Lectures on Gluon and Graviton Scattering Amplitudes

York-Peng Yao

Lecture 2

4. Review of Lecture 1

For a particle with zero mass, we can use two component spinors or twistors representation, which are solutions to the massless Dirac equation

$$< i|_b = \bar{u}_-(p_i),\ \ |i >^b = u_+(p_i),\ \ [i|_{\dot{a}} = \bar{u}_+(p_i),\ \ |i]^{\dot{a}} = u_-(p_i).$$

Then, the momentum p of a massless particle is written as

$$P^{\dot{a}b} = (\tilde{\sigma} \cdot P)^{\dot{a}b} = -|p]^{\dot{a}} < p|^b,$$

and

$$P_{b\dot{a}} = (\sigma \cdot P)_{b\dot{a}} = |p >_b [p|_{\dot{a}},$$

where

$$\sigma_\mu = (I, \vec{\sigma}) \quad \tilde{\sigma}_\mu = (I, -\vec{\sigma}).$$

We use them to form scalar products of spinors

$$< p_i p_j >=< p_i|^b|p_j >_b= - < p_j p_i >,$$

and

$$[p_j p_i] = [p_j|_{\dot{a}}|p_i]^{\dot{a}} = -[p_i p_j],$$

from which the scalar product of two vectors is

$$-2P_i \cdot P_j =< p_i p_j > [p_j p_i].$$

Also, we use them to build polarization vectors for gauge particles of momentum K_i

$$\epsilon^{h=+}(q_i, k_i)^\mu = \frac{< q_i|\sigma^\mu|k_i]}{\sqrt{2} < q_i k_i >}, \ \epsilon^{h=-}(q_i, k_i)^\mu = \frac{[q_i|\tilde{\sigma}^\mu|k_i >}{-\sqrt{2}[q_i k_i]}$$

in which Q_i is a reference momentum, which can be individually assigned for each K_i. Changing Q_i is a change of gauge.

Because the four matrices σ_μ or $\tilde{\sigma}^\mu$ are complete, we have a completeness relation for them

$$(\sigma_\mu)_{\gamma\dot{\delta}}(\tilde{\sigma}^\mu)^{\dot{\alpha}\beta} = -2\delta^{\dot{\alpha}}_{\dot{\delta}}\delta^{\beta}_{\gamma}. \tag{1.1}$$

(5) QCD and Color Ordering

There have been great advances in our understanding of QCD amplitudes in the last decade, besides being able to compute them much more efficiently. One outcome of this is that some remarkable relations (KLT) exist between gauge and gravity amplitudes. To arrive at that, we shall begin here with some notational and basic preparation of QCD and the procedure of color ordering. We shall use the four gluon amplitudes as an example to make our manipulations concrete.

The QCD Lagrangian is

$$L = -\frac{1}{4} F_c^{\mu\nu} F_{c\mu\nu}, \tag{5.1}$$

where the field strength is

$$F_c^{\mu\nu} = \partial^\mu A_c^\nu - \partial^\nu A_c^\mu + g f_{abc} A_a^\mu A_b^\nu. \tag{5.2}$$

The totally anti-symmetric structure constants appear in the Lie algebra

$$[T_a,\ T_b] = i\sqrt{2} f_{abc} T_c, \tag{5.3}$$

in which we are primarily interested in $SU(N_c)$, with the generators in the fundamental representation normalized to

$$Tr(T_a T_b) = \delta_{ab}. \tag{5.4}$$

From eqs. (5.3-4), we obtain

$$f_{abc} = -\frac{i}{\sqrt{2}} Tr([T_a,\ T_b]T_c). \tag{5.5}$$

We also note that

$$(F_b)_{ac} = i\sqrt{2} f_{abc},$$

is also a representation of the algebra, which when writing out and relabeling is the Jacobi identity

$$f_{a_1 a_2 b} f_{a_3 a_4 b} + f_{a_1 a_3 b} f_{a_4 a_2 b} + f_{a_1 a_4 b} f_{a_2 a_3 b} = 0. \tag{5.6}$$

