## Note 1

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## 1 Introduction

### 1.1 Necessity of field theory in relativistic system

In non-relativistic quantum mechanics, Schrodinger equation implies conservation of particle number. This can be seen as follows. Schrodinger Equation gives

$$
H \psi=i \hbar \frac{\partial \psi}{\partial t} \quad \Rightarrow \quad \int d^{3} x\left(\psi^{\dagger} H \psi\right)=i \hbar \int d^{3} x\left(\psi^{\dagger} \frac{\partial \psi}{\partial t}\right)
$$

and its complex conjugate

$$
\psi^{\dagger} H=-i \hbar \frac{\partial \psi^{\dagger}}{\partial t} \quad \Rightarrow \quad \int d^{3} x\left(\psi^{\dagger} H \psi\right)=-i \hbar \int d^{3} x\left(\psi \frac{\partial \psi^{\dagger}}{\partial t}\right)
$$

Here we have used the fact that Hamiltonian is hermitian, $H=H^{\dagger}$. Then

$$
\Longrightarrow \frac{d}{d t} \int d^{3} x \psi^{\dagger} \psi=0 \rightarrow \int d^{3} x\left(\psi^{\dagger} \psi\right) \quad \text { independent of time }
$$

Thus the number of particles is conserved and no particle creation or annihilation. But from the canonical commutation relation,

$$
[x, p]=-i \hbar
$$

we get the uncertainty relation

$$
\triangle x \triangle p \geqslant \hbar
$$

For relativistic system, the uncertainty in $p$ will induce uncertainty in energy $E$ through the relation,

$$
p^{2} c^{2}+m^{2} c^{4}=E^{2}
$$

to give

$$
\triangle E=\frac{p \triangle p}{E} c^{2} \geqslant \frac{p \hbar c^{2}}{E \triangle x} \quad \text { or } \quad \triangle x \geqslant \frac{p c}{E}\left(\frac{\hbar c}{\triangle E}\right)
$$

To avoid new particle creation we require $\triangle \mathrm{E} \leqslant \mathrm{mc}^{2}$. Then we get a lower bound on $\triangle x$

$$
\Delta x \geqslant \frac{p c}{E} \frac{\hbar}{m c}=\left(\frac{v}{c}\right)\left(\frac{\hbar}{m c}\right)
$$

(a) Non-relativistic particles,

Here the velocity is small compared with $c$,

$$
\frac{v}{c} \ll 1
$$

Then $\Delta \mathrm{x}$ can be made arbitrary small and wavefunction $\psi(\mathrm{x})$ has the meaning that $|\psi(\mathrm{x})|^{2}$ is the probability density for finding the particle in a volume $d^{3} x$ around $\vec{x}$. In other words, particles can be localized for arbitrary small volume in space.
(b) Relativistic particle

In this case, we have

$$
\frac{v}{c} \approx 1,
$$

and

$$
\Delta x \geqslant\left(\frac{\hbar}{m c}\right)
$$

This means that particles can not be localized to a distance smaller than the Compton wavelength ot the particle. In other words, if we use the wavefunction to describe particles, then particle creation can not be avoided at distance $\triangle x$ smaller than theCompton wavelength ot the particle

Note that the usual relativistic wave equations in the forms of Klein-Gordon equations or Dirac equations run into difficulties precisely because particle creations are not inclused.

### 1.2 Non-relativistic field theory-many body problem

Field theory is also a very useful tool for the study of collective phenomena in the non-relativistic many body problem in the form of quasi-particles e.g. superfluidity,superconductivity........ The formalism here is usually called the Second quantization where one creation and annihilation operators $\mathrm{a}_{k}, \mathrm{a}_{k}^{\dagger}$ for each energy level. For example the transisition from state $l$ to state $k$ is described by the operator ( $\mathrm{a}_{k}^{\dagger} \mathrm{a}_{k}$ ) annihilates a particle in state $l$ and creates a particle in state k .

### 1.3 Gauge principle

The development of high energy physics in recent years has converged to the description of all fundamental Interactions in terms of gauge theories;
(a) Strong Interaction-QCD;
gauge theory based on $\mathrm{SU}(3)$ symmetry
(b) Electromagnetic and Weak interaction-

Electroweak theory-gauge theory based on $S U(2) \times U(1)$ symmetry
(c) Gravitational interaction-

Einstein's theory-gauge theory of local coordinate transformation.

### 1.4 Natural unit

In high energy physics it is very convient to us the natural unit in which

$$
\hbar=c=1
$$

so that many formulae simplified. Recall that in MKS units

$$
\hbar=1.055 \times 10^{-34} J \mathrm{sec}
$$

Thus $\hbar=1$ implies that energy has the same dimension as (time) $)^{-1}$. Also

$$
c=2.99 \times 10^{8} \mathrm{~m} / \mathrm{sec} \Longrightarrow(\text { time })=(\text { length })
$$

$c=1$ will have the effect that time and length will have the same dimension. In this unit, at the end of the calculation one puts back the factors of $\hbar$ and $c$ depending on the physical quantities in the problem. For example, the quantity $m_{e}$ can have following different meanings depending on the contexts;
(a) Reciprocal length

$$
m_{e}=\frac{1}{\frac{\hbar}{m_{e} c}}=\frac{1}{3.86 \times 10^{-11} \mathrm{~cm}}
$$

