Lectures on Gluon and Graviton Scattering Amplitudes

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Introduction:

Although non-abelian gauge theory and gravity share one common principle *local gauge invariance*, their connection seemed to be lacking in the Lagrangian formulation. If one went on to perform perturbative calculation, their shared distinct feature *in the past* was an enormous amount of algebraic misery to arrive at some physical scattering amplitudes, which turns out to be quite simple.

The gauge symmetry we are referring to for QCD for a spin $\frac{1}{2}$ field is

$$\psi \rightarrow e^{i\sum_a T_a \delta\lambda_a}\psi,$$

while that for gravity is local spin rotation

$$\psi \rightarrow e^{i\sum_{c,d}\frac{\sigma_{cd}}{2}\delta\omega^{cd}}\psi.$$

The index a above refers to internal gauge group, whereas the indices c d refer to the local frame at a point in a general coordinate manifold. To be more specific, the Lagrangian for non-abelian gauge theory is

$$L = -\frac{1}{4} F_c^{\mu\nu} F_{c\mu\nu},$$

where the field strength is

$$F_c^{\mu\nu} = \partial^\mu A_c^\nu - \partial^\nu A_c^\mu + g f_{abc} A_a^\mu A_b^\nu$$

The totally anti-symmetric structure constants appear in the Lie algebra

$$[T_a,\ T_b] = i\sqrt{2} f_{abc} T_c.$$

Note that there are products of only three or four fields.

For pure gravity, the Lagrangian is

$$L = \int d^4x \sqrt{(-g)} R,$$

where the Riemann scalar is a function of the metric tensor $g_{\mu\nu}$ and the graviton field is given by $h_{\mu\nu}$ in

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}.$$

Here κ is the gravitational constant, which has the dimension of $mass^2$ and the flat space background metric is $\eta_{\mu\nu} = (-1, 1, 1, 1)$. When we expand L in $h_{\mu\nu}$, there will be infinite number of terms. Even for a fixed number of $h_{\mu\nu}$, the number of vertices is very large. Furthermore, the propagator for gravitons is quite tedious.

Since the seventies, we know that QCD is renormaliable. Also, there had been formal argument, based on the dimensionful nature of the gravitational constant, that quantum gravity would not work at loops. The surprise was provided by 't Hooft and Veltman that in fact pure gravity is finite at one loop, although when individual matter is included it ceases to be so. This demonstrates that formal argument can be misleading and that actual loop evaluation may be necessary in order to confirm or dispel speculations. The underlying message is that

there is a lot more symmetry in the amplitudes

than just by looking at the Lagrangian. This brings back to the issue of how to perform a calculation more effectively.

What is wrong with the old way of performing QCD calculations?

(i) Too many terms: There are six terms in each three point vertex in the Feynman gauge

$$ig\epsilon_{a_1\mu_1}\epsilon_{a_2\mu_2}\epsilon_{a_3\mu_3}f_{a_1a_2a_3}V_3(p_1\ p_2\ p_3)^{\mu_1\mu_2\mu_3},$$

with

$$\begin{aligned}V_3(p_1\ p_2\ p_3)^{\mu_1\mu_2\mu_3} = &[g^{\mu_1\mu_2}(p_1-p_2)^{\mu_3} \\ &+ g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3-p_1)^{\mu_2}],\end{aligned}$$

and one can absorb the four point contact term into the three point vertices. For a n-particle process, we have

$$6^{(n-2)} \times \textit{topologically inequivalent diagrams}$$

the last factor is $\sim n!$. We are talking about a huge number for even reasonable n.

(II) Plenty of cancellations: The final answer is much much simpler than what one starts with for any scattering amplitudes. This makes naive numerical simulation completely unreliable.

(iii) Too much emphasis on manifest covariance: There are only two degrees of freedom for each gluon or graviton. Insisting on manifest Lorentz covariance at every stage, one forces four degrees of freedom at the beginning, but the gluons know that they have only two relevant polarizations and the cancellations are to get rid of the unwanted degrees of freedom. A choice of gauge with only two degrees of freedom will help, at the expense of manifest Lorentz covariance. Of course, the final result is Lorentz invariant for scattering amplitudes.

(iv) No recycling of what has already been done: When we went on to calculate processes with more gluons, we had to start from the scratch all over again. The old efforts could not be recycled.