We should point out that repeated color indices are to be summed over from 1 to $N_c^2 - 1$. We work temporarily in the Feynman gauge, in which the rules are for the three incoming gluons $p_1 + p_2 + p_3 = 0$

$$ig \epsilon_{a_1 \mu_1} \epsilon_{a_2 \mu_2} \epsilon_{a_3 \mu_3} f_{a_1 a_2 a_3} V_3(p_1\ p_2\ p_3)^{\mu_1 \mu_2 \mu_3},$$

with

$$\begin{aligned} V_3(p_1\ p_2\ p_3)^{\mu_1 \mu_2 \mu_3} =& [g^{\mu_1 \mu_2}(p_1 - p_2)^{\mu_3} + g^{\mu_2 \mu_3}(p_2 - p_3)^{\mu_1} \\ & + g^{\mu_3 \mu_1}(p_3 - p_1)^{\mu_2}], \end{aligned}$$

for the four gluons $p_1 + p_2 + p_3 + p_4 = 0$

$$
\begin{aligned}
ig^2 \epsilon_{a_1\mu_1} \epsilon_{a_2\mu_2} \epsilon_{a_3\mu_3} \epsilon_{a_4\mu_4} V_4(p_1\ p_2\ p_3\ p_4)^{\mu_1\mu_2\mu_3\mu_4}_{a_1a_2a_3a_4} \\
&= ig^2 \epsilon_{a_1\mu_1} \epsilon_{a_2\mu_2} \epsilon_{a_3\mu_3} \epsilon_{a_4\mu_4} \\
&\times [f_{a_1a_2b} f_{a_3a_4b} (g^{\mu_1\mu_3} g^{\mu_2\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3}) \\
&+ f_{a_1a_3b} f_{a_2a_4b} (g^{\mu_1\mu_2} g^{\mu_3\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3}) \\
&+ f_{a_1a_4b} f_{a_2a_3b} (g^{\mu_1\mu_2} g^{\mu_3\mu_4} - g^{\mu_1\mu_3} g^{\mu_2\mu_4})],
\end{aligned}
$$

and for the propagator

$$
\frac{1}{i} \frac{g_{\mu\nu} \delta_{ab}}{p^2}.
$$

Let us now consider the invariant amplitude for four incoming gluons. With the coupling constant suppresed, we write it as

$$
M = M_s + M_u + M_t, \tag{5.7}
$$

where

$$
M_s = f_{a_1a_2b} f_{a_3a_4b} A_s, \quad M_u = f_{a_1a_3b} f_{a_2a_4b} A_u,
$$

$$
M_t = f_{a_1a_4b} f_{a_3a_2b} A_t,
$$

with

$$
\begin{aligned}
A_s = \epsilon_{a_1\mu_1}\epsilon_{a_2\mu_2}\epsilon_{a_3\mu_3}\epsilon_{a_4\mu_4} & \\
& \times [V_3(p_1\ p_2\ q)^{\mu_1\mu_2\lambda} g_{\lambda\kappa} V_3(p_3\ p_4\ \ -q)^{\mu_4\mu_4\kappa} \\
& + g^{\mu_1\mu_2} g^{\mu_3\mu_4}(p_1+p_2)^2] \frac{1}{(p_1+p_2)^2}, \\
& q = -(p_1+p_2),
\end{aligned}
$$

and

$$
A_u = A_s(2 \leftrightarrow 3), \quad A_t = A_s(2 \leftrightarrow 4).
$$

We have stripped out the color factors from $M_{s,u,t}$ to define $A_{s,u,t}$ and may consider this as a way of color ordering the indices $a_{1,2,3,4}$. However, there is another way, which is more frequently used.

