Introduction to Group Theory
Note 5 Tensor Method in $SU(n)$

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Tensor Analysis

The analysis we have discussed for $SU(2)$ and $SU(3)$ shows that, as thr group gets larger, the elmentary techniques used to dissect the representation structure becomes very complicate. The tensor method we will discuss here provides a handle which is very useful for low rank representations.

1 Tensor analysis in O(3)

1.1 Rotation in R_3

Rotation: coordinate axes are fixed and the physical system is undergoing a rotation. Let x'_a, x_b be the components of new and old vectors. Then we have

$$x'_a = \sum_b R_{ab} x_b$$

where R_{ab} are elements of matrix which represents rotation. For example, the matrix for the rotation about $z-$axis is of the form,

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that the relation between x'_a and x_b is linear and homogeneous.

Important properties of transformation:

1. R is an orthogonal matrix,

$$RR^T = R^T R = 1, \qquad \text{or} \qquad R_{ab}R_{ac} = \delta_{bc}, \quad R_{ab}R_{cb} = \delta_{ac}$$

More conviently, we will write the orthogonality relations as,

$$R_{ab}R_{cd}\delta_{ac} = \delta_{bd}, \quad R_{ab}R_{cd}\delta_{bd} = \delta_{ac}$$

i.e. in the product of 2 rotation matrix elements, making row(or coulm) indices the same and summed over will give Kronecker δ.

2. The combination $\vec{x}^2 = x_a x_a$ is invariant under rotations,

$$x'_a x'_a = R_{ac} R_{ab} x_c x_b = x_b x_b$$

This can be generalized to the case of 2 arbitrary vectors, $\vec{A}$, $\vec{B}$ with transformation property,

$$A'_a = R_{ab} A_b, \qquad B'_c = R_{cd} B_d$$

Then $\vec{A} \cdot \vec{B} = A_a B_a$ is invariant under rotation. It is more convienent to write this as

$$\vec{A} \cdot \vec{B} = A_a B_a = A_a B_b \delta_{ab}$$

which is sometimes called the **contraction** of indices.

3. Transfomation of the gradient operators,

$$\frac{\partial}{\partial x'_a} = \frac{\partial}{\partial x_c} \frac{\partial x_c}{\partial x'_a}$$

From $x_b = \left(R^{-1}\right)_{ba} x'_a$, we get then

$$\frac{\partial}{\partial x'_a} = \left(R^{-1}\right)_{ca} \frac{\partial}{\partial x_c}$$

Thus gradient operator transforms by $\left(R^{-1}\right)^T$. However, for rotations, R is orthogonal, $\left(R^{-1}\right)^T = R$,

$$\frac{\partial}{\partial x'_a} = R_{ac} \frac{\partial}{\partial x_c}$$

i.e. $\partial_\alpha = \dfrac{\partial}{\partial x_a}$ tranforms the same way as x_a.

1.2 Tensors

Suppose we have two vectors, i.e. they have the transformation properties,

$$A_a \rightarrow A'_a = R_{ab} A_b, \qquad B_c \rightarrow B'_c = R_{cd} B_d$$

then

$$A'_a B'_c = R_{ab} R_{cd} A_b B_d$$

The **second rank tensors** are those objects which have the same transformation properties as the product of 2 vectors, i. e.,

$$T_{ac} \rightarrow T'_{ac} = \left(R_{ab} R_{cd}\right) T_{bd}$$

<u>**Definition of n−th rank tensors**</u> (Cartesian tensors)

$$T_{i_1 i_2 \ldots} \rightarrow T'_{i_1 i_2 \ldots i_n} = \left(R_{i_1 j_1}\right)\left(R_{i_2 j_2}\right) \cdots \left(R_{i_n j_n}\right) T_{j_1 j_2 \cdots j_n}$$

Note again that these transformations are linear and homogeneous which implies that

$$if \qquad T_{j_1 j_2 \cdots j_n} = 0, \qquad for \text{ all } j_m$$

then they all zero in other coordinate system.

1.3 Tensor operations

1. Mulplication by constants

$$(cT)_{i_1 i_2 \ldots i_n} = c T_{i_1 i_2 \ldots i_n}$$

2. Add tensors of same rank

$$(T_1 + T_2)_{i_1 i_2 \ldots i_n} = (T_1)_{i_1 i_2 \ldots i_n} + (T_2)_{i_1 i_2 \ldots i_n}$$

3. Multiplication of 2 tensors

$$(ST)_{i_1 i_2 \ldots i_n j_1 j_2 \cdots j_m} = S_{i_1 i_2 \ldots i_n} T_{j_1 j_2 \cdots j_m}$$

This will give a tensor of rank which is the sum of the ranks of 2 constituent tensors.

