$
   Then
   $$D^j_{m'm}(\alpha,0,0) = e^{-im'\alpha}\delta_{mm'}$$
   and Eq(10) implies that
   $$\left(e^{-im\alpha} - e^{-im'\alpha}\right)M_{mm'} = 0$$
   Thus $M_{mm'} = 0$ if $m \neq m'$, i.e. $M$ is diagonal.

   (b) $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$
   Since we have already shown that $M$ is diagonal, Eq(10) gives
   $$M_{mm}D^j_{m'm} = D^j_{m'm}M_{m'm'} \qquad \text{(no sum)}$$
   Or
   $$D^j_{m'm}(M_{mm} - M_{m'm'}) = 0$$
   But for arbitrary $(\alpha,\beta,\gamma)$, $D^j_{m'm}$ is not zero. Thus $M_{mm} = M_{m'm'}$, or $M$ is a multiple of identity. Schur's lemma implies that $D^j_{m'm}(\alpha,\beta,\gamma)$ is irreducible.

5. If we replace $\beta$ by $\beta + 2\pi$, we see that from Eq(5) that the $2\times 2$ matrix $U$ has the property,
$$U \to -U$$
For the representation matrice this implies that
$$d^j(\beta + 2\pi) = (-1)^{2j}d^j(\beta), \qquad \text{or} \qquad D^j(-U) = (-1)^{2j}D^j(U)$$
Thus for $j$ =integer $\quad D^j(-U) = D^j(U) \quad$ single valued representation
$j$ = half integer $\quad D^j(-U) = D^j(U) \quad$ double valued representation

6. Representation of the generators $J_i$, $i = 1,2,3$
Recall that
$$R(\alpha,\beta,\gamma) = \exp\left(-\frac{iJ_z}{\hbar}\alpha\right)\exp\left(-\frac{iJ_y}{\hbar}\beta\right)\exp\left(-\frac{iJ_z}{\hbar}\gamma\right)$$
Take $\beta = \gamma = 0$, $\alpha \ll 1$, we get
$$R(\alpha,\beta,\gamma) \simeq \left(1 - \frac{iJ_z}{\hbar}\alpha\right)$$
From
$$D^j_{m'm}(\alpha,0,0) \simeq (1 - im'\alpha)\delta_{mm'}$$
we get
$$D^j_{m'm}(J_z) = \hbar m\delta_{mm'}$$
Similarly, take $\alpha = -\gamma = -\dfrac{\pi}{2}$, and $\beta$ small, we can get $D(J_x)$
$\alpha = \gamma = 0 \quad$ and $\beta$ small, we can get $D(J_y)$
In particular, for $j = 1/2$, we have
$$D^{1/2}(j_x) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad D^{1/2}(j_y) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad D^{1/2}(j_x) = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
which are just the Pauli matrices up to the factor $\dfrac{\hbar}{2}$.

7. Great Orthogonality Theorem reads

$$\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta d\beta \int_0^{\pi} d\gamma D^j_{m'm}(\alpha,\beta,\gamma)\, D^{k*}_{n'n}(\alpha,\beta,\gamma) = \delta_{jk}\delta_{mn}\delta_{m'n'}\frac{\int_0^{2\pi} d\alpha \int_0^{\pi}\sin\beta d\beta \int_0^{\pi} d\gamma}{(2j+1)}$$

Using $D^j_{m'm}(\alpha,\beta,\gamma)$ given in Eq(7) we can integrate over angles $\alpha$ and $\beta$ to reduce the integral to orthogonality relation on

$$\int_0^{\pi} \sin\beta d\beta d^j_{m'm}(\beta)\, d^k_{m'm}(\beta) = \frac{2\delta_{jk}}{(2j+1)}$$

8. Character of irreps
Since all rotations of same angle about any axis are in the same class, we can choose the rotation about the z-axis to compute the trace

$$\chi^{(j)}(\theta) = Tr\left[D^j\left(\vec{n},\theta\right)\right] = Tr\left[D^j(\theta,0,0)\right] = \sum_{m=-j}^{j} e^{-im\theta} = \frac{\sin\left(j+\dfrac{1}{2}\right)\theta}{\sin\dfrac{\theta}{2}}$$