(b) Reciprocal time

$$
m_{e}=\frac{1}{\frac{\hbar}{m_{e} c^{2}}}=\frac{1}{1.29 \times 10^{-21} \sec }
$$

(c) energy

$$
m_{e}=m_{e} c^{2}=0.511 \mathrm{Mev}
$$

(d) momentum

$$
m_{e}=m_{e} c=0.511 \mathrm{Mev} / c
$$

The following conversion relations

$$
\hbar=6.58 \times 10^{-22} \mathrm{Mev}-\mathrm{sec} \quad \hbar c=1.973 \times 10^{-11} \mathrm{Mev}-\mathrm{cm}
$$

are quite useful in getting the physical quantities in the right units.
Example: Thomson cross section

$$
\sigma=\frac{8 \pi \alpha^{2}}{3 m_{e}^{2}}=\frac{8 \pi \alpha^{2}(\hbar c)^{2}}{3 m_{e}^{2} c^{4}}=\left(\frac{1}{137}\right)^{2} \times \frac{\left(1.973 \times 10^{-11} \mathrm{Mev}-c m\right)^{2}}{(0.5 \mathrm{Mev})^{2}} \times\left(\frac{8 \pi}{3}\right) \simeq 6.95 \times 10^{-25} \mathrm{~cm}^{2}
$$

Another exercise is to relate Newton constant

$$
G_{N}=6.67 \times 10^{-11} \mathrm{~m}^{3} \mathrm{~kg}^{-1} \mathrm{sec}^{-2}
$$

to some energy scale, Planck scale. Use

$$
\hbar c=3.16 \times 10^{-26} \mathrm{Jm}
$$

we can write

$$
G_{N}=6.67 \times 10^{-11}\left(\frac{m^{2} k g}{s^{2}}\right) \frac{m}{k g^{2}}=6.67 \times 10^{-11} J \frac{m}{k g^{2}}
$$

Then we get for the combination

$$
\left(\frac{\hbar c}{G_{N}}\right)=3.16 \times 10^{-26} \mathrm{Jm} \times \frac{1}{6.67 \times 10^{-11} \mathrm{Jm}} \mathrm{~kg}^{2}=4.73 \times 10^{-16}(\mathrm{~kg})^{2}
$$

Use

$$
\sqrt{\frac{\hbar c}{G_{N}}}=2.176 \times 10^{-8}(\mathrm{~kg})
$$

and

$$
1 \mathrm{kgc}^{2}=1 \mathrm{~kg} \times\left(3 \times 10^{8} \mathrm{~m} / \mathrm{sec}\right)^{2}=9 \times 10^{16} \frac{\mathrm{~m}^{2} \mathrm{~kg}}{\mathrm{~s}^{2}}=9 \times 10^{16} \mathrm{~J} \quad \text { or } \quad 1 \mathrm{~kg}=9 \times 10^{16} \mathrm{~J} / \mathrm{c}^{2}
$$

we have

$$
\Longrightarrow \sqrt{\frac{\hbar c}{G_{N}}}=1.96 \times 10^{9} \mathrm{~J} / \mathrm{c}^{2}
$$

Use the conversion factor

$$
1 e v=1.6 \times 10^{-19} J, \quad 1 G e v=1.6 \times 10^{-10} J \quad \text { or } \quad 1 J=\frac{1}{1.6} \times 10^{10} G e v
$$

we finally get

$$
m_{p} \equiv \sqrt{\frac{\hbar c}{G_{N}}}=\frac{1.96 \times 10^{9}}{1.6} \times 10^{10} \mathrm{Gev}=1.225 \times 10^{19} \mathrm{Gev} / \mathrm{c}^{2}
$$

This is the usual statement that the energy associated with gravity (Planck scale) is ${ }^{\sim} 10^{19} \mathrm{Gev}$. Another way to express the result is

$$
G_{N}=6.07 \times 10^{-39}(\hbar c)\left(\mathrm{Gev} / \mathrm{c}^{2}\right)^{-2}
$$

## 2 Review of Special Relativity

The basic principles of special relativity are:
(a) The speed of light has the same value in all inertial frames.
(b) Physical laws take the same forms in all inertial frames.

### 2.1 Lorentz transformation

The coordinate transformation which relates 2 different inertial frames is called the Lorentz transformation. The reason for using Lorentz transformation is to simplify the analysis because in some situation it is more convienent to choose one frame than the other. Suppose the coordinate system $O^{\prime}$ is moving with velocity $v$ in x-direction with respect to $O$, then their coordinates are related by

$$
\begin{equation*}
x^{\prime}=\frac{x-v t}{\sqrt{1-v^{2}}} \quad y^{\prime}=y, \quad z^{\prime}=z, \quad t^{\prime}=\frac{t-v x}{\sqrt{1-v^{2}}} \tag{1}
\end{equation*}
$$

Or

$$
x^{\prime}=\gamma(x-v t), \quad y^{\prime}=y, \quad z^{\prime}=z, \quad t^{\prime}=\gamma(t-v x), \quad \gamma=\frac{1}{\sqrt{1-v^{2}}}
$$