New way of performing QCD calculations:

There are several big advances in making QCD and for that matter graviton scattering amplitudes much simpler. Furthermore, the exciting thing is that we now have learned a lot more about the extra symmetry and the structure of these two theories, which turn out to be intricately related. We shall list the big steps forward:

(i) Using spinors: One writes the polarization vectors as products of the spinors carrying the momenta of the particles with another set of spinors, known as reference spinors. Each polarization vector in principle can have its own reference spinor and by judicious choice, many diagrams in a process do not contribute. Clearly, these reference spinors make the intermediate expressions seemingly non-covariant, but the final results are independent on them. In fact, in some cases one can make use of this independence to obtain the whole amplitudes if parts of them are known.

(ii) Color decomposition: It turns out that it is not just convenient to separate the color structure of the amplitudes from the momentum dependence, there is duality in their respective symmetries. The parts with the color factors taken out are called the color-stripped amplitudes.

(iii) Analytic continuation in spinor space (BCFW): By making the amplitude complex, one can use residue theorem to relate the original amplitude to various products of physical amplitudes with smaller number of particles.

(iv) Existence of eigenvectors of zero eigenvalue in propagating matrices: One can use the minimal set of color factors in a process and write the fully color dressed amplitude as

$$\mathbf{A}^n =< C^n|\mathcal{M}^n|N^n >,$$

where $< C^n|$ and $|N^n >$ are row and column vectors made of the $(n-2)!$ c's and n's in the KK basis and are ordered in exactly the same way. One can then show that there are$(n-3)(n-3)!$ eigenvectors with zero eigenvalue. They give rise to gauge freedoms, which allow us to change components in $|N^n >$, and their elimination will reduce the number of independent color ordered amplitudes to $(n-2)!-(n-3)(n-3)! = (n-3)!$. This means that there are $(n-3)(n-3)!$ relations among the color-stripped amplitudes

$$A^n = \mathcal{M}^n |N^n >,$$

which are known as BCJ relations. The more interesting aspect is that we now solve for the relevant entries in the vector $|N^n >$ in terms of color stripped amplitudes, which will be used to relate color dynamics with gravity.

(v) KLT relations: The most interesting part of this whole development is that up to powers of coupling constants, the n graviton amplitude is given by

$$\mathbf{A_{gr}}^n =< N^n|\mathcal{M}^n|N^n > .$$

Here, one substitutes the relevant N entries in terms of color stripped amplitudes and QCD and gravity are joined at the tree level

There are four lectures in this series. The aim is to introduce the audience to some current activities in QCD and graviton amplitudes, the most intriguing of which is a better understanding of some celebrated relations between them at the tree level. This may point to a way to modify or add ingredients to gravity to make it into a renormalizable theory. In the first two lectures, we shall develop spinor or helicity technique to show how it simplifies amplitude calculation. The remaining two lectures will be devoted to QCD and graviton amplitudes.

Lecture 1

(1) SL(2,C)

The tensorial analysis of the Lorentz group usually begins with real transformations on a four vector x_μ, such that

$$x'_\mu = l_\mu{}^{\nu} x_\nu,$$

retains the invariance of the norm

$$x'_\mu x'^\mu = x_\mu x^\mu,$$

where we raise or lower indices with the metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = (-1, 1, 1, 1),$$

so that

$$x^\mu = g^{\mu\nu} x_\nu.$$

This will be satisfied if

$$l^t g l = g,$$

where we consider l and g as 4x4 matrices and t stands for transpose.

However, the fundamental representation of the Lorentz group is a spinor and we want to accommodate it. Let us define a set of basic 2x2 matrices

$$\sigma_\mu = (\sigma_0,\ \vec{\sigma}),$$

with

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let us consider the linear combination

$$Q = v_\mu \sigma^\mu = v^0 + \vec{v} \cdot \vec{\sigma} = \begin{pmatrix} v^0 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & v^0 - v_3 \end{pmatrix},$$

the determinant of which is

$$det(Q) = -v_\mu v^\mu.$$

Consider now transformations by a set of complex 2x2 matrices A

$$Q' = AQA^\dagger.$$

which will maintain the reality of the transformed v_μ, as we shall see, if they are real to begin with. This is because

$$Q'^\dagger = Q'.$$

Let us find the transformed 4 vectors: because the four matrices form a complete set for any 2x2 matrix, we can write

$$Q' = v'^0 + \vec{v}' \cdot \vec{\sigma},$$

the determinant of which is

$$det(Q') = |det(A)|^2 det(Q),$$

or

$$-v'_\mu v'^\mu = |det(A)|^2(-v_\mu v^\mu),$$

which shows that v'_μ are real for real v_μ. In fact, if we demand that A is a unimodular matrix, i. e.

