Quantum Field Theory I

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December 21, 2010

Homework set 5, Due Tue Jan 4

1. In the $\lambda\phi^4$ theory, the interaction is given by
$$H_I = \frac{\lambda}{4!}\phi^4$$
   (a) Show that to lowest order in λ, the differential cross section for 2 particles elastic scattering in the center of mass frame is given by
$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{128\pi^2 S}$$
   where $S = (q_1 + q_2)^2$, q_1, q_2 are the momenta of the particles in the initial state.

   (b) Show that the 2-point Green's function satisifies the relation,
$$\left(\Box_x + \mu^2\right)\langle 0\left|T\left(\phi(x)\phi(y)\right)\right|0\rangle = \lambda\left\langle 0\left|T\left(\phi^3(x)\phi(y)\right)\right|0\right\rangle - i\delta^4(x-y)$$
   Verify this relation diagramatically to first order in λ.

2. Given a sequence of numbers $g_0, g_1, g_3, \cdots$, the most economical way to encompass them is through the generating function,
$$Z(j) = \sum_{m=0}^{\infty}\frac{g_m j^m}{m!}$$
   so that
$$g_m = \left.\frac{\partial^m Z(j)}{\partial j^m}\right|_{j=0}$$
   The generating functional for the Green's functions of scalar fields is defined by,
$$Z[J] = \sum_{m=0}^{\infty}\frac{i^m}{m!}\int d^4x_1\cdots d^4x_m J(x_1)\cdots J(x_m)\, G^{(m)}(x_1,\cdots x_m)$$
   where
$$G^{(m)}(x_1,\cdots x_m) = \langle 0\left|T\left(\phi(x_1)\cdots\phi(x_m)\right)\right|0\rangle$$
   and can be expressed in terms of generating functional as
$$G^{(m)}(x_1,\cdots x_m) = (-i)^m\left.\frac{\delta^m Z[J]}{\delta J(x_1)\cdots\delta J(x_m)}\right|_{J=0}$$
   (a) Show that $Z[J]$ can be written as
$$Z[J] = \left\langle 0\left|T\exp\left(i\int\phi(x)J(x)\,d^4x\right)\right|0\right\rangle$$
   (b) Show that for the case of free scalar field $Z[J]$ can be written as
$$Z[J] = \exp\left[-\frac{i}{2}\int d^4x d^4y J(x)\,\Delta_F(x-y)\,J(y)\right]$$
   where
$$i\Delta_F(x-y) = \langle 0\left|T\left(\phi(x)\phi(y)\right)\right|0\rangle$$

3. Transistion amplitude for a free particle with mass m moving in one dimension is given by

$$\langle q', t' | q, t \rangle = \left\langle q' | e^{-iH(t'-t)} | q \right\rangle$$

and

$$H = \frac{p^2}{2m}$$

(a) Compute this amplitude directly from the Hamiltonian given above.

(b) Compute this amplitude by using path integral method.

4. The Lagrangian for a simple harmonic oscillator is of the form

$$L = \frac{m}{2} \dot{q}^2 - \frac{m\omega^2}{2} q^2$$

(a) Show that the transistion amplitude has the form,

$$\langle q_f, t_f | q_i, t_i \rangle = \left[\frac{m\omega}{2\pi i \sin \omega (t_f - t_i)} \right]^{1/2} \exp \left\{ \frac{im\omega}{2 \sin \omega (t_f - t_i)} \left[\left(q_f^2 + q_i^2 \right) \cos \omega (t_f - t_i) - 2 q_f q_i \right] \right\}$$

(b) Show that for an initial wave packet of the Gaussian form

$$\psi_a (q, 0) = \left(\frac{m\omega}{\pi} \right)^{1/4} \exp \left[-\frac{m\omega}{2} (q - a)^2 \right]$$

the wave function for $t \neq 0$ satisfies

$$|\psi_a (q, 0)|^2 = \left(\frac{m\omega}{\pi} \right)^{1/2} \exp \left[-\frac{m\omega}{2} (q - a \cos \omega t)^2 \right]$$

Namely, there is no spread of wave packet.

5. The Lagrangian for electromagnetic field is of the form,

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Suppose we add a mass term to the Lagrangian so that

$$L' = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + m^2 A_\mu A^\mu$$

(a) Show that the equation of motion is of the form,

$$\left(\partial^\mu \partial_\mu + m^2 \right) A^\nu - \partial^\nu \partial_\mu A^\mu = 0$$

(b) Define the generating functional by

$$W[J] = \int [dA^\mu] \exp \left\{ i \int d^4x \left(L' + J^\mu A_\mu \right) \right\}$$

Show that

$$W[J] = \exp \left\{ \frac{i}{2} \int d^4x d^4y J^\mu (x) G_{\mu\nu} (x - y) J^\nu (y) \right\}$$

where Green's function by

$$\left[\left(\partial^\alpha \partial_\alpha + m^2 \right) g_{\mu\nu} - \partial_\nu \partial_\mu \right] G^{\nu\beta} (x - y) = g_\mu^\beta \delta^4 (x - y)$$

Find the Green's function in momentum space.