4. Contaction

$$S_{abc}T_{ae} = S_{abc}T_{de}\delta_{ad} \rightarrow \text{3rd rank tensor}$$

This will reduce the rank of tensor by 2.

5. Symmetrization

$$\text{if } T_{ab} \text{ 2nd rank tensor} \Rightarrow \qquad T_{ab} \pm T_{ba} \quad \text{are also 2nd rank tensors}$$

6. Special numerical tensors

$$RR^T = 1, \qquad \Rightarrow R_{ij}R_{kj} = \delta_{ik}, \qquad or \qquad R_{ij}R_{kl}\delta_{jl} = \delta_{ik}$$

This means that δ_{ij} can be treated as 2nk rank tensor. Similarly,

$$(\det R)\,\varepsilon_{abc} = \varepsilon_{ijk}R_{ai}R_{bj}R_{ck}$$

ε_{abc} a 3rd rank tensor. Useful identities for ε_{abc}

$$\varepsilon_{ijk}\,\varepsilon_{ijl} = 2\delta_{kl}, \qquad \varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

More general notation for tensor transformation (Jackson),

$$x'_a = R_{ab}x_b, \qquad \Rightarrow R_{ab} = \frac{\partial x'_a}{\partial x_b}$$

Then we can wrtie

$$x'_a = \frac{\partial x'_a}{\partial x_b}x_b$$

2 Transformation Law of Tensors in SU(N)

The $SU(n)$ group consists of $n \times n$ unitary matrices with unit determinant. We can regard these matrices as linear transformations on an $n-$dimensional complex vector space C_n. Thus any vector

$$\psi_i = (\psi_1, \psi_2, \cdots \psi_n)$$

in C_n is mapped by an $SU(n)$ transformation U_{ij}, as

$$\psi_i \rightarrow \psi'_i = U_{ij}\psi_j \tag{1}$$

Thus ψ'_i also belong to C_n, with $UU^\dagger = U^\dagger U = 1$ and with $\det U = 1$. Clearly for any two vectors we can define a scalar product

$$(\psi, \phi) \equiv \psi_i^* \phi_i$$

which is invariant under $SU(n)$ transformation. The transformation law for the conjugate vector is given by,

$$\psi_i^* \rightarrow \psi_i'^* = U_{ij}^*\psi_j^* = \psi_j^* U_{ji}^\dagger \tag{2}$$

It is convenient to introduce upper and lower indices to write

$$\psi^i \equiv \psi_i^*, \qquad U_i^{\ j} \equiv U_{ij}, \qquad U^i_{\ j} \equiv U_{ij}^*$$

Thus complex conjugation just change the lower indices to upper ones, and vice versa. In these notation, the transformation law in Eqs(1,2) become

$$\psi_i \rightarrow \psi'_i = U_i^{\ j}\psi_j$$

The

$$\psi^i \rightarrow \psi'^i = U^i_{\ j}\psi^j$$

The $SU(n)$ invariant scalar product is then

$$(\psi, \phi) = \psi^i \phi_i$$

and the unitary condition becomes

$$U^i_{\ k}U^k_{\ j} = \delta^i_{\ j} \tag{3}$$

where the Kronecker delta is defined as

$$\delta^i_{\ j} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Note that in this notation the summation is always over a pair of upper and lower indices. We call this a **contraction of indices**. The ψ_i are the basis for the $SU(n)$ **defining representation (**also called the **fundamental** or **vector representation** and denoted as **n**), while the ψ^i are the basis for the conjugate representation, $\mathbf{n}^*$.

Higher-rank tensors are defined as those quantities which have the same transformations properties as the direct products of vectors. Thus tensors generally have both upper and lower indices with the transfomation law,

$$\psi'^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} = \left(U^{i_1}_{k_1} U^{i_2}_{k_2} \cdots U^{i_p}_{k_p}\right)\left(U^{l_1}_{j_1} U^{l_2}_{j_2} \cdots U^{l_q}_{j_q}\right)\psi^{k_1 k_2 \cdots k_p}_{l_1 l_2 \cdots l_q}$$

They correspond to the basis for higher-dimensional representations.