Note that

$$\chi^{(j)}(\theta) - \chi^{(j-1)}(\theta) = 2\cos j\theta$$

**Theorem:** There are no irreps of $SU(2)$ group other than $D^j(\alpha,\beta,\gamma)$.

Proof: Suppose $D$ is another irrep with character$\chi(\theta)$ and not contained in $D^j$. Then we have $\chi(\theta) = \chi(-\theta)$ since both rotations having the same angle are in the same class. Then from orhogonality theorem $\chi(\theta)$ is perpendicular to $\chi^{(j)}(\theta)$ for all $j$. This also implies that $\chi(\theta)$ is perpendicular to $\chi^{(j)}(\theta) - \chi^{(j-1)}(\theta) = 2\cos j\theta$. From the property of Fourier series, we know that $\cos j\theta$, $2j = 1, 2, 3, \cdots$ form a complete set of even functions in the range $0 < \theta < \pi$. Thus $\chi(\theta) = 0$. for all $\theta$. ■

## 3.6 Basis Functions

Suppose we define for $j = l =$integer

$$D^{(l)}_{0m}(\alpha,\beta,\gamma) = \sqrt{\frac{4\pi}{2l+1}} Y^m_l(\beta,\gamma)$$

then we have

$$D^{(l)}_{m0}(\alpha,\beta,\gamma) = (-1)^m \sqrt{\frac{4\pi}{2l+1}} Y^{m*}_l(\beta,\alpha)$$

Suppose we have the following relation for the multiplication of group elements

$$R(\alpha_2,\beta_2,\gamma_2)\, R(\alpha_1,\beta_1,\gamma_1) = R(\alpha,\beta,\gamma)$$

The corresponding representation matrices will satisfy

$$D^l_{m'm}(\alpha,\beta,\gamma) = \sum_{m"} D^l_{m'm"}(\alpha_2,\beta_2,\gamma_2)\, D^l_{m"m}(\alpha_1,\beta_1,\gamma_1)$$

Setting $m' = 0$, the relation is

$$Y^m_l(\beta,\gamma) = \sum_{m"} Y^{m"}_l(\beta_2,\gamma_2)\, D^l_{m"m}(\alpha_1,\beta_1,\gamma_1)$$

Thus the functions $\{Y^m_l(\beta,\gamma)\}$ form the basis for the irrep $D^l(\alpha,\beta,\gamma)$. We can rewrite this in a more familiar spherical angles notation as,

$$P_R Y^m_l(\theta,\phi) = Y^m_l\left(\theta',\phi'\right) = \sum_{m"} Y^{m"}_l(\theta,\phi)\, D^l_{m"m}(\alpha,\beta,\gamma) \tag{11}$$

where

$$\cos\theta' = \cos\theta\cos\beta + \sin\theta\sin\phi\cos(\phi-\gamma)$$

**Example :** $l = 1$

$$Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} = -\sqrt{\frac{3}{8\pi}}\frac{(x+iy)}{r}, \qquad Y_1^0(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\cos\theta = \sqrt{\frac{3}{8\pi}}\frac{z}{r},$$

$$Y_1^{-1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} = -\sqrt{\frac{3}{8\pi}}\frac{(x-iy)}{r}$$

For the special case of $m = 0$ we have

$$Y_l^0(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$

and from Eq(11) we get the addition theorem,

$$P_l(\cos\theta') = \left(\frac{4\pi}{2l+1}\right)\sum_m Y_l^m(\theta,\phi)\,Y_l^{m*}(\beta,\gamma)$$

where

$$\cos\theta' = \cos\theta\cos\beta + \sin\theta\sin\phi\cos(\phi-\gamma)$$

# 4 Product Representations

## 4.1 Addition of Angular Momentum

Suppose $D^{(j_1)}$ and $D^{(j_2)}$ are 2 irreps of $SU(2)$ group. We want to know how to reduce the product representation $D^{(j_1)} \otimes D^{(j_2)} \equiv D^{(j_1\times j_2)}$, i.e. write $D^{(j_1\times j_2)}$ as sum of irreps. Using the characters of irreps (take $j_1 > j_2$), we have