$$|det(A)| = 1,$$

we find that we have a Lorentz transformation on v for each A. As before we write this as

$$v'^\mu = l_\nu{}^\mu v^\nu,$$

or

$$Av^{\mu}\sigma_{\mu}A^{\dagger} = l_{\nu}{}^{\mu}v^{\nu}\sigma_{\mu} = l_{\mu}{}^{\nu}v^{\mu}\sigma_{\nu},$$

which gives

$$A\sigma_{\mu}A^{\dagger} = l_{\mu}{}^{\nu}\sigma_{\nu}.$$

Let us introduce

$$\tilde{\sigma}_{\nu} = (I,\ -\vec{\sigma})$$

which has a - sign for the space components, as contrasted with σ_{ν}. Then, one can easily check that

$$Tr(\sigma_{\mu}\tilde{\sigma}_{\nu}) = -2g_{\mu\nu},$$

the use of which gives

$$\frac{1}{2}Tr(\tilde{\sigma}^{\nu}A\sigma_{\mu}A^{\dagger}) = -l^{\nu}{}_{\mu},$$

which is to say that given a complex unimodular matrix A we can construct a Lorentz transformation. It can be shown that given a Lorentz transformation, we can find two A, which differ by a sign:

$$A = N^{-1}[l_{\mu}{}^{\mu} + (l_0^k + l_k^0 - i\epsilon^{klm}l_{ml})\sigma_k],$$

with

$$N^2 = 4 + (l_\mu{}^\mu)^2 - l^\mu{}_\nu l^\nu{}_\mu + i\epsilon^{\mu\nu\lambda\kappa} l_{\mu\nu} l_{\lambda\kappa}, \quad \epsilon^{0123} = \epsilon^{123} = 1.$$

Note that the transformation of interest to us is $A\sigma A^\dagger$ and therefore the phase of $det(A)$ is of no consequence. We shall take

$$det(A) = 1.$$

The matrices A and $A^\dagger$ act on something from the left or from the right. These 'something' we call spinors or twistors, which carry different indices. Thus, we assign the following

$$\eta' = A\eta \rightarrow \eta_\alpha,$$

$$\tilde{\eta}' = A^\star \tilde{\eta} = \tilde{\eta} A^\dagger \rightarrow \tilde{\eta}_{\dot{\alpha}},$$

$$\xi' = \xi A^{-1} \rightarrow \xi^\alpha,$$

and

$$\tilde{\xi}' = \tilde{\xi}(A^{-1})^\star = (A^{-1})^\dagger \tilde{\xi} \rightarrow \tilde{\xi}^{\dot{\alpha}}.$$

From the list above, we can immediately infer that a complex conjugated spinor will have its index changed from a dotted into an undotted one and vice versa. Because these objects have only two components, we see that α denotes either the upper or the lower entry or

$$\alpha = 1,\ 2.$$

Then, since σ_μ are transformed by

$$A\sigma_\mu A^\dagger = l_\mu{}^\nu \sigma_\nu,$$

by implication they carry the indices

$$(\sigma_\mu)_{\alpha\dot{\beta}}$$

and similarly for the reason

$$(A^\dagger)^{-1}\tilde{\sigma}_\mu A^{-1} = l_\mu{}^\nu \tilde{\sigma}_\nu,$$

$\tilde{\sigma}_\mu$ carry indices

$$(\tilde{\sigma}_\mu)^{\dot{\alpha}\beta}.$$

It is obvious that scalars are formed by contracting an upper dotted index with a lower dotted index, or an upper undotted index with a lower undotted index, such as

$$\tilde{\eta}\cdot\tilde{\xi} = \tilde{\eta}_{\dot{\alpha}}\tilde{\xi}^{\dot{\alpha}}, \quad \xi\cdot\eta = \xi^\alpha \eta_\alpha.$$