This implies that

$$
t^{2}-x^{2}-y^{2}-z^{2}=t^{\prime 2}-x^{\prime 2}-y^{\prime 2}-z^{\prime 2} \quad \text { or } \quad t^{2}-\vec{r}^{2}=t^{\prime 2}-{\overrightarrow{r^{\prime}}}^{2} \quad \text { Lorentz } \quad \text { invariant }
$$

In other words, the combination in the form of proper time

$$
\tau^{2}=t^{2}-\vec{r}^{2}
$$

is invariant under Lorentz transfomation. This relation ensures that the speeds of light are the same in all inertial frames and can be seen as follows.Suppose $\overrightarrow{r_{1}}\left(t_{1}\right)$ and $\overrightarrow{r_{2}}\left(t_{2}\right)$ are 2 points on a trajectory of a free particle.Then the speed of the particle is

$$
|\vec{v}|=\frac{1}{\left|t_{2}-t_{1}\right|} \sqrt{\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}+\left(z_{1}-z_{2}\right)^{2}}
$$

For the case, $|\vec{v}|=1$, we get

$$
\left(t_{1}-t_{2}\right)^{2}=\left|\overrightarrow{r_{1}}-\overrightarrow{r_{2}}\right|^{2}
$$

Since this is invariant under Lorentz transformation, we get the speed of light is the same in all inertial frames. Another common way to paremetrize the Lorentz transformation in $\operatorname{Eq}(1)$ is given by

$$
x^{\prime}=\cosh \omega x-\sinh \omega t, \quad y^{\prime}=y, \quad z^{\prime}=z, \quad t^{\prime}=\sinh \omega x-\cosh \omega t
$$

where

$$
\tanh \omega=v
$$

For the case of infinitesmal interval ( $d t, d x, d y, d z$ ), we can define the infinitesmal proper time as

$$
(d \tau)^{2}=(d t)^{2}-(d x)^{2}-(d y)^{2}-(d z)^{2}
$$

It is convient to combine space time coordiantes into Minkowski space,

$$
x^{\mu}=(t, x, y, z)=\left(x^{0}, x^{1}, x^{2}, x^{3}\right)
$$

and $x^{\mu}$ is called 4 - vector. Define Lorentz invariant product as

$$
x^{2}=\left(x_{0}\right)^{2}-\left(x_{1}\right)^{2}-\left(x_{2}\right)^{2}-\left(x_{3}\right)^{2}
$$

We can introduce a metric $g_{\mu \nu}$ to write this as,

$$
x^{2}=x_{\mu} x_{\nu} g_{\mu \nu}
$$

where

$$
g_{\mu \nu}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{array}\right)
$$

For convenience, we can define another 4 -vector

$$
x_{\mu}=g_{\mu \nu} x^{\nu}=\left(t,-x^{1},-x^{2},-x^{3}\right)=(t,-\vec{r})
$$

so that

$$
x^{2}=x^{\mu} x_{\mu}
$$

Or for infinitesmal coordinates, we have

$$
(d x)^{2}=(d x)^{\mu}(d x)_{\mu}=\left(d x^{0}\right)^{2}-(d \vec{x})^{2}
$$

Write the Lorentz transformation as

$$
x^{\mu} \rightarrow x^{\prime \mu}=\Lambda_{\nu}^{\mu} x^{\nu}
$$

For example for Lorentz transformation in the $x$-direction, we have

$$
\Lambda_{\nu}^{\mu}=\left(\begin{array}{cccc}
\frac{1}{\sqrt{1-\beta^{2}}} & \frac{-\beta}{\sqrt{1-\beta^{2}}} & 0 & 0 \\
\frac{-\beta}{\sqrt{1-\beta^{2}}} & \frac{1}{\sqrt{1-\beta^{2}}} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

Introduce generalized Lorentz transformationa as linear transformation of $x^{\mu}$ which leaves $x^{2}=x^{\mu} x_{\mu}$ invariant. Write

$$
x^{\prime 2}=x^{\prime \mu} x^{\prime \nu} g_{\mu \nu}=\Lambda_{\alpha}^{\mu} \Lambda_{\beta}^{\nu} g_{\mu \nu} x^{\alpha} x^{\beta}
$$

then $x^{2}=x^{2}$ implies

$$
\begin{equation*}
\Lambda_{\alpha}^{\mu} \Lambda_{\beta}^{\nu} g_{\mu \nu}=g_{\alpha \beta} \tag{2}
\end{equation*}
$$

and is called pseudo-orthogonality relation.

### 2.2 Energy and Momentum

We now want to write the usual physical quantities like energy and momentum in Minkowski space. Start with infintesma coordinate 4 - vector

$$
d x^{\mu}=\left(d x^{0}, d x^{1}, d x^{2}, d x^{3}\right)
$$

Since the proper time

$$
(d \tau)^{2}=d x^{\mu} d x_{\mu}=\left(d x^{0}\right)^{2}-(d \vec{x})^{2}=(d t)^{2}-\left(\frac{d \vec{x}}{d t}\right)^{2}(d t)^{2}=\left(1-v^{2}\right)(d t)^{2}
$$

is Lorentz invariant, we can get the 4 - velocity by taking the ratio,

$$
u^{\mu}=\frac{d x^{\mu}}{d \tau}=\left(\frac{d x^{0}}{d \tau}, \frac{d \vec{x}}{d \tau}\right)
$$