We use an identity of completeness, which states that since the identity matrix and the $N_c^2 - 1$ T_a matrices in the fundamental representation can be used as basis for any $N_c \times N_c$ matrix, we have

$$
(T_a)_{i_1}{}^{j_1}(T_a)_{i_2}{}^{j_2} = \delta_{i_1}{}^{j_2}\delta_{i_2}{}^{j_1} - \frac{1}{N_c}\delta_{i_1}{}^{j_1}\delta_{i_2}{}^{j_2}
$$

Together with eq.(5.5), we can write the color factors as

$$
\begin{aligned}
f_{a_1a_2b}f_{a_3a_4b} = & -\frac{1}{2}[Tr(T_{a_1}T_{a_2}T_{a_3}T_{a_4}) - Tr(T_{a_1}T_{a_2}T_{a_4}T_{a_3}) \\
& - Tr(T_{a_1}T_{a_3}T_{a_4}T_{a_2}) + Tr(T_{a_1}T_{a_4}T_{a_3}T_{a_2})],
\end{aligned}
$$

$$f_{a_1a_3b}f_{a_2a_4b} = -\frac{1}{2}[Tr(T_{a_1}T_{a_3}T_{a_2}T_{a_4}) - Tr(T_{a_1}T_{a_3}T_{a_4}T_{a_2})$$
$$- Tr(T_{a_1}T_{a_2}T_{a_4}T_{a_3}) + Tr(T_{a_1}T_{a_4}T_{a_2}T_{a_3})],$$

$$f_{a_1a_4b}f_{a_3a_2b} = -\frac{1}{2}[Tr(T_{a_1}T_{a_4}T_{a_3}T_{a_2}) - Tr(T_{a_1}T_{a_4}T_{a_2}T_{a_3})$$
$$- Tr(T_{a_1}T_{a_3}T_{a_2}T_{a_4}) + Tr(T_{a_1}T_{a_2}T_{a_3}T_{a_4})].$$

With the above color decomposition, we write eq.(5.7) as

$$M = \sum_{j\neq k\neq l=}^{2,3,4} M(1jkl)Tr(T_{a_1}T_{a_j}T_{a_k}T_{a_l}),$$

where we find

$$M(1432) = M(1234) = \frac{1}{2}(-A_s - A_t),$$
$$M(1243) = M(1342) = \frac{1}{2}(A_s + A_u),$$
$$M(1324) = M(1423) = \frac{1}{2}(-A_u + A_t). \qquad (5.8)$$

Note that the sum of the right hand sides of the last three equations add up to zero, which means that there are only two independent color-stripped amplitudes $(1jkl)$. This can also be seen if we order the color according to $f_{a_ia_jb}f_{a_ka_lb}$

with $i \neq j \neq k \neq l = 1, 2, 3, 4$, in view of the Jacobi identity eq.(5.6). We can check that each of the independent color-stripped amplitudes is gauge invariant.

It is customary to absorb the factor $\frac{1}{2}$ in eqs. (5.8) into $V_3 \to \frac{1}{\sqrt{2}} V_3$ and $V_4 \to \frac{1}{2} V_4$.

(6) Four Gluon Scattering

Now that we have developed the necessary preliminaries, we shall calculate the color-ordered scattering amplitude $M(1234)$. It can be shown that we must have two spins up and two spins down for the amplitude not to vanish. Thus, we consider the spin assignment $p_1^- + p_2^- + p_3^+ + p_4^+ = 0$.

From eq.(5.8), we see that we need to calculate $A_s,\ A_t$. We can choose our reference momenta judiciously to simplify the chore. It turns out that if we take $q_1 = q_2 = p_4,\ q_3 = q_4 = p_1$, V_4 and A_t give no contribution and there will be just one term for A_s. To proceed, we have

$$\begin{aligned} A_t =& (g^{\mu_1\mu_4}(p_1 - p_4)^\mu + g^{\mu_4\mu}(p_4 - q)^{\mu_1} + g^{\mu\mu_1}(q - p_1)^{\mu_4}) \\ & \qquad \times \frac{1}{(p_1 + p_4)^2} \\ & \times (g^{\mu_3\mu_2}(p_3 - p_2)_\mu + g_\mu^{\mu_2}(p_2 + q)^{\mu_3} + g_\mu^{\mu_3}(-q - p_3)^{\mu_2}) \\ & \times \epsilon(p_1, p_4)_{\mu_1}^- \epsilon(p_2, p_4)_{\mu_2}^- \epsilon(p_3, p_1)_{\mu_3}^+ \epsilon(p_4, p_1)_{\mu_4}^+, \\ & \quad q = -(p_1 + p_4) = p_2 + p_3. \end{aligned}$$

