Quantum Field Theory I

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Homework set 4, Due Fri Dec 17

1. Dirac equation for electron moving in the electromagnetic field can be obtained from the free Dirac equation by the replacement $i\partial_\mu \longrightarrow i\partial_\mu - eA_\mu$,

$$\left[\gamma^\mu \left(i\partial_\mu - eA_\mu\right) - m\right]\psi\left(\vec{x}, t\right) = 0$$

Then the equation for the positron is

$$\left[\gamma^\mu \left(i\partial_\mu + eA_\mu\right) - m\right]\psi_c\left(\vec{x}, t\right) = 0$$

Assume that ψ_c is related to ψ by

$$\psi_c = \tilde{C}\psi^*$$

$\tilde{C}$ is called the charge conjugation matrix.

(a) Find $\tilde{C}$ in terms of Dirac γ matrices.

(b) For the $v-$spinor of the form,

$$v\left(p, s\right) = N\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_s$$

Compute its charge conjugate $v_c\left(p, s\right) = \tilde{C}v^*\left(p, s\right)$

(c) To implement the charge conjugation for the fermion field, we write

$$\psi_c = C\psi C^{-1} = \tilde{C}\psi^*$$

where C is the charge conjugation operator. Find the relation between $\bar{\psi}_c\gamma^\mu\psi_c$ and $\bar{\psi}\gamma^\mu\psi$.

2. The parity operator P is defined as

$$\psi_p = P\psi P^{-1} = \gamma_0\psi\left(t, -\vec{x}\right)$$

(a) Find the relation between $\bar{\psi}_p\psi_p$ and $\bar{\psi}\psi$.

(b) Repeat part (a) for $\bar{\psi}_p\gamma_5\psi_p$, $\bar{\psi}_p\gamma_\mu\psi_p$,$\bar{\psi}_p\gamma_\mu\gamma_5\psi_p$,$\bar{\psi}_p\sigma_{\mu\nu}\psi_p$.

3. The left-handed and right-handed components of a Dirac particle are defined by,

$$\psi_L \equiv \frac{1}{2}\left(1-\gamma_5\right)\psi, \qquad \psi_R \equiv \frac{1}{2}\left(1+\gamma_5\right)\psi$$

(a) Show that they are eigenstates of γ_5 matrix. What are the eigenvalues?

(b) Are they eigenstates of parity operator?

(c) Write the u spinor in the form,

$$u\left(p, s\right) = N\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s$$

where N is some normalization constant and χ_s is an arbitrary 2 component spinor. Show that if we choose χ_s to be eigenstate of $\vec{\sigma}\cdot\vec{p}$,

$$\left(\vec{\sigma}\cdot\hat{p}\right)\chi_s = \frac{1}{2}\chi_s$$

then $u(p,s)$ is an eigenstate of the helicity operator $\lambda = \vec{S}\cdot\hat{p}$ where $\vec{S}$ is the spin operator given by

$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

(d) Show that in the limit $m \to 0$, the helicity eigenstate is also eigenstate of γ_5.

4. Consider a free scalar field $\phi(x)$ where the 4-momentum operator is of the form,

$$P^\mu = \int d^3k\, k^\mu a^\dagger(k)\, a(k)$$

(a) As a useful tool show that for two operators A and B, the following identity holds

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots$$

(b) Use this identity to show that

$$e^{iP\cdot x} a(k)\, e^{-iP\cdot x} = a(k)\, e^{-ik\cdot x}$$

and

$$[P^\mu, \phi(x)] = i\partial^\mu \phi(x)$$

(c) Let $|K\rangle$ be an eigenstate of P^μ, satisfying $P^\mu |K\rangle = K^\mu |K\rangle$. Show that

$$\langle K|\phi(x)\,\phi(y)|K\rangle = \langle K|\phi(x-y)\,\phi(0)|K\rangle$$

5. (Optional) Consider the following physical processes,

$$e^+(p_1) + e^-(p_2) \longrightarrow \gamma(k_1) + \gamma(k_2)$$

(a) To lowest order in e, draw all Feynman diagrams which contribute to this processes.

(b) Write out the matrix element for this processes

(c) Compute the unpolarized differential cross section in the center of mass frame.