$$\begin{aligned}
\chi^{(j_1\times j_2)}(\theta) &= \chi^{(j_1)}(\theta)\,\chi^{(j_2)}(\theta) = \frac{\sin\left(j_1+\frac{1}{2}\right)\theta\sin\left(j_2+\frac{1}{2}\right)\theta}{\sin^2\frac{\theta}{2}} = \frac{\cos(j_1-j_2)\theta - \cos(j_1+j_2+1)\theta}{\sin^2\frac{\theta}{2}} \\
&= \frac{1}{\sin^2\frac{\theta}{2}}\left\{\begin{array}{c}[\cos(j_1-j_2)\theta - \cos(j_1-j_2+1)\theta] + [\cos(j_1-j_2+1)\theta - \cos(j_1-j_2+2)\theta] \\ +\cdots[\cos(j_1+j_2)\theta - \cos(j_1+j_2+1)\theta]\end{array}\right\} \\
&= \frac{1}{\sin\frac{\theta}{2}}\left\{\sin\left(j_1-j_2+\frac{1}{2}\right)\theta + \sin\left(j_1-j_2+\frac{3}{2}\right)\theta + \cdots\sin(j_1+j_2)\theta\right\}
\end{aligned}$$

We see that the character of the product representation $D^{(j_1)} \otimes D^{(j_2)}$ has the decomposition,

$$\chi^{(j_1\times j_2)}(\theta) = \sum_{J=|j_1-j_2|}^{j_1+j_2}\chi^{(J)}(\theta)$$

which implies the following reduction,

$$D^{(j_1\times j_2)} = D^{(j_1)} \otimes D^{(j_2)} = \sum_{J=|j_1-j_2|}^{j_1+j_2} D^{(J)} \tag{12}$$

We can relate this to the addition of angular momenta as follows. Let $f_{j_1}^{m_1}(\hat{r})$ be the basis for $D^{(j_1)}$ irrep and $f_{j_2}^{m_2}(\hat{r})$ be the basis for $D^{(j_2)}$. Then the product $f_{j_1}^{m_1}(\hat{r})\,f_{j_2}^{m_2}(\hat{r})$ are the basis for the representation $D^{(j_1)} \otimes D^{(j_2)}$, i.e.

$$P_R\left[f_{j_1}^{m_1}f_{j_2}^{m_2}\right] = \left[P_R f_{j_1}^{m_1}\right]\left[P_R f_{j_2}^{m_2}\right] = \left(\sum_{m_1'} f_{j_1}^{m_1'}D_{m_1m_1'}^{(j_1)}\right)\left(\sum_{m_2'} f_{j_2}^{m_2'}D_{m_2m_2'}^{(j_2)}\right)$$

For infinitesmal rotation around, say z-axis, we can write, (setting $\hbar = 1$ )

$$P_R = e^{-i\alpha J_z} \simeq (1-i\alpha J_z), \qquad \alpha \ll 1$$

Let $\vec{J}_1$ be the operator acting on $f_{j_1}^{m_1}$ and leaving $f_{j_2}^{m_2}$ alone and $\vec{J}_2$ be the operator acting on $f_{j_2}^{m_2}$ and leaving $f_{j_1}^{m_1}$ alone.

Then we can write

$$\begin{aligned}
P_R\left[f_{j_1}^{m_1}f_{j_2}^{m_2}\right] &= \left[(1-i\alpha J_z)\,f_{j_1}^{m_1}\right]\left[(1-i\alpha J_z)\,f_{j_2}^{m_2}\right] \simeq [1-i\alpha(J_{1z}+J_{2z})]\,f_{j_1}^{m_1}f_{j_2}^{m_2} \\
&= (1-i\alpha J_z)\,f_{j_1}^{m_1}f_{j_2}^{m_2}
\end{aligned}$$

where

$$J_z = J_{1z} + J_{2z}$$

We can extend this to other components to write

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

which is just the total angular momentum. This means that the infinitesmal transformation of the product of the basis can be written in terms of total angular momenta. From Eq(12) we see that decomposition of the product of irreps is related to the additon of angular momenta.