The first factor gives expressions

$$\begin{aligned} g^{\mu_1\mu_4}\epsilon(p_1, p_4)_{\mu_1}^- \epsilon(p_4, p_1)_{\mu_4}^+ \to & [p_4|\tilde{\sigma}^\mu|p_1 >< p_1|\sigma_\mu|p_4] \\ & = -2[p_4\ p_4] < p_1\ p_1 >= 0, \end{aligned} \tag{6.1}$$

where use has been made of eq.(1.1)

$$(\sigma_\mu)_{\gamma\dot{\delta}}(\tilde{\sigma}^\mu)^{\dot{\alpha}\beta} = -2\delta^{\dot{\alpha}}_{\dot{\delta}}\delta^{\beta}_{\gamma}.$$

$$(p_4 - q)^{\mu_1}\epsilon(p_1, p_4)^-_{\mu_1} \to [p_4|\tilde{\sigma}\cdot(2p_4 + p_1)|p_1 >= 0,$$

and

$$(q - p_1)^{\mu_4}\epsilon(p_4, p_1)^+_{\mu_4} \to < p_1|\sigma\cdot(-2p_1 - p_4)|p_4] = 0.$$

As for A_s

$$\begin{aligned} A_s =& (g^{\mu_1\mu_2}(p_1 - p_2)^\mu + g^{\mu_2\mu}(p_2 - q)^{\mu_1} + g^{\mu\mu_1}(q - p_1)^{\mu_2}) \\ & \times \frac{1}{(p_1 + p_2)^2}(g^{\mu_3\mu_4}(p_3 - p_4)_\mu \\ & \qquad + g^{\mu_4}_{\mu}(p_4 + q)^{\mu_3} + g^{\mu_3}_{\mu}(-q - p_3)^{\mu_4}) \\ & \times \epsilon(p_1, p_4)^-_{\mu_1}\epsilon(p_2, p_4)^-_{\mu_2}\epsilon(p_3, p_1)^+_{\mu_3}\epsilon(p_4, p_1)^+_{\mu_4}, \\ & q = -(p_1 + p_2) = p_4 + p_3, \end{aligned}$$

we have for the first factor

$$\begin{aligned} g^{\mu_1\mu_2}\epsilon(p_1, p_4)^-_{\mu_1}\epsilon(p_2, p_4)^-_{\mu_2} \to & [p_4|\tilde{\sigma}_\mu|p_1 > [p_4|\tilde{\sigma}^\mu|p_2 > \\ & = - < p_1|\sigma_\mu|p_4][p_4|\tilde{\sigma}^\mu|p_2 > \\ & = 2 < p_1\ p_2 > [p_4\ p_4] = 0. \end{aligned}$$

Similar arguments give

$$\epsilon_3 \cdot \epsilon_4 = \epsilon_2 \cdot \epsilon_4 = \epsilon_1 \cdot \epsilon_4 = \epsilon_1 \cdot \epsilon_3 = 0.$$

We are then left with

$$A_s = \frac{-4\epsilon_2 \cdot \epsilon_3 p_3 \cdot \epsilon_4 p_2 \cdot \epsilon_1}{(p_1 + p_2)^2},$$

which is now evaluated by

$$\epsilon_2 \cdot \epsilon_3 = -\frac{1}{2} \frac{[p_4|\tilde{\sigma}_\mu|p_2 >< p_1|\sigma_\mu|p_3]}{[p_4 \ p_2] < p_1 \ p_3 >} = \frac{[p_4 \ p_3] < p_1 \ p_2 >}{[p_4 \ p_2] < p_1 \ p_3 >},$$

$$p_3 \cdot \epsilon_4 = \frac{< p_1|\sigma \cdot p_3|p_4]}{\sqrt{2} < p_1 \ p_4 >} = \frac{< p_1 \ p_3 > [p_3 \ p_4]}{\sqrt{2} < p_1 \ p_4 >},$$

and

$$p_2 \cdot \epsilon_1 = \frac{[p_4 \ p_2] < p_2 \ p_1 >}{\sqrt{2}[p_4 \ p_1]}.$$