2.1 Invariant tensors

The Kronecker delta and Levi-Civita symbol are invariant tensors under $SU(n)$ transformations. The play important role in the study of irreducible tensors.

1. From the unitarity condition of Eq(3) we immediately have

$$\delta^i_j = U^i_{\ k} U^l_{\ j} \delta^k_l \tag{4}$$

   Hence δ^i_j ,even though do not change under the $SU(n)$ transformations, behaves as if they are second rank tensors. They can be used to contract indices of other tensor to produce a tensor of lower rank. For example, if ψ^k_{ij} is a 3rd rank tensor,

$$\psi^k_{ij} \to \psi'^k_{ij} = U^k_a U^b_i U^c_j \psi^a_{bc}$$

   the contracting with δ^k_i gives

$$\psi'^k_{ij}\delta^k_i = \delta^k_i U^k_a U^b_i U^c_j \psi^a_{bc} = U^c_j \delta^b_a \psi^a_{bc}$$

   where we have used Eq(4). This gives a tensor of rank 1(vector).

2. The Levi-Civita symbol is defined as the totally antisymmetric quantity,

$$\varepsilon^{i_1 i_2 \cdots i_n} = \varepsilon_{i_1 i_2 \cdots i_n} = \begin{cases} 1 & \text{if } (i_1 i_2 \cdots i_n) \text{ is an even permutation of } (1, 2, \cdots n) \\ -1 & \text{if } (i_1 i_2 \cdots i_n) \text{ is an odd permutation of } (1, 2, \cdots n) \\ 0 & \text{otherwise} \end{cases}$$

   This is also an invariant tensor, because from the property of the determinant we have

$$(\det U)\,\varepsilon_{i_1 i_2 \cdots i_n} = U^{j_1}_{i_1} U^{j_2}_{i_2} \cdots U^{j_n}_{i_n} \varepsilon_{j_1 j_2 \cdots j_n}$$

   Since $\det U$ in $SU(n)$, $\varepsilon_{i_1 i_2 \cdots i_n}$ can be treated as $n-th$ rank tensor. We can use this to change the rank of a tensor. For example,

$$\psi^{i_2 \cdots i_n} \varepsilon_{i_1 i_2 \cdots i_n} \sim \psi_{i_1}$$

   is a vector.

3 Irreducible Representations and Young Tableaux

3.1 Permutation symmetry and tensors

Generally the tensors we have just define are basis for reducible representation of $SU(n)$. To decompose them into irreducible representations we use the following property of these tensors. The permutation of upper(or lower) indices commutes with the $SU(n)$ transformations, as the latter consists of product of identical U_{ij}'s(or U^*_{ij}'s). We will illustrate this with a simple example. Consider the second rank tensor ψ_{ij} whose transformation is given by

$$\psi'_{ij} = U^a_i U^b_j \psi_{ab}$$

Since U's are the same, we can relabel the indices to get

$$\psi'_{ji} = U^b_j U^a_i \psi_{ba}$$

Thus the permutation of indices in the tensor does not change the transformation law. If P_{12} is the permutation opeator which interchanges the first two indices,

$$P_{12}\psi_{ij} = \psi_{ji},$$

then P_{12} commutes with the group transformation,

$$P_{12}\psi'_{ij} = U_i^a U_j^b \left(P_{12}\psi_{ab}\right)$$

This property can be used to decompose ψ_{ij} as follows. First we form eigenstates of the permutation operator P_{12} by symmetrization or antisymmetrization,

$$S_{ij} = \frac{1}{2}\left(\psi_{ij} + \psi_{ji}\right), \qquad A_{ij} = \frac{1}{2}\left(\psi_{ij} - \psi_{ji}\right)$$

so that

$$P_{12}S_{ij} = S_{ij}, \qquad P_{12}A_{ij} = -A_{ij}$$

In group theory language, S_{ij} form basis of an one-dimensional representation of the permutation group S_2 and A_{ij} the basis for another representation. It is clear that S_{ij}, and A_{ij} will not mix under the $SU(n)$ transformations,

$$S'_{ij} = U_i^a U_j^b S_{ab}, \qquad A'_{ij} = U_i^a U_j^b A_{ab}$$

This shows that the second rank tensor ψ_{ij} decomposes into S_{ij}, and A_{ij} in such a way that group transformations never mix parts with different symmetries. It turns out that S_{ij}, and A_{ij} can not be decomposed any further and they thus form the basis of irreducible representations of $SU(n)$. This can be generalized to tensors of higher rank (hence the possibility of mixed symmetries) with the result that the basis for irreducible representations of $SU(n)$ correspond to tensors with definite permutation symmetry among (the positions of) its indices.