which is also a 4 -vector in Minkowski space. It is easy to see that there is a constraint on these 4 components of $u^{\mu}$,

$$
u^{\mu} u_{\mu}=\frac{d x^{\mu}}{d \tau} \frac{d x_{\mu}}{d \tau}=1
$$

Note that the spatial components are

$$
\vec{u}=\frac{d \vec{x}}{d \tau}=\frac{d \vec{x}}{d t}\left(\frac{d t}{d \tau}\right)=\vec{v}\left(\frac{d t}{d \tau}\right)=\frac{1}{\sqrt{1-v^{2}}} \vec{v}
$$

and for $v \ll 1$, we see that

$$
\vec{u} \approx \vec{v}
$$

In other words, $u^{\mu}$ is just the generalization of the velocity $\vec{v}$ in ordinary space. From 4 - velocity, we can form the 4 - momentum as

$$
p^{\mu}=m u^{\mu}=\left(\frac{m}{\sqrt{1-v^{2}}}, \frac{m \vec{v}}{\sqrt{1-v^{2}}}\right)
$$

For $\quad v \ll 1$,

$$
p^{0}=\frac{m}{\sqrt{1-v^{2}}}=m\left(1+\frac{1}{2} v^{2}+\ldots\right)=m+\frac{m}{2} v^{2}+\ldots
$$

which is just the total energy and

$$
\vec{p}=m \vec{v} \frac{1}{\sqrt{1-v^{2}}}=m \vec{v}+\ldots
$$

is the usual momentum. Thus energy and momentum combine into a 4 - vector,

$$
p^{\mu}=(E, \vec{p})
$$

Note that

$$
p^{2}=E^{2}-\vec{p}^{2}=\frac{m^{2}}{1-v^{2}}\left[1-v^{2}\right]=m^{2}
$$

which is a constriant between energy and momentum.

### 2.3 Tensor analysis

To implement the second part of the special relativity, physical laws take the same forms in all inertial frames, we need to express physical laws in terms of tensors in Minkowski space. We now give a simple introduction to tensor analysis in Minkowski space. Basically, tensors have the same transformation properties as product of vectors. In Minkowski space, we have two different types of vectors with transformation properties,

$$
x^{\prime \mu}=\Lambda_{\nu}^{\mu} x^{\nu}, \quad x_{\mu}^{\prime}=\Lambda_{\mu}^{\nu} x_{\nu}
$$

We can multiply these vectors to produce 3 different type of $2 n d$ rank tensors with transformation properties,

$$
T^{\prime \mu \nu}=\Lambda_{\alpha}^{\mu} \Lambda_{\beta}^{\nu} T^{\alpha \beta}, \quad T_{\mu \nu}^{\prime}=\Lambda_{\mu}^{\alpha} \Lambda_{\nu}^{\beta} T_{\alpha \beta}, \quad T_{\nu}^{\mu}=\Lambda_{\alpha}^{\mu} \Lambda_{\nu}^{\beta} T_{\beta}^{\alpha}
$$

The most general tensor will have the transformation property,

$$
\begin{equation*}
T_{\nu_{1} \cdots \nu_{m}}^{\mu_{1} \cdots \mu_{n}}=\Lambda_{\alpha_{1}}^{\mu_{1}} \cdots \Lambda_{\alpha_{n}}^{\mu_{n}} \Lambda_{\nu_{1}}^{\beta_{1}} \cdots \Lambda_{\nu_{m}}^{\beta_{m}} T_{\beta_{1} \cdots \beta_{m}}^{\alpha_{1} \cdots \alpha_{n}} \tag{3}
\end{equation*}
$$

Note the transformation of tensor components is linear and homogeneous.
Tensor operations; operation which preserves the tensor property
(a) Multiplication by a constant, $(c T)$ has the same tensor properties as $T$
(b) Addition of tensor of same rank
(c) Multiplication of two tensors
(d) Contraction of tensor indices. For example,$T_{\mu}^{\mu \alpha \beta \gamma}$ is a tensor of rank 3 while $T_{\nu}^{\mu \alpha \beta \gamma}$ is a tensor or rank 5. This follows from the psudo-orthogonality relation in $\mathrm{Eq}(2)$.
(e) Symmetrization or anti-symmetrization of indices. This can be seen as follows. Suppose $T^{\mu \nu}$ is a second rank tensor,

$$
T^{\prime \mu \nu}=\Lambda_{\alpha}^{\mu} \Lambda_{\beta}^{\nu} T^{\alpha \beta}
$$

Then interchanging the indices we get

$$
T^{\prime \nu \mu}=\Lambda_{\alpha}^{\nu} \Lambda_{\beta}^{\mu} T^{\alpha \beta}=\Lambda_{\beta}^{\nu} \Lambda_{\alpha}^{\mu} T^{\beta \alpha}
$$

Adding these two equations, we get

$$
T^{\prime \mu \nu}+T^{\prime \nu \mu}=\Lambda_{\alpha}^{\mu} \Lambda_{\beta}^{\nu}\left(T^{\alpha \beta}+T^{\beta \alpha}\right)
$$