Altogether, they yield

$$A_s = 2 \frac{[p_3 \ p_4]^2 < p_1 \ p_2 >}{< p_1 \ p_4 > [p_4 \ p_1][p_2 \ p_1]},$$

which will be rearranged by writing as

$$A_s = 2 \frac{[p_3 \ p_4]^2 < p_1 \ p_2 >^3}{< p_1 \ p_4 > ([p_4 \ p_1] < p_1 \ p_2 >)(< p_1 \ p_2 > [p_2 \ p_1])}.$$

Then, we apply momentum conservation to simplify:

$$\begin{aligned}[p_4\ p_1] < p_1\ p_2 > &= -[p_4|\tilde{\sigma}\cdot p_1|p_2 > \\ &= [p_4|\tilde{\sigma}\cdot(p_2+p_3+p_4)|p_2 > \\ &= [p_4|\tilde{\sigma}\cdot p_3|p_2 >= -[p_4\ p_3] < p_3\ p_2 >,\end{aligned}$$

and

$$\begin{aligned}< p_1\ p_2 > [p_2\ p_1] &= -(p_1+p_2)^2 = -(p_3+p_4)^2 \\ &=< p_3\ p_4 > [p_4\ p_3],\end{aligned}$$

which give

$$A_s = -2\frac{< p_1\ p_2 >^4}{< p_1\ p_2 >< p_2\ p_3 >< p_3\ p_4 >< p_4\ p_1 >},$$

or

$$M(1^-2^-3^+4^+) = \frac{< 12 >^4}{< 12 >< 23 >< 34 >< 41 >},$$

where we skip writing p for each momentum. This structure is an example of what is called the maximal helicity violating amplitude

$$\begin{aligned}&M(1^+\dots(i-1)^+i^-(i+1)^+\dots(j-1)^+j^-(j+1)^+\dots n^+) \\ &= \frac{< ij >^4}{< 12 >< 23 > \dots < (n-1)n >< n1 >},\end{aligned} \tag{6.1}$$

$$M(1^- \ldots (i-1)^- i^+ (i+1)^- \ldots (j-1)^- j^+ (j+1)^- \ldots n^-) \\ = \frac{[ij]^4}{[12][23]\ldots[(n-1)n][n1]}, \tag{6.2}$$

which can be proved recursively. The easiest way to go about it is to complexify the momenta, which we shall do, and then use Cauchy integration to obtain it.

We also want to point out that

$$\begin{aligned} M(1^+2^+ \ldots n^+) &= M(1^-2^- \ldots n^-) \\ &= M(1^+2^+ \ldots (i-1)^+ i^- (i+1)^+ \ldots n^+) \\ &= M(1^-2^- \ldots (i-1)^- i^+ (i+1)^- \ldots n^-) \\ &= 0. \end{aligned}$$

(7) Complex Momenta

We have given examples, using spinor method to simplify calculations. That is of course a big step forward already. However, there are still short comings, the worst of which is that for each new process, we have to start from scratch and apply Feynman rules to construct all the diagrams. The algebra gets much more complicated as we add another particle to a process. There is also the feeling that much effort may be wasted, because there is a huge amount of cancellations to make the final answers much simpler than the number of terms we need to handle at the beginning. It will be of great help if we can find a procedure to recycle some of the old results so arduously obtained. This is where the method of complex momenta comes in. To extend kinematics into complex domain is something that the S-matrix theorists did in the sixties and seventies. The success was limited, because the focus then was to use dispersion technique to obtain amplitudes from some spectral densities, hoping that one can avoid perturbation. We are not going to recount the difficulties they were confronted with in analytic continuation and gauge invariance. The method through complexification we are going to discuss now is quite different, in that at least for tree amplitudes, its aim is to obtain higher point amplitudes from lower point ones recursively. There is no ambiguity as to how to do analytic continuation, because we shall be dealing with rational functions of a complex variable. Gauge invariance is also automatic, because at the end we shall have only on-shell amplitudes.