## 4.2 Clebsch-Gordon Coefficients

In the decomposition of product representations we have

$$D^{(j_1 \times j_2)} = D^{(j_1)} \otimes D^{(j_2)} = D^{(|j_1-j_2|)} + D^{(|j_1-j_2|+1)} + \cdots + D^{(j_1+j_2)}$$

This means that the representation matrices $D^{(j_1 \times j_2)}$ can be transform by a similarity transformation into a direct sum of irreps $D^j$. In terms of matrice elements we have

$$D^{(j_1 \times j_2)}_{m_1' m_2'; m_1 m_2} = D^{(j_1)}_{m_1 m_1'} D^{(j_2)}_{m_2 m_2'}$$

On the other hand,

$$\sum_J D^J \equiv \Delta = \begin{pmatrix} D^{(|j_1-j_2|)} & & & \\ & D^{(|j_1-j_2|+1)} & & \\ & & \ddots & \\ & & & D^{(j_1+j_2)} \end{pmatrix} \equiv \Delta_{J'M';JM} = \delta_{JJ'} D^J_{MM'}$$

where $J, J'$ label the boxes and $M, M'$ labels the row and colums within each box. Let $A$ be the unitary matrix which transform $\Delta$ into $D^{(j_1)} \otimes D^{(j_2)}$,

$$D^{(j_1)} \otimes D^{(j_2)} = A^\dagger \Delta A$$

Writing out in terms of matrix elements this reads,

$$D^{(j_1 \times j_2)}_{m_1' m_2'; m_1 m_2} = \left[A^\dagger\right]_{m_1' m_2'} {}_{;J'M'} \Delta_{J'M;JM} \left[A\right]_{JM; m_1 m_2} \tag{13}$$

We adopt a new notation for the matrix elements of the similarity transformation,

$$[A]_{JM; m_1 m_2} \equiv \langle JM | j_1 m_1 j_2 m_2 \rangle$$

These are usually called **Clebsch-Gordon coefficients**. Since $A$ is unitary, $AA^{\dagger :} = 1$, and $A^\dagger A = 1$ we have

$$\sum_{J'M'} \langle j_1 m_1' j_2 m_2' | J'M' \rangle \langle J'M' | j_1 m_1 j_2 m_2 \rangle = \delta_{m_1' m_1} \delta_{m_2' m_2}$$

$$\sum_{m_1 m_2} \langle JM | j_1 m_1 j_2 m_2 \rangle \langle j_1 m_1 j_2 m_2 | J'M' \rangle = \delta_{JJ'} \delta_{MM'}$$

Note that the unitary matrix $A$ is not uniquely defined. Let $B$ be a unitary matrix which commutes with $\Delta$,

$$B\Delta = \Delta B, \qquad \text{or} \qquad \Delta = B^\dagger \Delta B$$

Then

$$D = A^\dagger \Delta A = (BA)^\dagger \Delta BA$$

This means that if $A$ block diagonalizes $D$, so does $A' = BA$. Since $\Delta$ is of the form

$$\Delta = \begin{pmatrix} D^{(J_1)} & & & \\ & D^{(J_2)} & & \\ & & \ddots & \\ & & & D \end{pmatrix}$$

From Schur's lemma, $B$ must be of the form,

$$B = \begin{pmatrix} c_1 I_1 & & & \\ & c_2 I_2 & & \\ & & \ddots & \\ & & & c_n I_n \end{pmatrix}, \qquad \text{where} \quad c_i = e^{i\theta_r}$$

Or

$$B_{JM;J'M'} = \delta_{JJ'}\delta_{MM'}e^{i\theta_j}$$

and

$$[A']_{JM;m_1m_2} = e^{i\theta_J}\,[A]_{JM;m_1m_2}$$

Thus the similarity transformation is defined up to $J$ dependent phases.