This means the symmetric combinations transforms into symmetric tensor. Similarly by substracting these two equations, the anti-symmetric tensor transforms into antisymmetic one.
(f) Numerical tensors : $g_{\mu \nu}$, and $\varepsilon^{\alpha \beta \gamma \delta}$ have the property

$$
\Lambda_{\alpha}^{\mu} \Lambda_{\beta}^{\nu} g_{\mu \nu}=g_{\alpha \beta}, \quad \varepsilon^{\alpha \beta \gamma \delta} \operatorname{det}(\Lambda)=\Lambda_{\mu}^{\alpha} \Lambda_{\nu}^{\beta} \Lambda_{\rho}^{\gamma} \Lambda_{\sigma}^{\delta} \varepsilon^{\mu \nu \rho \sigma}
$$

These mean that $g_{\mu \nu}$, and $\varepsilon^{\alpha \beta \gamma \delta}$ transform in the same way as tensors if $\operatorname{det}(\Lambda)=1$.
Example: $M^{\mu \nu}=x^{\mu} p^{\nu}-x^{\nu} p^{\mu}, \quad F^{\mu \nu}=\partial^{\mu} A^{\nu}-\partial^{\nu} A^{\mu}$ are all second rank antisymmetric tensor.
The most important property of tensors in Minkowski space is that if all components of a tensor vanish in one inertial frame they vanish in all inertial frame. This follows from the linear and homogeneous nature of the tensor transformation given in $\mathrm{Eq}(3)$. From this property we see that if we express physical laws in terms of tensors in Minkowski space then they will take the same form in all inertial frame. For example, if we have in one inertial frame the relation,

$$
f^{\mu}=m a^{\mu}
$$

then we can define a new tensor

$$
t^{\mu}=f^{\mu}-m a^{\mu}
$$

and all components of $t^{\mu}$ vanish in this inertial frame. So all components of $t$ vanish in all other inertial frame, i.e.

$$
t^{\prime \mu}=f^{\mu^{\prime}}-m a^{\prime \mu}=0
$$

Or

$$
f^{\mu^{\prime}}=m a^{\prime \mu}
$$

This is content of the statement that physical laws take the same form in all inertial frames.

## 3 Action principle

All classical mechanics can be reformulated in terms of action principle which states that the actual trajectory of a partilce is the one which minimizes the action.

### 3.1 Particle mechanics

Consider a simple case where a particle moves from $\mathrm{x}_{1}$ at $\mathrm{t}_{1}$ to $\mathrm{x}_{2}$ at $\mathrm{t}_{2}$. Write the action as

$$
S=\int_{t_{1}}^{t_{2}} L(x, \dot{x}) d t \quad L: \text { Lagrangian }
$$

Then the action principle says the least action

$$
\delta S=0
$$

will give the equation of motion. To get the least action, we make a small change in the trajectory $x(t)$,

$$
x(t) \rightarrow x^{\prime}(t)=x(t)+\delta x(t)
$$

with end points fixed

$$
\text { i.e. } \quad \delta x\left(t_{1}\right)=\delta x\left(t_{2}\right)=0 \quad \text { initial condition }
$$

Then the change in the action is

$$
\delta S=\int_{t_{1}}^{t_{2}}\left[\frac{\partial L}{\partial x} \delta x+\frac{\partial L}{\partial \dot{x}} \delta(\dot{x})\right] d t
$$

Note that

$$
\delta \dot{x}=\dot{x^{\prime}}(t)-\dot{x}(t)=\frac{d}{d t}[\delta(x)]
$$

i.e. the change in derivative is the derivative of the change. Then

$$
\delta S=\int_{t_{1}}^{t_{2}}\left[\frac{\partial L}{\partial x} \delta x+\frac{\partial L}{\partial \dot{x}} \frac{d}{d t}(\delta x)\right] d t=\int_{t_{1}}^{t_{2}}\left[\frac{\partial L}{\partial x}-\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{x}}\right)\right] \delta x d t
$$

where we have integrated by parts and used the initial condition. For $S$ to be minimum, we require

$$
\begin{aligned}
& \frac{\delta S}{\delta x}=0, \\
& \text { i.e. } \frac{\partial L}{\partial x}-\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{x}}\right)=0 \quad \text { Euler-Lagrange equation }
\end{aligned}
$$

The conjugate momentum is defined as

$$
p \equiv \frac{\partial L}{\partial \dot{x}}
$$

The Hamiltonian is defined by Legendre transform to change the variable $\dot{x}$ for $p$,

$$
H(p, q)=p \dot{x}-L(x, \dot{x})
$$

For the simple case, where the force is derivable from a potential $V(x)$, Newton's equation of motion is

$$
m \frac{d^{2} x}{d t^{2}}=-\frac{\partial V}{\partial x}
$$

We can write a Lagrangian of the form

$$
L=\frac{m}{2}\left(\frac{d x}{d t}\right)^{2}-V(x)
$$

so that

$$
\frac{\partial L}{\partial x}=-\frac{\partial V}{\partial x}, \quad \frac{d}{d t}\left(\frac{\partial L}{\partial \dot{x}}\right)=m \frac{d^{2} x}{d t^{2}}
$$