Consistency requires the following index assignments

$$A_\alpha{}^\beta, \ (A^\dagger)^{\dot\beta}{}_{\dot\alpha}, \ (A^{-1})_\beta{}^\alpha, \ ((A^{-1})^\dagger)^{\dot\alpha}{}_{\dot\beta}.$$

There is a two dimensional alternating symbol which we want to introduce and assign spinor indices:

$$\epsilon \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon^{-1}.$$

By direct multiplication, one verifies the property

$$\epsilon\sigma_\mu^t\epsilon^{-1} = \tilde\sigma_\mu.$$

If we compare the spinor indices of σ_μ and $\tilde\sigma_\mu$, we see that ϵ and ϵ^{-1} change an upper index into a lower (upper)or vice versa. Therefore, they carry indices

$$\epsilon_{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta}, \quad \epsilon^{-1}{}^{\alpha\beta} = \epsilon^{-1}_{\dot\alpha\dot\beta},$$

We should be careful with this raising or lowering operation, because of the antisymmetric nature. Therefore, if we agree to lower an index by

$$\zeta_\alpha = \epsilon_{\alpha\beta}\zeta^\beta,$$

then we have to raise it by

$$\zeta^\beta = \epsilon^{-1}{}^{\beta\kappa}\zeta_\kappa,$$

so that

$$\zeta_\alpha = \epsilon_{\alpha\beta}{\epsilon^{-1}}^{\beta\kappa}\zeta_\kappa = \delta_\alpha^\kappa\zeta_\kappa = \zeta_\alpha.$$

There is an identity that, for any non-singular 2x2 matrix M,

$$\epsilon M^t\epsilon^{-1} = M^{-1}det(M).$$

The proof of this is quite simple. We express

$$M = b^\mu\sigma_\mu,$$

and find

$$Mb^\nu\tilde{\sigma}_\nu = b^\mu\sigma_\mu b^\nu\tilde{\sigma}_\nu = -b_\mu b^\mu = det(M),$$

after using

$$\sigma_\mu\tilde{\sigma}_\nu + \sigma_\nu\tilde{\sigma}_\mu = -2g_{\mu\nu}.$$

Thus

$$b^\mu\tilde{\sigma}_\mu = M^{-1}det(M).$$

However,

$$M^t = b^\mu\sigma_\mu^t,$$

which gives the wanted result

$$\epsilon M^t\epsilon^{-1} = b^\mu\epsilon\sigma_\mu^t\epsilon^{-1} = b^\mu\tilde{\sigma}_\mu = M^{-1}det(M).$$

An consequence of this identity is for example

$$\epsilon A^t \epsilon^{-1} = A^{-1},$$

which shows the consistency of our index assignments. Another example is that under a Lorentz transformation

$$\begin{aligned}\epsilon'_{\alpha\beta} &= A_\alpha{}^\delta A_\beta{}^\kappa \epsilon_{\delta\kappa} = (A\epsilon A^t)_{\alpha\beta} = (A\epsilon A^t \epsilon^{-1}\epsilon)_{\alpha\beta} \\ &= (AA^{-1})_{\alpha\beta} = \epsilon_{\alpha\beta},\end{aligned}$$

which is a statement that ϵ and therefore ϵ^{-1} are invariant tensors.

Because the four matrices σ_μ or $\tilde{\sigma}^\mu$ are complete, we have a completeness relation for them

$$(\sigma_\mu)_{\gamma\dot{\delta}} (\tilde{\sigma}^\mu)^{\dot{\alpha}\beta} = -2\delta^{\dot{\alpha}}_{\dot{\delta}} \delta^\beta_\gamma. \tag{1.1}$$

This is equivalent to the Fierz identity of gamma matrices.