3.2 Young tableaux

The task of finding irreducible tensors of an arbitrary rank f (i.e. number of indices) involves forming a complete set permutation operations on these indices. The problem of finding irreducible representations of the permutation group has a complete solution in terms of the **Young tableaux.** They are pictorial representations of the permutation operations of f objects as a set of f boxes each with an index number in it. For example, for the second rank tensors, the symmetrization of indices i and j in S_{ij} is represented by $\begin{array}{|c|c|}\hline i & j \\ \hline\end{array}$; the antisymmetrization operation in A_{ij} is represented by $\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}$. For the third rank tensors ψ_{ijk}, we have

$$\begin{array}{|c|c|c|}\hline i & j & k \\ \hline\end{array} \quad \text{for totally symmetric combination } S_{ijk} = \frac{1}{6}\left(\psi_{ijk} + \psi_{jki} + \psi_{kij} + \psi_{jik} + \psi_{kji} + \psi_{ikj}\right) \tag{5}$$

$$\begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline\end{array}, \quad \text{for the totally antisymmetric combination } A_{ijk} = \frac{1}{6}\left(\psi_{ijk} + \psi_{jki} + \psi_{kij} - \psi_{jik} - \psi_{kji} - \psi_{ikj}\right) \tag{6}$$

$$\begin{array}{|c|c|}\hline i & j \\ \hline k & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \quad \text{for the tensor with mixed symmetry } \chi_{ij;k} = \frac{1}{4}\left(\psi_{ijk} + \psi_{jik} - \psi_{jki} - \psi_{kji}\right) \tag{7}$$

A general Young tableau is shown in the following figure. It is an arrangement of f boxes in rows and columns such that the length of rows should not increase from top to bottom: $f_1 \geq f_2 \geq f_3 \cdots$ and $f_1 + f_2 + f_3 \cdots = f$. Each box has an index $i_k = 1, 2, \cdots n$.

$$\begin{array}{|c|c|c|c|c|}\hline i_1 & i_2 & \cdots & \cdots & i_{f_1} \\ \hline i_{f_1+1} & \cdots & \cdots & i_{f_2} & \multicolumn{1}{c}{} \\ \cline{1-4} \vdots & & \multicolumn{3}{c}{} \\ \cline{1-2} i_{\cdots} & \multicolumn{4}{c}{} \\ \cline{1-1}\end{array}$$

To this tableau we associate the tensor,

$$\psi_{i_1 \cdots i_{f_1}; i_{f_1+1} \cdots}$$

with the following operations.

1. First apply symmetrization operators which leave invariant the indices appearing in each row.

2. Then apply antisymmetrization operators which leave indices in each column invariant.

A tableau where the index numbers do not decrease when going from left to right in a row and always increase from top to bottom is a **standard tableaux.** , For example, the $n = 3$ mixed symmetry tensors are given here with the respective Young tableaux.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad 2\left(\psi_{112} - \psi_{211}\right), \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \quad 2\left(\psi_{113} - \psi_{311}\right), \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \left(\psi_{122} - \psi_{221}\right)$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad (\psi_{133}-\psi_{331}), \qquad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad (\psi_{233}-\psi_{332}), \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad 2(\psi_{223}-\psi_{322})$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad (\psi_{123}+\psi_{213}-\psi_{321}-\psi_{312}), \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad (\psi_{132}+\psi_{312}-\psi_{231}-\psi_{213})$$

where we have make used of the tensor $\chi_{ij;k}$ given in Eq (7). Thus we have 8 indepedent tensors which correspond to the 8−dimensional irrep we discussed before. Note that non-standard tableaux give tensors that, by symmetrization or antisymmetrization, either vanish or are not independent of the standard tableaux. Thus for a given pattern of the Young tableaux the number of independent tensors is equal to the number of standard tableaux which can be formed. It is not hard to see that this number for the simplest case of a tensor with k antisymmetric indices is

$$\left.\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \multicolumn{1}{c}{\vdots} \\ \hline \\ \hline\end{array}\right\} k, \qquad \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n(n-1)\cdots(n-k+1)}{1\times 2\times 3\cdots k}$$

and that for a tensor with k symmetric indices the number is

$$\underbrace{\begin{array}{|c|c|c|c|c|}\hline & & & \multicolumn{1}{c|}{\cdots} & \\ \hline\end{array}}_{k} \qquad \begin{pmatrix} n+k-1 \\ k \end{pmatrix} = \frac{n(n+1)\cdots(n+k-1)}{1\times 2\times 3\cdots k}$$

One should note thath because of antisymmetrization there are not more than n rows in any Young tableaux. Also if there are n rows, we can use $\varepsilon_{i_1\cdots i_n}$ to contract the indices in the columns with n entries. Pictorially, we can simply cross out anu columns with n rows without changing the transformation property of the tensor.