**Condon-Shortly convention** : If we choose

$$\langle JJ|j_1 j_1 j_2 J - j_1\rangle$$

to be real and positive, then it turns out that all Clebsch-Gordon coefficients are real.

In terms of Clebsch-Gordon coefficients Eq 13) becomes

$$D^{(j_1)}_{m_1m_1'}D^{(j_2)}_{m_2m_2'} = \sum_{J,M,M'} D^J_{M'M}\,\langle JM'|j_1m_1'j_2m_2'\rangle\,\langle JM|j_1m_1j_2m_2\rangle$$

**Theorem**: Let $\psi^{m_1}_{j_1}(\hat{r})$ be the basis for $D^{(j_1)}$ irrep and $\psi^{m_2}_{j_2}(\hat{r})$ be the basis for $D^{(j_2)}$. Then

$$\phi^J_M = \sum_{m_1m_2}\langle JM|j_1m_1j_2m_2\rangle\,\psi^{m_1}_{j_1}\psi^{m_2}_{j_2}$$

are basisi for $D^J$.

Proof:

$$\begin{aligned} P_R\phi^J_M &= \sum_{m_1m_2}\langle JM|j_1m_1j_2m_2\rangle\left(P_R\psi^{m_1}_{j_1}\right)\left(P_R\psi^{m_2}_{j_2}\right) = \sum_{m_1m_2}\langle JM|j_1m_1j_2m_2\rangle\,D^{(j_1)}_{m_1m_1'}D^{(j_2)}_{m_2m_2'}\psi^{m_1'}_{j_1}\psi^{m_2'}_{j_2} \\ &= \sum_{m_1m_2}\langle JM|j_1m_1j_2m_2\rangle\sum_{J,M,M'}D^{J'}_{M'M"}\,\langle J'M'|j_1m_1'j_2m_2'\rangle\,\langle J'M'|j_1m_1j_2m_2\rangle\,\psi^{m_1'}_{j_1}\psi^{m_2'}_{j_2} \\ &= \sum_{M"m_1m_2}D^J_{M"M}\,\langle JM"|j_1m_1'j_2m_2'\rangle\,\psi^{m_1'}_{j_1}\psi^{m_2'}_{j_2} \end{aligned}$$

Thus $\phi^J_M$ does transform according to the representation $D^J$.■

As a consequence if $\vec{j}_3 = \vec{j}_1 + \vec{j}_2$ then

$$\sum_{m_1m_2m_3}\psi^{m_1}_{j_1}\psi^{m_2}_{j_2}\psi^{m_3}_{j_3}\,\langle j_1m_1j_2m_2|j_3,-m_3\rangle\,\langle j_3,-m_3j_3m_3|00\rangle$$

is invariant under $SU(2)$ transformations.

## 4.3 Rotation group and Quantum Mechanics

In quantum mechanics we implement symmetry transformations by unitary operator $U$ on the state $|\psi\rangle$

$$|\psi\rangle \longrightarrow |\psi'\rangle = U\,|\psi\rangle\,, \qquad \text{for all states}$$

so that

$$\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle$$

Alternatively we can put the unitary operator on the hermitian operators which correspond to physical observables,

$$A \longrightarrow A' = UAU^\dagger$$

If $U' = U$, or $[U, A] = 0$, we say that $A$ is invariant under the transformation $U$. In particular if Hamiltonian $H$ is invariant under the symmetry transformation $U$ we have

$$[U, H] = 0 \tag{14}$$

Suppose $|\psi\rangle$ is an eigenstate of $H$ with energy $E$,

$$H|\psi\rangle = E|\psi\rangle$$

Then from Eq(14) we see that

$$HU|\psi\rangle = UH|\psi\rangle = E(U|\psi\rangle)$$

which means that $U|\psi\rangle$ is also an eigenstate of $H$ with same energy $E$. If we run the operator $U$ through the whole group $G$ the states we obtain will form the basis of irreducible representation of the group $G$. In other words, the degenercy of a energy eigenstate correspponds to the dimensionality of some irrep.