(2) Spin $\frac{1}{2}$

We define the four component wave function and matrices as

$$\psi = \begin{pmatrix} \xi^{\dot{\beta}} \\ \eta_\alpha \end{pmatrix},$$

and

$$\gamma_\mu = \begin{pmatrix} 0 & -(\tilde{\sigma}_\mu)^{\dot{\alpha}\beta} \\ -(\sigma_\mu)_{\alpha\dot{\beta}} & 0 \end{pmatrix}.$$

Then the Dirac equation is

$$(m + \gamma \cdot p)\psi = 0,$$

with

$$\{\gamma_\mu,\ \gamma_\nu\} = -2g_{\mu\nu}.$$

For the massless situation, we have

$$\sigma_\mu p^\mu \xi = (p^0 + \vec{\sigma} \cdot \vec{p})\xi = 0,$$

or

$$\frac{\vec{\sigma} \cdot \vec{p}}{|p|}\xi = -\xi,$$

for positive energy particle $|\vec{p}| = |p| = p^0$, which means that it has negative helicity or left-handed. Likewise

$$\tilde{\sigma}_\mu p^\mu \eta = (p^0 - \vec{\sigma}\cdot\vec{p})\eta = 0,$$

or

$$\frac{\vec{\sigma}\cdot\vec{p}}{|p|}\eta = \eta,$$

for positive energy particle, which means that it has positive helicity or right-handed.

We can project these two different spinors out from the 4-component wave function by defining

$$\gamma_5 = \gamma^0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix},$$

which gives

$$\frac{1}{2}(1 - i\gamma_5)\psi = \xi, \quad \frac{1}{2}(1 + i\gamma_5)\psi = \eta.$$

We would like to point out that in the above discussion, we use the Weyl representation, which is related to the Dirac representation by a similarity transformation

$$(\gamma_\mu)_{Weyl} = S(\gamma_\mu)_{Dirac}S^{-1},$$

where

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}, \quad S^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}.$$

We can check that

$$(\gamma_i)_{Weyl} = (\gamma_i)_{Dirac},$$

but

$$(\gamma^0)_{Dirac} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

which allows us to write

$$S = \frac{1}{\sqrt{2}}(I - i\gamma^0\gamma_5)$$

in either representation.

We need a different normalization for the solution of the Dirac equation in the massless case from the massive case. Let us briefly discuss it. In the Dirac representation, the equation

$$(\gamma \cdot p)_{Dirac} u(p) = 0,$$

gives

$$-p^0 a + \vec{\sigma} \cdot \vec{p} b = 0, \quad -\vec{\sigma} \cdot \vec{p} a + p^0 b = 0,$$

where we have written

$$u = \begin{pmatrix} a \\ b \end{pmatrix},$$

and a and b each has two components. This gives two choices: $a = b$ or $a = -b$. For the first, which corresponds to positive helicity solution for positive energy $p^0 > 0$, we have

$$u_+(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{p_+} \\ \sqrt{p_-} e^{i\phi_p} \\ \sqrt{p_+} \\ \sqrt{p_-} e^{i\phi_p} \end{pmatrix},$$

where

$$p_+ = p^0 + p_3, \; p_- = p^0 - p_3,$$

and

$$e^{i\phi} = \frac{p_1 + ip_2}{\sqrt{p_1^2 + p_2^2}} = \frac{p_1 + ip_2}{\sqrt{p_+ p_-}}.$$

For $b = -a$, we have

$$u_-(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{p_-} e^{-i\phi_p} \\ -\sqrt{p_+} \\ -\sqrt{p_-} e^{-i\phi_p} \\ \sqrt{p_+} \end{pmatrix},$$

which is the negative helicity solution. We shall choose the phase such that the negative energy solutions are

$$v_+(p) = u_-(p), \quad v_-(p) = u_+(p).$$

Futhermore, the charge conjugation matrix is

$$C = i\gamma_2 = C^{-1} = C^\dagger.$$

such that

$$Cu_-^\star = u_+, \quad Cu_+^\star = u_-.$$

The Weyl solutions are obtained by

$$u_+^{Weyl}(p) = Su_+(p) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{p_+} \\ \sqrt{p_-}e^{i\phi_p} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \eta(p)_1 \\ \eta(p)_2 \end{pmatrix},$$

and

$$u_-^{Weyl}(p) = Su_-(p) = \begin{pmatrix} \sqrt{p_-}e^{-i\phi_p} \\ -\sqrt{p_+} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \xi(p)^{\dot{1}} \\ \xi(p)^{\dot{2}} \\ 0 \\ 0 \end{pmatrix}.$$