Fundamental theorem (See for example, Hammermesh 1963.) A tensor corresponding to the Young tableaux of a given pattern forms the basis of an irrep of $SU(n)$. Moreover if we enumerate all possible Young tableaux under the restriction that there should be no more than $n-1$ rows, the corresponding tensors form a complete set, in the sense that all finite-dimensonal irreps of the group are counted only once.

We next give two formulae of the **dimensionalitof irreps**. If the Young tableaux is characterized by the length of its rows $(f_1, f_2, \cdots f_{n-1})$, define the length differences of adjacent rows as

$$\lambda_1 = f_1 - f_2, \qquad \lambda_2 = f_2 - f_3, \qquad \cdots \lambda_{n-2} = f_{n-2} - f_{n-1}, \qquad \lambda_{n-1} = f_{n-1},$$

The dimension of an $SU(n)$ irreps will then be the number of standard tableaux for a given pattern and is given by

$$\begin{aligned} d(\lambda_1, \lambda_2, \cdots \lambda_{n-1}) &= (1+\lambda_1)(1+\lambda_2)\cdots(1+\lambda_{n-1}) \\ &\times \left(1+\frac{\lambda_1+\lambda_2}{2}\right)\left(1+\frac{\lambda_2+\lambda_3}{2}\right)\cdots\left(1+\frac{\lambda_{n-2}+\lambda_{n-1}}{2}\right) \\ &\times \left(1+\frac{\lambda_1+\lambda_2+\lambda_3}{3}\right)\left(1+\frac{\lambda_2+\lambda_3+\lambda_4}{3}\right)\cdots\left(1+\frac{\lambda_{n-3}+\lambda_{n-2}+\lambda_{n-1}}{3}\right) \\ &\cdots \\ &\times \left(1+\frac{\lambda_1+\lambda_2+\cdots+\lambda_{n-1}}{n-1}\right) \end{aligned} \tag{8}$$

On can easily check that the special results of totally symmetric and anti-symmetric tensors are recovered from this formaula. Here we give some simple examples.

Example 1. $SU(2)$ group. The Young tableaux can have only one row

$$d(\lambda_1) = (1+\lambda_1)$$

Thus $\lambda_1 = 2j$. It follows that

$$\begin{array}{|c|}\hline \\ \hline\end{array} \quad \text{doublet}, \qquad \begin{array}{|c|c|}\hline & \\ \hline\end{array} \quad \text{triplet}, \cdots$$

Example 2. $SU(3)$ group. The Young tableaux can now have two rows and

$$d(\lambda_1, \lambda_2) = (1+\lambda_1)(1+\lambda_2)\left(1+\frac{\lambda_1+\lambda_2}{2}\right)$$

This equivalent to the other formula for the dimensionality of $SU(n)$ representations with the identification, $\lambda_1 = p$ and $\lambda_2 = q$. Simple cases are given below

$$\begin{array}{|c|}\hline \\ \hline\end{array}\,(1,0)\ \mathbf{3}, \qquad \begin{array}{|c|c|}\hline & \\ \hline\end{array}\,(2,0)\ \mathbf{6}, \qquad \begin{array}{|c|c|c|}\hline & & \\ \hline\end{array}\,(3,0)\ \mathbf{10},$$

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}(0,1)\ \mathbf{3}^*, \qquad \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}(0,2)\ \mathbf{6}^*, \qquad \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}(1,1)\ \mathbf{8}\ ,$$

The formula given in Eq(8) is rather cumbersome to use for large value of n; in such case the second formula is perhaps more useful. For this we need to introduce two definitions–**hook length** and **distance to the first box**. For any box in the tableaux, draw two perpendicular lines, in the shape of a hook, one going to the right and another going downward. The total number of boxes that this hook passes, including the original box itself, is the hook length (h_i) associated with ith box. For example,

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}\ h_i = 3, \qquad \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ h_i = 1$$