# 5 Wigner-Eckart Theorem

## 5.1 Tensor operators

Suppose $D^{(\alpha)}$ is an irrep of group $G$ and dimension of $D^{(\alpha)}$ is $d_\alpha$. A set of operators $T_i^{(\alpha)}, i = 1, 2, \cdots d_\alpha$, transforming under the symmetry group $G$ as

$$P_R T_i^{(\alpha)} P_R^{-1} = \sum_j T_j^{(\alpha)} D_{ji}^{(\alpha)}(R)$$

is said to be irreducible tensor operators corresponding to $D^{(\alpha)}$ irreps.

For the case of $SO(3)$ (or $SU(2)$ group) , if the operators $T_j^m, m = -j, -j+1, \cdots j$ satisfy the relation,

$$P_R(\alpha,\beta,\gamma) T_i^m P_R^{-1}(\alpha,\beta,\gamma) = \sum_{m'} T_i^{m'} D_{m'm}^j(\alpha,\beta,\gamma)$$

then we say $T_j^m$ are irreducible tensors of rank $j$ in $SO(3)$. In terms of generators, we have

$$\left[J_z, T_j^k\right] = kT_j^k$$

$$\left[J_\pm, T_j^k\right] = \sqrt{(j \mp k)(j \pm k + 1)} T_j^{k\pm 1}$$

**Remarks :**

1. If $T_{j_1}^{m_1}$ are irreducible tensor of rank $j_1$ and $T_{j_2}^{m_2}$ are irreducible tensor of rank $j_2$ then

$$\sum_{m_1,m_2} \langle jm|j_1m_1j_2m_2\rangle T_{j_1}^{m_1} T_{j_2}^{m_2}$$

are irreducible tensor of rank $j$. In particular, since

$$\langle 00|jmj-m\rangle = \frac{(-1)^{j-m}}{\sqrt{2j+1}}$$

we see that the combination

$$\sum_m T_j^m S_j^{-m} (-1)^m$$

is an invariant operator under rotations. Here $S_j^m$ is another tensor operator of rank $j$.

2. If $T_j^m$ are irreducible tensor operator of rank $j$, so is

$$S_j^m = (-1)^m \left[T_j^{-m}\right]^\dagger$$

This follows from the fact that

$$D_{m'm}^{j*}(\alpha,\beta,\gamma) = (-1)^{m'-m} D_{-m',-m}^{j*}(\alpha,\beta,\gamma)$$

3. Combining remarks (1) and (2) we get that

$$\sum_m T_j^m \left(T_j^m\right)^\dagger$$

is a $SO(3)$ invariant operator.

## 5.2 Theorem$(Wigner - Eckart)$

If $\phi_j^m$ are basis for $D^j$ , $\phi_{j'}^{m'}$ are basis for $D^{j\prime}$ and $T_k^q$ are irreducible tensor operator of rank $k$, then

$$\left\langle \phi_{j'}^{m'}|T_k^q|\phi_j^m\right\rangle = \langle j'm'|kqjm\rangle \frac{\left\langle \phi_{j'}||T_k||\phi_j\right\rangle}{\sqrt{2j+1}}$$

where $\left\langle \phi_{j'}||T_k||\phi_j\right\rangle$ are the <u>reduced matrix elements</u> and are independent of $m, m', q$.

Proof:

Since $T_k^q$ are irreducible tensor operator, we can write

$$\begin{aligned}\left\langle \phi_{j'}^{m'}|T_k^q|\phi_j^m\right\rangle &= \left\langle P_R\phi_{j'}^{m'}|P_R T_k^q P_R^{-1}|P_R\phi_j^m\right\rangle \\ &= \sum_{l,l'q',} D_{l'm'}^{j*}(R) D_{lm}^j(R) \left\langle \phi_{j'}^{l'}|T_k^{q'}|\phi_j^l\right\rangle D_{q'q}^k(R) \\ &= \sum_{J,M,M'l,l'q',} D_{l'm'}^{j*}(R) D_{M'M}^J(R) \langle JM'|kq'jl'\rangle \langle JM|kqjm\rangle \left\langle \phi_{j'}^{l'}|T_k^{q'}|\phi_j^l\right\rangle\end{aligned}$$