For the scalar products, we find that using

$$\begin{aligned} \bar{u}(p_i)_- =& (\sqrt{(p_i)_-}e^{i\phi_{p_i}}, -\sqrt{(p_i)_+}, \sqrt{(p_i)_-}e^{i\phi_{p_i}}, -\sqrt{(p_i)_+}) \\ & \times \frac{1}{\sqrt{2}} \end{aligned}$$

we have

$$\begin{aligned} < i\ j > \equiv & \ \bar{u}(p_i)_- u(p_j)_+ \\ & = \sqrt{p_{i-}}\sqrt{p_{j+}}\ e^{i\phi_{p_i}} - \sqrt{p_{i+}}\sqrt{p_{j-}}\ e^{i\phi_{p_j}} \\ & = -\epsilon^{\alpha\beta}\eta(p_i)_\alpha\eta(p_j)_\beta = \eta(p_i)^\beta\eta(p_j)_\beta \\ & = -\eta(p_i)_\alpha\eta(p_j)^\alpha = -< j\ i > . \end{aligned}$$

Likewise, we have

$$\begin{aligned}
[i\ j] &\equiv \bar{u}_+(p_i)u_-(p_j) = -\epsilon^{-1}_{\dot{\alpha}\dot{\beta}}\xi(p_i)^{\dot{\alpha}}\xi(p_j)^{\dot{\beta}} = -\xi(p_i)^{\dot{\alpha}}\xi(p_j)_{\dot{\alpha}} \\
&= \xi(p_i)_{\dot{\beta}}\xi(p_j)^{\dot{\beta}} = \xi(p_i)^{\dot{1}}\xi(p_j)^{\dot{2}} - \xi(p_i)^{\dot{2}}\xi(p_j)^{\dot{1}} \\
&= \sqrt{p_{i\,-}}e^{-i\phi_{p_i}}(-\sqrt{p_{j\,+}}) - \sqrt{p_{j\,-}}e^{-i\phi_{p_j}}(-\sqrt{p_{i+}}) \\
&= -[j\ i],
\end{aligned}$$

and thus

$$[i\ j] = < j\ i >^\star .$$

We should caution that this complex conjugation property holds only for real momenta. As we shall see, to continue the kinematics to complex values is very useful in calculating amplitudes and in investigating their properties. In the above, we have taken

$$|j>_\beta = \eta(p_j)_\beta, \quad < i|^\beta = \eta(p_i)^\beta = (\epsilon^{-1})^{\beta\alpha}\eta(p_i)_\alpha,$$

and

$$|j]^{\dot{\beta}} = \xi(p_j)^{\dot{\beta}}, \quad [i|_{\dot{\beta}} = \xi(p_i)_{\dot{\beta}} = (\epsilon^{-1})_{\dot{\beta}\dot{\alpha}}\xi(p_i)^{\dot{\alpha}}.$$

When we multiply explicit expressions for the spinors together, we obtain

$$< i\ j > [j\ i] = 2(p_i^0 p_j^0 - \vec{p}_i\cdot\vec{p}_j) = -2p_i\cdot p_j = s_{ij},$$

$$(\sigma \cdot p)_{\beta\dot{\alpha}} = |p>_\beta \, [p|_{\dot{\alpha}}, \quad (-\tilde{\sigma} \cdot p)^{\dot{\alpha}\beta} = |p]^{\dot{\alpha}} < p|^\beta,$$

and

$$\frac{1}{2} < p|\sigma^\mu|p] = p^\mu, \quad -\frac{1}{2}[p|\tilde{\sigma}^\mu|p >= p^\mu,$$

which tell us that we can do all the kinematics with spinors.

Because of the antisymmetry of the ϵ symbol in two dimensions, the product

$$\epsilon^{ab}\epsilon^{cd} = \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc},$$

which gives rise to

$$\epsilon^{ab}\epsilon^{cd} = \epsilon^{ad}\epsilon^{cb} + \epsilon^{ac}\epsilon^{bd}.$$

When we multiply this identity to four spinors, we obtain

$$< ij >< kl > + < jk >< il > + < ki >< jl >= 0,$$

and

$$[\ ij\ ][\ kl\ ] + [\ jk\ ][\ il\ ] + [\ ki\ ][\ jl\ ] = 0,$$

which are known as Schouton identities.