Thus Euler-Lagrange equation gives usual Newton's equation of motion. Also, the Hamiltonian given by

$$
H=p \dot{x}-L=\frac{m}{2}(\dot{x})^{2}+V(x) \quad \text { where } \quad p=\frac{\partial L}{\partial \dot{x}}=m \dot{x}
$$

is just the total energy. Generalization to more than one degree of freedoms is straightforward,

$$
\begin{gathered}
x(t) \rightarrow q_{i}(t), \quad i=1,2, \ldots, n \\
S=\int_{t_{1}}^{t_{2}} L\left(q_{i}, \dot{q}_{i}\right) d t
\end{gathered}
$$

Euler-Lagrange equations are

$$
\begin{gathered}
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{q}_{i}}\right)-\frac{\partial L}{\partial q_{i}}=0 \quad i=1,2, \ldots, n \\
p_{i}=\frac{\partial L}{\partial \dot{q}_{i}}, H=\Sigma p_{i} \dot{q}_{i}-L
\end{gathered}
$$

Example: Simple harmonic oscillator in 3-dimensions
Here the Lagrangian is given by

$$
L=T-V=\frac{m}{2}\left({\dot{x_{1}}}^{2}+{\dot{x_{2}}}^{2}+{\dot{x_{3}}}^{2}\right)-\frac{m w^{2}}{2}\left(x_{1}^{2}+x_{2}^{2}+x_{3}^{2}\right)
$$

and

$$
\frac{\partial L}{\partial x_{i}}=-m w^{2} x_{i} \quad \frac{\partial L}{\partial \dot{x_{i}}}=m \dot{x_{i}}
$$

Euler-Langarange equation gives

$$
m \ddot{x}_{i}=-m w^{2} x_{i}
$$

which is just the usual equation for simple harmonic oscillator derived from Newton's second law.

### 3.2 Field Theory

Field theory can be viewed as the limiting case of particle mechanics where number of degrees of freedom is infinite. We will use a continuous function $\phi(\vec{x}, t)$ to describe this system $q_{i}(t) \rightarrow \phi(\vec{x}, t)$. We will write the action as

$$
S=\int L\left(\phi(\vec{x}, t), \partial_{\mu} \phi\right) d^{3} x d t \quad L: \text { Lagrangian density }
$$

Variation of action gives

$$
\begin{aligned}
\delta S & =\int\left[\frac{\partial L}{\partial \phi} \delta \phi+\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)} \delta\left(\partial_{\mu} \phi\right)\right] d x^{4} \\
& =\int\left[\frac{\partial L}{\partial \phi}-\partial_{\mu} \frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)}\right] \delta \phi d x^{4}
\end{aligned}
$$

where we have used $\delta\left(\partial_{\mu} \phi\right)=\partial_{\mu}(\delta \phi)$ and do the integration by part. Then $\delta S=0$ implies

$$
\Longrightarrow \frac{\partial L}{\partial \phi}=\partial_{\mu}\left(\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)}\right) \quad \text { Euler-Lagrange equation }
$$

Conjugate momentum density is defined by

$$
\pi(\vec{x}, t)=\frac{\partial L}{\partial\left(\partial_{0} \phi\right)}
$$

and Hamiltonian density is

$$
H=\pi \dot{\phi}-L
$$

Generalization to more than one field is striaghtforward,

$$
\phi(\vec{x}, t) \rightarrow \phi_{i}(\vec{x}, t), i=1,2, \ldots, n
$$

The equations of motion are

$$
\frac{\partial L}{\partial \phi_{i}}=\partial_{\mu}\left(\frac{\partial L}{\partial\left(\partial_{\mu} \phi_{i}\right)}\right) \quad i=1,2, \ldots, n
$$

and conjugate momentum is

$$
\pi_{i}(\vec{x}, t)=\frac{\partial L}{\partial\left(\partial_{0} \phi_{i}\right)}
$$

The Hamiltonian density is

$$
H=\Sigma_{i} \pi_{i} \dot{\phi}_{i}-L
$$

## 4 Symmetry and Noether's Theorem

Continuous symmetry implies conservation law. This is the content of Noether's theorem. For example, invariance under time translation

$$
t \rightarrow t+a, \quad \mathrm{a} \quad \text { is arbitrary constant }
$$

gives energy conservation. This can be illustrated as follows. Newton's equation of motion for a force derived from a potential $V(\vec{x}, t)$ is of the form,

$$
m \frac{d^{2} \vec{x}}{d t^{2}}=-\vec{\nabla} V(\vec{x}, t)
$$

Suppose $\mathrm{V}(\vec{x}, \mathrm{t})=\mathrm{V}(\vec{x})$ is independent of time,i. e. invariant under time translation. Then

$$
m \frac{d \vec{x}}{d t} \cdot\left(\frac{d^{2} \vec{x}}{d t^{2}}\right)=-\left(\frac{d \vec{x}}{d t}\right) \cdot \vec{\nabla} V=-\frac{d}{d t}[V(\vec{x})]
$$

Or

$$
\frac{d}{d t}\left[\frac{1}{2} m\left(\frac{d \vec{x}}{d t}\right)^{2}+V(\vec{x})\right]=0
$$

which is just the energy conservation. Similarity, invariance under spatial translation

$$
\vec{x} \rightarrow \vec{x}+\vec{a}
$$

gives momentum conservation and invariance under rotations gives angular momentum conservation. Noether's theorem is a unified treatment of symmetries in the Lagrangian formalism.