Summing over all group elements in $SU(2)$, we get

$$\int dR \left\langle \phi_{j'}^{m'}|T_k^q|\phi_j^m\right\rangle = \left\langle \phi_{j'}^{m'}|T_k^q|\phi_j^m\right\rangle \int dR = \int dR \sum_{J,M,M'l,l'q',} D_{l'm'}^{j*}(R) D_{M'M}^J(R) \langle JM'|kq'jl'\rangle \langle JM|kqjm\rangle \left\langle \phi_{j'}^{l'}|T_k^{q'}|\phi_j^l\right\rangle$$

we have used the fact $\left\langle \phi_{j'}^{m'}|T_k^q|\phi_j^m\right\rangle$ is independent of $R$. Using great orthogonality theorem

$$\int dR D_{l'm'}^{j*}(R) D_{M'M}^J(R) = \delta_{Jj'}\delta_{l'M'}\delta_{m'M} \int dR$$

we get

$$\begin{aligned}\left\langle \phi_{j'}^{m'}|T_k^q|\phi_j^m\right\rangle &= \langle j'm'|kqjm\rangle \sum_{l,l'q'} \langle j'l'|kq'jl\rangle \left\langle \phi_{j'}^{l'}|T_k^{q'}|\phi_j^l\right\rangle \\ &= \langle j'm'|kqjm\rangle \frac{\left\langle \phi_{j'}||T_k||\phi_j\right\rangle}{\sqrt{2j+1}}\end{aligned}$$

where

$$\left\langle \phi_{j'}||T_k||\phi_j\right\rangle = \sqrt{2j+1} \sum_{l,l'q'} \langle j'l'|kq'jl\rangle \left\langle \phi_{j'}^{l'}|T_k^{q'}|\phi_j^l\right\rangle$$

is the reduced matrix element and is independent of $m, m\prime, q$.

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**Remarks**:

1. The $m, m\prime, q$ dependence in the matrix element $\left\langle \phi_{j'}^{m'}|T_k^q|\phi_j^m\right\rangle$ are all contained in the Clebsch-Gordon coefficients $\langle j'm'|kqjm\rangle$ which are universal and independent of the details of $\phi_j^m, \phi_{j'}^{m'}$ and $T_k^q$. In other words, in the ratios of matrix elements we have

$$\frac{\left\langle \phi_{j'}^{m_2}|T_k^q|\phi_j^{m_1}\right\rangle}{\left\langle \phi_{j'}^{m_4}|T_k^q|\phi_j^{m_3}\right\rangle} = \frac{\langle j'm_2|kqjm_1\rangle}{\langle j'm_4|kqjm_3\rangle}$$

   Thus , we can just calculate only one such matrix element explicitly and use Clebsch-Gordon coefficients to obatin other matrix elements with same $j, j'$ and $k$.

2. Since Clebsch-Gordon coefficients $\langle j'm'|kqjm\rangle$ has the property that it vanishes unless,$|k-j| \le j' \le j+k$, and $m' = q+m$, the matrix elements will satify the same selection rules independent of the nature of the wavefunctions $\phi_j^m, \phi_{j'}^{m'}$ or the operator $T_k^q$ .

**Example**: Dipole matrix elements $\left\langle \phi_{j'}^{m'}|\vec{r}|\phi_j^m\right\rangle$, $\vec{r} = (x, y, z)$. Note that the linear combinations,

$$r_+ = -\frac{1}{\sqrt{2}}(x+iy), \qquad r_0 = z, \qquad r_- = \frac{1}{\sqrt{2}}(x-iy)$$

are the basis for the irrep $D^1_{mm'}$. Thus we have the selection rules: matrix elements vanish unless

$$j' = j \pm 1, \qquad \text{but if } j = 0 \quad \text{then } j' = 1$$

and

$$m' = m \text{ or } m \pm 1$$