Quantum Field Theory
Homework set 2, Due Fri Nov 5

1. The Dirac Hamiltonian for free particle is given by

$$H = \vec{\alpha} \cdot \vec{p} + \beta m$$

The angular momentum operator is of the form,

$$\vec{L} = \vec{r} \times \vec{p}$$

(a) Compute the commutators,

$$\left[\vec{L}, H\right]$$

Is $\vec{L}$ conserved?

(b) Define $\vec{S} = -\frac{i}{4}\left(\vec{\alpha} \times \vec{\alpha}\right)$ and show that

$$\left[\vec{L} + \vec{S}, H\right] = 0$$

(c) Show that $\vec{S}$ satisfy the angular momentum algebra, i.e.

$$[S_i, S_j] = i\varepsilon_{ijk} S_k$$

and

$$\vec{S}^2 = \frac{3}{4}.$$

2. The Dirac spinors are of the form,

$$u\left(p, s\right) = \sqrt{E+m} \begin{pmatrix} 1 \\ \dfrac{\vec{\sigma} \cdot \vec{p}}{E+m} \end{pmatrix} \chi_s, \qquad v\left(p, s\right) = \sqrt{E+m} \begin{pmatrix} \dfrac{\vec{\sigma} \cdot \vec{p}}{E+m} \\ 1 \end{pmatrix} \chi_s \qquad s = 1, 2$$

where

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a) Show that

$$\bar{u}\left(p, s\right) u\left(p, s'\right) = 2m\delta_{ss'}, \qquad \bar{v}\left(p, s\right) v\left(p, s'\right) = 2m\delta_{ss'}$$
$$\bar{v}\left(p, s\right) u\left(p, s'\right) = 0, \qquad \bar{u}\left(p, s\right) v\left(p, s'\right) = 0$$

(b) Show that

$$\sum_s u_\alpha\left(p, s\right) \bar{u}_\beta\left(p, s\right) = \left(\not{p} + m\right)_{\alpha\beta}$$
$$\sum_s v_\alpha\left(p, s\right) \bar{v}_\beta\left(p, s\right) = \left(\not{p} - m\right)_{\alpha\beta}$$

3. Suppose a free Dirac particle at t=0, is described by a wavefunction,

$$\psi\left(0, \vec{x}\right) = \frac{1}{\left(\pi d^2\right)^{3/4}} \exp\left(-\frac{r^2}{2d^2}\right) \omega$$

where d is some constant and

$$\omega = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Compute the wavefunction for $t \neq 0$. What happens when d is very small.

4. One-dimensional Schrodinger equation is given by

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V\left(x\right)\right]\psi\left(x,t\right)=i\hbar\frac{\partial\psi}{\partial t}$$

   (a) Find the Lagrangian density which will give Schrodinger equation as the equation of motion.
   (b) From the Lagrangian density find the conjugate momenta and impose the quantization conditions. Also find the Hamiltonian.
   (c) Suppose the potential $V\left(x\right)$ is a square-well potential such that the energy eigenvalues are

$$E_1, E_2, \cdots$$

   and corresponding eigenfunctions are given by

$$\phi_1, \phi_2, \cdots$$

   Expand the field operator $\psi\left(x,t\right)$ in terms of the eigenfunctions of this potential

$$\psi\left(x,t\right)=\sum_n a_n\phi_n\left(x\right)e^{iE_nt/\hbar}$$

   where $\phi_n\left(x\right)$ is the normalized eigenfunction of this potential. Compute the commutors

$$\left[a_n, a_m\right], \qquad \left[a_n, a_m^\dagger\right]$$

   (d) Find the eigenvalues of the Hamiltonian.

5. (Optional) Consider a one-dimensional string with length L which satisfies the wave equaiton,

$$\frac{\partial^2\phi}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}$$

   (a) Find the solutions of this wave equation with the boundary conditions,

$$\phi\left(0,t\right)=\phi\left(L,t\right)=0$$

   (b) Find the Lagrangian density which will give this wave equation as the equation of motion.
   (c) From the Lagrangian density find the conjugate momenta and impose the quantization conditions. Also find the Hamiltonian.
   (d) Find the eigenvalues of the Hamiltonian.

6. (Optional) Consider a 2×2 hermitian matrix defined by

$$X=x_0+\vec{\sigma}\cdot\vec{x}$$

where $\vec{\sigma}=\left(\sigma_1,\sigma_2,\sigma_3\right)$ are Pauli matrices and $\left(x_0,\vec{x}\right)$ are space-time coordinates.

   (a) Compute the determinant of X
   (b) Suppose U is a 2×2 matrix with $\det U=1$. Define a new 2×2 matrix by a similarity transformation,

$$X'=UXU^\dagger$$

   Show that X' can be written as

$$X'=x_0'+\vec{\sigma}\cdot\vec{x}'$$

   (c) Show that the relation between $\left(x_0,\vec{x}\right)$ and $\left(x_0',\vec{x}'\right)$ is a Lorentz transformation.
   (d) Suppose U is of the form,

$$U=\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

   Find the relation between $\left(x_0,\vec{x}\right)$ and $\left(x_0',\vec{x}'\right)$.