3. Spin 1

We want to build spin one polarization vectors by using spinors. For a particle with momentum k if we limit ourselves to gamma matrices and the wave functions $u(k)_{\pm}$ and their conjugates, then the only vector we shall end up with is k^{μ}. Thus, we need to introduce an external vector and its wave functions to assist. The situation is not new, because we encounter this before when we work in for example axial guage, light cone gauge etc. While manifest Lorentz covariance is lost in a spinor calculation, this luxury is somewhat compensated by gauge invariance and simplicity, as we shall see. To begin, let us introduce a reference spin $\frac{1}{2}$ massless object moving in the -z direction with momentum q. This is not a restriction, because we can always orient our coordinate system to make it so, then its four momentum vector is

$$q^{\mu} = q(1,\ 0,\ 0,\ -1).$$

There, the wave functions are in the Dirac representation

$$u(q)_+ = \sqrt{q} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad u(q)_- = \sqrt{q} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

and

$$\bar{u}_+ = \sqrt{q}(0\ 1\ 0\ \ -1), \quad \bar{u}_- = \sqrt{q}(1\ 0\ 1\ 0).$$

We claim that the following

$$\epsilon(k,q)^\mu_+ \equiv \frac{\bar{u}(q)_-\gamma^\mu u(k)_-}{\sqrt{2}\bar{u}(q)_- u(k)_+},$$

and

$$\epsilon(k,q)^\mu_- \equiv \frac{\bar{u}(q)_+\gamma^\mu u(k)_+}{\sqrt{2}(\bar{u}(q)_- u(k)_+)^\star}.$$

are the polarization vectors. First, because of the wave equation, it is clear that the transversality condition

$$k_\mu \epsilon(k,q)^\mu_\pm = 0$$

is satisfied. In fact, we also have

$$q_\mu \epsilon(k,q)^\mu_\pm = 0.$$

Also, they are orthonomal which may be checked for example by using explicit expressions

$$\epsilon(k,q)^\mu_+ = \frac{1}{\sqrt{2}}(\sqrt{\frac{k_-}{k_+}}e^{-i\phi_k},\ 1,\ -i,\ -\sqrt{\frac{k_-}{k_+}}e^{-i\phi_k}),$$

$$\epsilon(k,q)^\mu_- = \frac{1}{\sqrt{2}}(\sqrt{\frac{k_-}{k_+}}e^{i\phi_k},\ 1,\ i,\ -\sqrt{\frac{k_-}{k_+}}e^{i\phi_k}) = (\epsilon(k,q)_+{}^\mu)^\star.$$

Then

$$\epsilon(k,q)^{\mu}_{+}(\epsilon(k,q)_{+\mu})^{\star} = \epsilon(k,q)^{\mu}_{-}(\epsilon(k,q)_{-\mu})^{\star} = 1,$$

$$\epsilon(k,q)^{\mu}_{+}(\epsilon(k,q)_{-\mu})^{\star} = \epsilon(k,q)^{\mu}_{-}(\epsilon(k,q)_{+\mu})^{\star} = 0.$$

Let us perform the polarization sum to show that it is the same as in the gauge $q_\mu A^\mu = 0$, where A^μ is a vector gauge field. We have

$$\epsilon(k,q)^{\mu}_{+}\epsilon(k,q)^{\star\ \nu}_{+} = \frac{N^{\mu\nu}_{+}}{D}, \quad \epsilon(k,q)^{\mu}_{-}\epsilon(k,q)^{\star\ \nu}_{-} = \frac{N^{\mu\nu}_{-}}{D},$$

the numerators of which are

$$\begin{aligned} N^{\mu\nu}_{+} &= \bar{u}(q)_{-}\gamma^{\mu}u(k)_{-}\bar{u}(k)_{-}\gamma^{\nu}u(q)_{-} \\ &= \bar{u}(q)_{-}\gamma^{\mu}(-\frac{1}{2}(1-i\gamma_5)\gamma\cdot k)\gamma^{\nu}u(q)_{-} \\ &= -\bar{u}(q)_{-}\gamma^{\mu}\gamma\cdot k\gamma^{\nu}(\frac{1}{2}(1-i\gamma_5))u(q)_{-} \\ &= -\bar{u}(q)_{-}\gamma^{\mu}\gamma\cdot k\gamma^{\nu}u(q)_{-} \\ &= -\bar{u}(q)_{-}\gamma^{\mu}(-2k^{\nu}-\gamma^{\nu}\gamma\cdot k)u(q)_{-}, \end{aligned}$$

