33-779 Nuclear and Particle Physics I

November 9, 2010

Homework set 3, Due Tue Nov 23

1. The Dirac equation for free particle is given by,

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0$$

   Under the parity transformation the space-time coordiante transform as

$$x^\mu \to x'^\mu = (x_0, -x_1, -x_2, -x_3)$$

   The Dirac equation in the new coordinate system is of the form,

$$\left(i\gamma^\mu \partial'_\mu - m\right)\psi'(x') = 0$$

   Find the relation between $\psi(x)$ and $\psi'(x')$.

2. Let ϕ be a free scalar field satisfying the field equation,

$$\left(\partial^\mu \partial_\mu + \mu^2\right)\phi(x) = 0$$

   (a) Show that the propagator defined by

$$\Delta_F(x-y) \equiv \langle 0\,|T(\phi(x)\,\phi(y))|\,0\rangle = \theta(x_0 - y_0)\,\phi(x)\,\phi(y) + \theta(y_0 - x_0)\,\phi(y)\,\phi(x)$$

   can be written as

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \frac{i}{k^2 - \mu^2 + i\varepsilon}$$

   (b) Show that the unequal time commutator for these free fields is given by

$$i\Delta(x-y) \equiv [\phi(x), \phi(y)] = \int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\left[e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right]$$

   (c) Show that $\Delta(x-y) = 0$ for space-like separation, i.e.

$$\Delta(x-y) = 0, \qquad \text{if} \quad (x-y)^2 < 0$$

3. Consider the Lagrangian density given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 + J(x)\,\phi, \qquad J(x) \quad \text{arbitray function}$$

   (a) Show that the equation of motion is of the form,

$$\left(\partial^\mu \partial_\mu + \mu^2\right)\phi(x) = J(x)$$

   (b) Find the conjugate momenta and impose the quantization conditions.

   (c) Find the creation and annihilation operators.

4. Dirac particle in the presence of electromagnetic field satisfies the equation,

$$\left[\gamma^{\mu}\left(i\partial_{\mu}-eA_{\mu}\right)-m\right]\psi\left(x\right)=0$$

Or

$$i\frac{\partial\psi}{\partial t}=\left[\vec{\alpha}\cdot\left(\vec{p}-e\vec{A}\right)+\beta m+e\Phi\right]\psi$$

In the non-relativistic limit, we can write

$$\psi\left(x\right)=e^{-imt}\begin{pmatrix}u\\ l\end{pmatrix}$$

Show that the upper component satisfies the equation,

$$i\frac{\partial u}{\partial t}=\left[\frac{1}{2m}\left(\vec{p}-e\vec{A}\right)^{2}-\frac{e}{m}\vec{\sigma}\cdot\vec{B}+e\Phi\right]u$$

For the case of weak uniform magnetic field $\vec{B}$ we can take $\vec{A}=\frac{1}{2}\vec{B}\times\vec{r}$. Show that

$$i\frac{\partial u}{\partial t}=\left[\frac{1}{2m}\left(\vec{p}\right)^{2}-\frac{e}{2m}\left(\vec{L}+2\vec{S}\right)\cdot\vec{B}\right]u.$$

5. (Optional) Consider Lagrangian density of the form,

$$\mathcal{L}=\frac{1}{2}\partial_{\mu}\phi^{\dagger}\partial^{\mu}\phi-\frac{1}{2}\mu^{2}\phi^{\dagger}\phi$$

(a) Find the conjugate momenta and impose the quantization conditions.

(b) Find the creation and annihilation operators.

(c) Show that the Lagrangian density is invariant under the transformation,

$$\phi\rightarrow\phi'=e^{i\alpha}\phi,\qquad\alpha\quad\text{some constant}$$

Use the Nother's theorem to find the conserved charge Q correspong to this symmetry

(d) Compute the following commutators,

$$\left[Q,\phi\left(x\right)\right],\qquad\left[Q,\phi^{\dagger}\left(x\right)\right]$$

(e) Compute the following vacuum vacuum expectation values,

$$\left\langle 0\left|T\left(\phi\left(x\right)\phi\left(y\right)\right)\right|0\right\rangle,\qquad\left\langle 0\left|T\left(\phi\left(x\right)\phi^{\dagger}\left(y\right)\right)\right|0\right\rangle,\qquad\left\langle 0\left|T\left(\phi^{\dagger}\left(x\right)\phi^{\dagger}\left(y\right)\right)\right|0\right\rangle$$

6. (Optional) Consider Lagrangian density of the form,

$$\mathcal{L}=\frac{1}{2}\partial_{\mu}\vec{\phi}\cdot\partial^{\mu}\vec{\phi}-\frac{1}{2}\mu^{2}\vec{\phi}\cdot\vec{\phi}$$

where $\vec{\phi}=\left(\phi_{1},\phi_{2},\phi_{3}\right)$ is a collection of 3 real scalar fields.

(a) Find the conjugate momenta and impose the quantization conditions.

(b) Find the creation and annihilation operators.

(c) Show that the Lagrangian density is invariant under the transformation,

$$\phi_{i}\rightarrow\phi_{i}'=R_{ij}\phi_{j},\qquad\text{with}\quad RR^{T}=R^{T}R=1$$

Use the Nother's theorem to find the conserved charges Q^{i}, $i=1,2,3$ correspong to these symmetries.

(d) Compute the commutators,

$$\left[Q^{i},\phi^{j}\right]$$

(e) Compute the commutators,

$$\left[Q^{i},Q^{j}\right]$$