The distance to the first box (D_i) is defined to be the number of steps going from the box in the upper left-handed corner of the tableaux to the ith box with each step toward the right counted as $+1$ unit and each downward step as -1 unit. For example, we have

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline -1 & 0 & \multicolumn{1}{c}{} \\ \cline{1-2} -2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

The dimension of the $SU(n)$ irrep associated with the Young tableaux is given by

$$d = \prod_i \left(\frac{n + D_i}{h_i} \right)$$

where the products are taken over all boxes in the tableaux. We illustrate this with a simple example

$$\text{Young tableaux } \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \qquad \text{hook lengths } \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \qquad \text{distance to the first box } \begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

and the dimension is

$$d = \left(\frac{n}{3}\right)\left(\frac{n+1}{1}\right)\left(\frac{n-1}{1}\right) = \frac{n\left(n^2-1\right)}{3}$$

This gives $d = 8$ for $n = 3$ as expected.

4 Reduction of the product representations

One of the most useful application of $SU(n)$ irrep with Young tableaux is the decomposition of the product representations. To find the irrep in the product of two factors,

1. In the smaller tableaux, assign the same symbol, say a, to all boxes in the first row, the same symbol b to all boxes in the second row, etc.

$$\begin{array}{|c|c|c|c|c|}\hline a & a & \cdots & \cdots & a \\ \hline b & b & \cdots & b & \multicolumn{1}{c}{} \\ \cline{1-4} c & \cdots & c & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3} \cdots & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

2. Attach boxes labelled by a to the second tableaux in all possible ways subjected to the rules that no two a's appear in the same column and that the resultant graph is still a Young tableaux (i.e. the length of rows does not increase from top to bottom and there are not more than n rows, etc.) Repeat this process with b's ..etc.

3. After all symbols have been added to the tableaux, these added symbols are then read from **right to letft** in the first row, then the second row in the same order,... and so forth. This sequence of symbols $aabbac\cdots$, must form a **lattice permtutation**. Thus, to left of any symbol there are not fewer a than b and no fewer b than $c, \cdots$ etc.

We consider two examples in the $SU(3)$ group.

Example 1.

$$\begin{array}{|c|}\hline a \\ \hline \end{array} \times \begin{array}{|c|}\hline \\ \hline \end{array} = \begin{array}{|c|}\hline \\ \hline a \\ \hline \end{array} + \begin{array}{|c|c|}\hline & a \\ \hline \end{array}$$

which corresponds to

$$\mathbf{3} \times \mathbf{3} = \mathbf{3}^{*}+\mathbf{6}$$

Example 2.

$$\begin{ytableau} a & a \\ b \end{ytableau} \times \begin{ytableau} {} & {} \\ {} \end{ytableau} = \mathbf{8} \times \mathbf{8}$$

Firs step:

$$\begin{ytableau} {} & {} & a \\ {} \end{ytableau} + \begin{ytableau} {} & {} \\ {} & a \end{ytableau} + \begin{ytableau} {} & {} \\ {} \\ a \end{ytableau}$$

Second step:

$$\begin{ytableau} {} & {} & a & a \\ {} \end{ytableau} + \begin{ytableau} {} & {} & a \\ {} & a \end{ytableau} + \begin{ytableau} {} & {} & a \\ {} \\ a \end{ytableau} + \begin{ytableau} {} & {} \\ {} & a \\ a \end{ytableau}$$

Third step:

$$\underset{\mathbf{27}}{\begin{ytableau} {} & {} & a & a \\ {} & b \end{ytableau}} + \underset{\mathbf{10}}{\begin{ytableau} {} & {} & a & a \\ {} \\ b \end{ytableau}} + \underset{\mathbf{10}^{*}}{\begin{ytableau} {} & {} & a \\ {} & a & b \end{ytableau}} + \underset{\mathbf{8}}{\begin{ytableau} {} & {} & a \\ {} & a \\ b \end{ytableau}} + \underset{\mathbf{8}}{\begin{ytableau} {} & {} & a \\ {} & b \\ a \end{ytableau}} + \underset{\mathbf{1}}{\begin{ytableau} {} & {} \\ {} & a \\ a & b \end{ytableau}} \tag{9}$$

This gives the usual result

$$\mathbf{8} \times \mathbf{8} = \mathbf{1} + \mathbf{8} + \mathbf{8} + \mathbf{10} + \mathbf{10}^{*} + \mathbf{27}$$