We shall use the four gluon amplitude $M(1^-2^-3^+4^+)$ as a concrete example. We shall re-derive the result by relating it to the on-shell three gluon amplitudes. There is one important remark we need to make immediately. If we confine ourselves to real momenta, the three gluon on-shell amplitudes are ill-defined. The reason is that if $p_1 + p_2 + p_3 = 0$, and if p's are real, then

$$[p_i\ p_j] =< p_j\ p_i >^\star .$$

Then

$$0 = -p_1^2 = -(p_2 + p_3)^2 = -2p_2 \cdot p_3 = | < p_2\ p_3 > |^2,$$

and similar algebra gives

$$| < p_3\ p_1 > |^2 = | < p_1\ p_2 > |^2 = 0,$$

which means

$$< p_i\ p_j >= 0, \quad [p_i\ p_j] = 0, \quad i, j = 1, 2, 3. \qquad (7.1)$$

However, we remarked before that we would like to have

$$M(1^-2^-3^+) = \frac{<12>^3}{<23><31>}, \quad M(1^+2^+3^-) = \frac{[12]^3}{[23][31]}. \tag{7.2}$$

We do not know what to make of eq.(7.2) if eq.(7.1) is true. However, if p's are allowed to be complex, we can choose $[p_i\ p_j] = 0$, but $< p_i\ p_j > \neq 0$, giving meaning to $M(1^-2^-3^+)$, or $[p_i\ p_j] \neq 0$, but $< p_i\ p_j >= 0$, giving meaning to $M(1^+2^+3^-)$. As for the derivation of eq.(7.2), if one takes V_3 and eq.(3.1) and tries to do it directly, one will find a lot of ambiguities of $\frac{0}{0}$ type. A much better way is to quantize QCD in the space cone gauge, then the results will fall out easily. Let us just accept eq.(7.2).

We expect to obtain the s-channel contribution as

$$M_s(1^-2^-3^+4^+) \sim M(1^-2^-k^+)\frac{1}{-s}M(-k^-3^+4^+),$$

$$s = -k^2, \quad k = -(p_1 + p_2).$$

Please note that a propagator can join only a negative helicity end to a positive helicity end, or vice versa. However, we don't know what the right hand side means for real momenta.

As said, we have at the back of our mind that these momenta are complex, so that M_3 can be defined. In fact, we make

$$1 = |1> [1| \rightarrow \hat{1} = |1> [\hat{1}|, \quad [\hat{1}| = [1| + z[4|, \quad |\hat{1}>= |1>, \tag{7.1}$$

$$2 \rightarrow 2, \qquad 3 \rightarrow 3,$$

$$4 = |4> [4| \rightarrow \hat{4} = |\hat{4}> [4|, \quad |\hat{4}>= |4> -z|1>, \quad [\hat{4}| = [4|, \tag{7.2}$$

where we have abbreviated our notation by using their numerical subscripts to denote the corresponding momentum vectors and also skip the σ^{μ} between $|\ >$ and $[\ |$ which converts vectors into bi-spinors. Because these hatted vectors are still products of spinors, their norms still vanish. In other words, they still represent massless particles. Also, it is easy to check that four momentum conservation is respected

$$\hat{p}_1 + p_2 + p_3 + \hat{p}_4 = p_1 + p_2 + p_3 + p_4 = 0.$$

We shall show that taking z as a complex variable, the analytic continuation can give rise to simple poles only but not cuts in $\hat{M}_s(\hat{1}^- 2^- 3^+ \hat{4}^+)$. In that case, if

$$\hat{M}_s(\hat{1}^- 2^- 3^+ \hat{4}^+) \rightarrow 0, \quad z \rightarrow \infty,$$

we can use contour integration to write

$$\begin{aligned}\frac{1}{2\pi i}\oint\frac{dz}{z}\hat{M}_s(\hat{1}^-2^-3^+\hat{4}^+) \\ &= 0 \\ &= M_s(1^-2^-3^+4^+) \\ &\quad + \sum_i \frac{1}{z_i} Residue\ of\, \hat{M}_s(\hat{1}^-2^-3^+\hat{4}^+)|_{z=z_i}.\end{aligned} \tag{7.3}$$

This is known as BCFW recursion.