### 4.1 Particle mechanics

We will first illustrate Noether's theorem in classical mechnics. We use the action given by

$$
S=\int L\left(q_{i}, \dot{q}_{i}\right) d t
$$

Suppose S is invariant under some continuous symmetry transformation,

$$
q_{i} \rightarrow q_{i}^{\prime}=f_{i j}(\alpha) q_{j}
$$

where $f_{i j}(\alpha)^{\prime} s$ are some functions of a parameter $\alpha$, with $f_{i j}(0)=\delta_{i j}$. Consider infinitesmal transformation,

$$
\alpha \ll 1
$$

then

$$
q_{i} \rightarrow q_{i}^{\prime} \simeq q_{i}+\alpha f_{i j}^{\prime}(0) q_{j}=q_{i}+\delta q_{i} \quad \text { with } \quad \delta q_{i}=\alpha f_{i j}^{\prime}(0) q_{j}
$$

The change of $S$ under the transformation is

$$
\delta S=\int\left[\frac{\partial L}{\partial q_{i}} \delta q_{i}+\frac{\partial L}{\partial \dot{q}_{i}} \delta \dot{q}_{i}\right] d t \quad \text { where } \quad \delta \dot{q}_{i} \rightarrow \frac{d}{d t}\left(\delta q_{i}\right)
$$

Using the equation of motion,

$$
\frac{\partial L}{\partial q_{i}}=\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{q}_{i}}\right)
$$

we can write $\delta \mathrm{S}$ as

$$
\delta S=\int\left[\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{q}_{i}}\right) \delta q_{i}+\frac{\partial L}{\partial \dot{q}_{i}} \frac{d}{d t}\left(\delta q_{i}\right)\right] d t=\int\left[\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{q}_{i}} \delta q_{i}\right)\right] d t
$$

Thus $\delta S=0$ will yield

$$
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{q}_{i}} \delta q_{i}\right)=0 \quad \text { or } \quad \frac{d}{d t}\left(\frac{\partial L}{\partial \dot{q}_{i}} \alpha f_{i j}^{\prime}(0) q_{j}\right)=0
$$

This can be written as

$$
\text { or } \quad \frac{d A}{d t}=0, \quad A=\frac{\partial L}{\partial \dot{q}_{i}} \alpha f_{i j}^{\prime}(0) q_{j}
$$

This combination $A$ is the conserved charge.
Example: rotational symmetry in 3-dimension
Write the action as

$$
S=\int L(\vec{x}, \dot{\vec{x}}) d t=\int L\left(x_{i}, \dot{x_{i}}\right) d t
$$

Suppose $S$ is invariant under rotation in 3-dimension,

$$
x_{i} \rightarrow x_{i}^{\prime}=R_{i j} x_{j}
$$

where $R$ is an orthogonal matrix, i.e.

$$
R R^{T}=R^{T} R=1 \quad \text { or } \quad R_{i j} R_{i k}=\delta_{j k}
$$

For infinitesmal rotations, we write

$$
R_{i j}=\delta_{i j}+\varepsilon_{i j} \quad\left|\varepsilon_{i j}\right| \ll 1
$$

Orthogonality requires,

$$
\left(\delta_{i j}+\varepsilon_{i j}\right)\left(\delta_{i k}+\varepsilon_{i k}\right)=\delta_{j k} \Longrightarrow \varepsilon_{j k}+\varepsilon_{k j}=0 \quad i, e, \quad \varepsilon_{j k} \quad \text { is } \quad \text { antisymmetric }
$$

We can compute the conserved charges as

$$
J=\frac{\partial L}{\partial \dot{x}} \varepsilon_{i j} x_{j}=\varepsilon_{i j} p_{i} x_{j}
$$

If we write $\varepsilon_{i j}=-\varepsilon_{i j k} \theta_{k}$

$$
\text { then } \quad J=-\theta_{k} \varepsilon_{i j k} p_{i} x_{j}=-\theta_{k} J_{k} \quad J_{k}=\varepsilon_{i j k} x_{i} p_{j}
$$

$J_{k}$ can be identified with k -th component of the usual angular momentum.

### 4.2 Field Theory

The generalization to field theory is straightforward. Start from the action wrtten in the form,

$$
S=\int L\left(\phi, \partial_{\mu} \phi\right) d^{4} x
$$

under the symmetry transformation,

$$
\phi(x) \rightarrow \phi^{\prime}\left(x^{\prime}\right),
$$

where we have included the transformations which involve change of coordinates,

$$
x^{\mu} \rightarrow x^{\prime \mu}
$$

For infinitesmal transformation, we write

$$
\delta \phi=\phi^{\prime}\left(x^{\prime}\right)-\phi(x), \quad \delta x^{\prime \mu}=x^{\prime \mu}-x^{\mu}
$$