$$N^{\mu\nu}_{-} = -\bar{u}(q)_{-}\gamma^{\nu}(-2k^{\mu}-\gamma^{\mu}\gamma\cdot k)u(q)_{-},$$

while the denominator

$$D = 2|\bar{u}(q)_{-}u(k)_{+}|^2 = 2(2qk_{+}) = -4q\cdot k.$$

Then, we have

$$
\begin{aligned}
N_{+}^{\mu\nu} + N_{-}^{\mu\nu} &= \bar{u}(q)_{(}2(k^\mu\gamma^\nu + k^\nu\gamma^\mu) \\
&\qquad + (\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\gamma\cdot k)u(q)_{-} \\
&= 4(k^\mu q^\nu + k^\nu q^\mu) - 4g^{\mu\nu} q\cdot k,
\end{aligned}
$$

where we have used

$$
\bar{u}(q)_{-}\gamma^\mu u(q)_{-} = 2q^\mu, \quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}.
$$

The result is

$$
\sum_{\lambda=\pm} \epsilon(k,q)_\lambda^\mu \epsilon(k,q)_\lambda^{\star\,\nu} = g^{\mu\nu} - \frac{k^\mu q^\nu + k^\nu q^\mu}{q\cdot k},
$$

which is the projector in the gauge $q\cdot A = 0$. Finally, to check that the helicity assignments are correct, we take a special case when the vector particle is moving in $+z$ direction

$$
k^\mu = k(1,\ 0,\ 0,\ 1),
$$

which acquires

$$
\epsilon(k,q)_+^\mu = \frac{1}{\sqrt{2}}(0,\ 1,\ -i, 0), \quad \epsilon(k,q)_-^\mu = \frac{1}{\sqrt{2}}(0,\ 1,\ i, 0),
$$

which are the correct answer.

In the language of spinors, we have

$$\epsilon(k,q)^\mu_+ = \frac{< q|\sigma^\mu|k\ ]}{\sqrt{2} < q\ k >}, \quad \epsilon(k,q)^\mu_- = \frac{[\ q|\tilde{\sigma}^\mu|k >}{-\sqrt{2}[q\ k[}. \qquad (3.1)$$

Please note that a change of basis vector, say from q_μ to q'_μ is a gauge transformation. One can show that

$$\epsilon(k,q')^\mu_\pm = (g^{\mu\nu} - \frac{k^\mu q'^\nu}{q'\cdot k})\epsilon(k,q)_{\pm\nu} = \epsilon(k,q)^\mu_\pm - k^\mu \Lambda_\pm,$$

where the gauge function is

$$\Lambda_\pm = \frac{q'^\nu \epsilon(k,q)_{\pm\nu}}{q'\cdot k}.$$

We point out that all the conditions for polarization vectors are satisfied

$$k_\mu \epsilon(k,q')^\mu_\pm = 0, \quad q'_\mu \epsilon(k,q')^\mu_\pm = 0,$$

$$\epsilon(k,q')^\mu_+ (\epsilon(k,q')_{+\mu})^\star = \epsilon(k,q')^\mu_- (\epsilon(k,q')_{-\mu})^\star = 1,$$

$$\epsilon(k,q')^\mu_+ (\epsilon(k,q')_{-\mu})^\star = \epsilon(k,q')^\mu_- (\epsilon(k,q')_{+\mu})^\star = 0,$$

and

$$\sum_{\lambda=\pm} \epsilon(k,q')^\mu_\lambda \epsilon(k,q')^{\star\,\nu}_\lambda = g^{\mu\nu} - \frac{k^\mu q'^\nu + k^\nu q'^\mu}{q'\cdot k}$$

in the above construction of $\epsilon(k, q')^{\mu}_{\pm}$.

For a physical process, all physical quantities should be invariant under gauge transformations. In particular, scattering amplitudes should be independent of the reference vector q. This is a very stringent test of a correct calculation and gauge independence can be used to relate different parts.