Under the continuation of eqs. (7.1) and (7.2), we have in the s-channel

$$\hat{M}_s(\hat{1}^-2^-3^+\hat{4}^+) = \hat{M}(\hat{1}^-2^-\hat{k}^+)\frac{1}{-\hat{s}}\hat{M}(-\hat{k}^-3^+\hat{4}^+),$$

$$\hat{k} = -(\hat{p}_1 + p_2).$$

We are going to show that the only pole is from the propagator. Now

$$\begin{aligned}&\hat{M}(\hat{1}^-2^-\hat{k}^+)\hat{M}((-\hat{k})^-3^+\hat{4}^+) \\ &\qquad = \frac{<12>^3\,[34]^3}{<2\hat{k}><\hat{k}1>[4(-\hat{k})][(-\hat{k})3]},\end{aligned}$$

the denominator of which is

$$
\begin{aligned}
&< 2\hat{k} > [(-\hat{k})4] < 1\hat{k} > [(-\hat{k})3] \\
&\qquad =< 2|\Big(|1 > ([1| + z[4|) + |2 > [2|\Big)|4] \\
&\qquad\quad \times < 1|\Big(|1 > ([1| + z[4|) + |2 > [2|\Big)|3 > \\
&\qquad =< 21 > [14] < 12 > [23],
\end{aligned}
\tag{7.4}
$$

which has no z dependence. As for

$$
\hat{s} = -(\hat{p}_1 + p_2)^2 =< \hat{1}2 > [2\hat{1}] =< 12 > ([21] + z[24]),
$$

which has a simple pole at

$$
z_s = -\frac{[21]}{[24]}.
$$

We want to take note that the intermediate vector at the position of the pole can be calculated, which turns out to be

$$
\hat{k}|_{z_s} = \frac{[34]}{[24]}|3 > [2|.
$$

We can choose for example

$$
|\hat{k} >_{z_s} = \frac{[34]}{[24]}|3 >, \quad [\hat{k}|_{z_s} = [2|.
$$

Where we put the factor $\frac{[34]}{[24]}$ is immaterial, because it will cancel out, as we have shown in eq.(7.4).

It is now easy to put everything together and evaluate the integral of eq.(7.3), which results in

$$M_s(1^-2^-3^+4^+) = \frac{< 12 >^4}{< 12 >< 23 >< 34 >< 41 > .} \qquad (7.5)$$

We can show easily that with the shift we have in eqs.(7.1) and (7.2), $M_t = 0$, and therefore eq.(7.5) is the answer for $M(1^-2^-3^+4^+)$.

This example demonstrated that if we can make sure that the asymptotic behavior $M(z \to \infty) \to 0$ is ascertained, or better yet if one can arrange to have all the z-poles coming from propagators only, then to obtain higher point amplitudes from lower point ones can be quite readily done. Again, it turns out that the space-cone gauge of QCD has these properties.

In general, the continuation is done by two properly chosen momenta P_i and P_j such that we make

$$P_i \to \hat{P}_i = |i > [\hat{i}|, \quad [\hat{i}| = [i| + z[j|,$$

and

$$P_j \to \hat{P}_j = |\hat{j} > [j|, \quad |\hat{j} >= |j > -z|i > .$$

Clearly, energy momenta are conserved, because

$$\hat{P}_i + \hat{P}_j = P_i + P_j.$$

Also $\hat{P}_i$ and $\hat{P}_j$ are massless, because they are the product of two spinors. (Think of a 2×2 matrix, which can be written as the product of one row and one column vector. Its determinant vanishes.)

The most important point to check is that the mentioned asymptotic behavior $\hat{M}(z) \to 0$ as $z \to \infty$ is satisfied. This is not a trivial matter, because if we have to know the analytic form of $\hat{M}(z)$ in order to find out, then it becomes futile. It turns out that one can use the space-cone gauge to show that this is true for any number of gluons at the Lagrangian level. The space-cone gauge is to impose a condition

$$N \cdot A_a = 0, \qquad N = |i > [j|.$$

One can show that the interaction part of the Lagrangian is invariant under the z shift discussed above, in other words, the vertices do not depend on z and only the propagators depend on it and therefore M has very good asymptotic behavior.

D. Vaman and Y.-P. Yao, JHEP **0604**, 030 (2006).