For the transformation involving changes of coordinates, we need to include the change in the volume element

$$
d^{4} x^{\prime}=J d^{4} x \quad \text { where } \quad J=\left|\frac{\partial\left(x_{0}^{\prime}, x_{1}^{\prime}, x_{2}^{\prime}, x_{3}^{\prime}\right)}{\partial\left(x_{0}, x_{1}, x_{2}, x_{3}\right)}\right|
$$

is the Jacobian for the coordinate transformation. For infinitesmal transformation we can write,

$$
J=\left|\frac{\partial x^{\prime \mu}}{\partial x^{\nu}}\right| \approx\left|g_{\nu}^{\mu}+\frac{\partial\left(\delta x^{\mu}\right)}{\partial x^{\nu}}\right| \approx 1+\partial_{\mu}\left(\delta x^{\mu}\right)
$$

where we have used the relation

$$
\operatorname{det}(1+\varepsilon) \approx 1+\operatorname{Tr}(\varepsilon) \quad \text { for } \quad|\varepsilon| \ll 1
$$

Then

$$
d^{4} x^{\prime}=d^{4} x\left(1+\partial_{\mu}\left(\delta x^{\mu}\right)\right)
$$

The change in the action is then

$$
\delta S=\int\left[\frac{\partial L}{\partial \phi} \delta \phi+\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)} \delta\left(\partial_{\mu} \phi\right)+L \partial_{\mu}\left(x^{\mu}\right)\right] d x^{4}
$$

It is useful to define the change of $\phi$ for fixed $\mathrm{x}^{\mu}$,

$$
\bar{\delta} \phi(x)=\phi^{\prime}(x)-\phi(x)=\phi^{\prime}(x)-\phi^{\prime}\left(x^{\prime}\right)+\phi^{\prime}\left(x^{\prime}\right)-\phi(x)=\phi^{\prime}(x)-\phi^{\prime}\left(x^{\prime \mu} \phi^{\prime}\right) \delta x_{\mu}+\delta \phi
$$

Note the operator $\bar{\delta}$ commutes with the derivative operator $\partial_{\mu}$.

$$
\text { or } \quad \delta \phi=\bar{\delta} \phi+\left(\partial_{\mu} \phi\right) \delta x^{\mu}
$$

Similarly,

$$
\delta\left(\partial_{\mu} \phi\right)=\bar{\delta}\left(\partial_{\mu} \phi\right)+\partial_{\nu}\left(\partial_{\mu} \phi\right) \delta x^{\nu}
$$

Then

$$
\delta S=\int\left[\frac{\partial L}{\partial \phi}\left(\bar{\delta} \phi+\left(\partial_{\mu} \phi\right) \delta x^{\mu}\right)+\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)}\left(\bar{\delta}\left(\partial_{\mu} \phi\right)+\partial_{\nu}\left(\partial_{\mu} \phi\right) \delta x^{\nu}\right)+L \partial_{\mu}\left(\delta x^{\mu}\right)\right] d x^{4}
$$

Use Euler-Lagrange equation of motion

$$
\frac{\partial L}{\partial \phi}=\partial^{\mu}\left(\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)}\right)
$$

we can write

$$
\left.\frac{\partial L}{\partial \phi} \bar{\delta} \phi+\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)}\right) \bar{\delta}\left(\partial_{\mu} \phi\right)=\partial^{\mu}\left(\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)} \bar{\delta} \phi+\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)} \partial_{\mu}(\bar{\delta} \phi)=\partial^{\mu}\left[\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)} \bar{\delta} \phi\right]\right.
$$

where we have used

$$
\partial_{\mu}(\bar{\delta} \phi)=\bar{\delta}\left(\partial_{\mu} \phi\right)
$$

We can also combine other terms as

$$
\left[\frac{\partial L}{\partial \phi}\left(\partial_{\nu} \phi\right)+\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)} \partial_{\nu}\left(\partial_{\mu} \phi\right)\right] \delta x^{\nu}+L \partial_{\nu}\left(\delta x^{\nu}\right)=\left(\partial_{\nu} L\right) \delta x^{\nu}+L \partial_{\nu}\left(\delta x^{\nu}\right)=\partial_{\nu}\left(L \delta x^{\nu}\right)
$$

Then we get

$$
\delta S=\int d x^{4} \partial_{\mu}\left[\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)} \bar{\delta} \phi+L \delta x^{\mu}\right]
$$

and if $\delta S=0$ under the symmetry ransformation of fields, then

$$
\partial^{\mu} J_{\mu}=\partial^{\mu}\left[\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)} \bar{\delta} \phi+L \delta x^{\mu}\right]=0 \quad \text { current conservation }
$$

Simple case: space-time translation
Here the coordinate transformation is,

$$
x^{\mu} \rightarrow x^{\prime \mu}=x^{\mu}+a^{\mu} \Longrightarrow \phi^{\prime}(x+a)=\phi(x)
$$

then

$$
\bar{\delta} \phi=\phi^{\mu} \partial_{\mu} \phi
$$

and the conservation laws take the form

$$
\partial^{\mu}\left[\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)}\left(-a^{\nu} \partial_{\nu} \phi\right)+L a^{\mu}\right]=-\partial^{\mu}\left(T_{\mu \nu} a^{\nu}\right)=0
$$

where

$$
T_{\mu \nu}=\frac{\partial L}{\partial\left(\partial_{\mu} \phi\right)} \partial_{\nu} \phi-g_{\mu \nu} L
$$

is the energy momentum tensor. In particular,

$$
T_{0 i}=\frac{\partial L}{\partial\left(\partial_{0} \phi\right)} \partial_{i} \phi
$$

and

$$
P_{i}=\int d x^{3} T_{0 i}
$$

is the total momentum of the fields.

