Note 2

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Contents

1 Klein Gordon Equation

Classically, the energy momentum relation is of the form

$$E = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r})$$

For the quantization of this system, we do the replacement $E \to i\frac{\partial}{\partial t}$, $\vec{p} \to -i\vec{\nabla}$ and act on a wavefunction ψ

$$i\frac{\partial\psi}{\partial t} = [-\frac{1}{2m}\nabla^2 + V(\vec{r})]\psi \quad \text{Schrodinger equation}$$

This equation does not work for relativistic system because spatial coordinate x and time t are not on equal footing. In other words, this is not invariant under Lorentz transformations. For relativistic free particle, we have instead

$$E^2 = \vec{p}^2 + m^2$$

The corresponding wave equation is then

$$(\nabla^2 + m^2)\psi = -\partial_0^2\psi$$

Or

$$(\Box + m^2)\psi = 0, \text{ where } \quad \Box = \partial_0^2 - \nabla^2 = \partial^\mu\partial_\mu = \partial^2$$

This is known as Klein-Gordon equation.

1.1 Probablity interpretation

From Klein-Gordon equation

$$(\partial_0^2 - \nabla^2 + m^2)\psi = 0$$

and its complex conjugate,

$$(\partial_0^2 - \nabla^2 + m^2)\psi^* = 0$$

we can derive the continuity equation,

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

where

$$\rho = i(\psi\partial_0\psi^* - \psi\partial_0\psi^*), \qquad \vec{j} = (\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi)$$

Then $P = \int \rho d^3x$ is conserved, i.e.

$$\frac{dP}{dt} = \int_V \frac{\partial \rho}{\partial t} d^3x = -\int_V \vec{\nabla} \cdot \vec{j} \, d^3x = -\oint_S \vec{j} \cdot \vec{ds} = 0 \qquad \text{if } \vec{j} = 0, \text{ on } S$$

Since P is conserved, we would like to interpret it as probability. However it is easy to see that P is not positive as required by the definition of probabilty. For example,

$$\text{if } \psi\tilde{\ } e^{iEt} \phi(x), \qquad \text{then} \qquad \rho = -2E \, |\phi(x)|^2 \leq 0$$

i. e. we get negative probability and it is not viable. On the other hand if we take the probabilty density to be $\rho = \psi\psi^*$ which is positive as in the case of Schrodinger equation, then it is easy to see that it is not conserved,

$$\frac{d}{dt} \int \psi\psi^* d^3x \neq 0$$

Thus it is not possible to have probability interpretation for Klein-Gordon equation.

1.2 Solutions to Klein-Gordon Equation

The Klein Gordon equation

$$(\Box + m^2)\psi(x) = (-\nabla^2 + \partial_0^2 + m^2)\psi(x) = 0,$$

is a differential equation with constant coefficients and has plain wave solution,

$$\phi(x) = e^{-ipx} \quad if \qquad p_0^2 - P^2 - m^2 = 0 \quad \text{or} \quad p_0 = \pm\sqrt{\vec{p}^2 + m^2}$$

(a) Positive energy solution: $P_0 = \omega_p = \sqrt{\vec{p}^2 + m^2}, \quad \vec{p} \, arbitrary$

$$\phi^{(+)}(x) = \exp\left(-i\omega_p t + i\vec{p} \cdot \vec{x}\right)$$

(b) Negative energy solution: $P_0 = -\omega_p = -\sqrt{\vec{p}^2 + m^2}$

$$\phi^{(-)}(x) = \exp\left(i\omega_p t - i\vec{p} \cdot \vec{x}\right)$$

Note that positive energy solutions $\phi^{(+)}(x)$ togather with the negative energy solution $\phi^{(-)}(x)$ form a complete set of solutions. The most general solution is a linear superposition of positve energy and negative energy soulutions,

$$\begin{aligned}
\phi(x) &= \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} [a(k)e^{i\vec{k}\vec{x} - i\omega_k t} + a(k)^+ e^{-i\vec{k}\vec{x} + i\omega_k t}] \\
&= \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} [a(k)e^{-ikx} + a(k)^+ e^{ikx}] \qquad (1)
\end{aligned}$$

where $kx = \omega_k t - \vec{k} \cdot \vec{x}$

2 Dirac Equation

Dirac(1928) want to construct a relativistic wave equation first order in time derivative just like Schrodinger equation which has conserved probabilty and positive. By special relativity the wave equation is also first order in spatial coordinates. He assume an Ansatz

$$E = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta m = \vec{\alpha} \cdot \vec{p} + \beta m \tag{2}$$

where α_i, β are assumed to be matrices. Then

$$E^2 = (\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta m)^2 = \frac{1}{2}(\alpha_i \alpha_j + \alpha_j \alpha_i) p_i p_j + \beta^2 p^2 + (\alpha_i \beta + \beta \alpha_i) m$$

To get relativistic energy momentum relation, we require

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \tag{3}$$

$$\alpha_i \beta + \beta \alpha_i = 0 \tag{4}$$

$$\beta^2 = 1 \tag{5}$$

From Eq(3) we get

$$\alpha_i^2 = 1 \tag{6}$$

Togather with Eq(5) we see that α_i, β all have eigenvalues ± 1. Eq(3) also implies

$$\alpha_1 \alpha_2 = -\alpha_2 \alpha_1 \implies \alpha_2 = -\alpha_1 \alpha_2 \alpha_1$$

Taking the trace

$$Tr \alpha_2 = -Tr\left(\alpha_1 \alpha_2 \alpha_1\right) = -Tr\left(\alpha_2 \alpha_1^2\right) = -Tr\left(\alpha_2\right)$$

Thus

$$Tr\left(\alpha_i\right) = 0 \tag{7}$$

Similarly,

$$Tr\left(\beta\right) = 0$$

From Eqs(6,7) we get the important result that α_i, β all have even dimension. Recall that Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are all traceless and anti-commuting. But here we need 4 such matrices. Thus α_i, β all have to be 4×4 matrices. One conveient choics is that used by Bjoken and Drell where matrices take the form

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Dirac equation is obtained from Eq(2) by the replacenments, $E \to i\dfrac{\partial}{\partial t}$, $\vec{p} \to -i\vec{\nabla}$

$$(-i\vec{\alpha} \cdot \nabla + \beta m)\psi = i\frac{\partial \psi}{\partial t}$$

Or

$$(-i\beta\vec{\alpha} \cdot \nabla - i\beta\partial_t + m)\psi = 0$$

For conveient, define a new set of matrices

$$\gamma^0 = \beta, \qquad \gamma^i = \beta \alpha_i$$

and in Bjorken and Drell notation,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{8}$$

Dirac equation is then

$$(-i\gamma^i\partial_i - i\gamma^0\partial_0 + m)\psi = 0, \qquad \text{or} \qquad (-i\gamma^\mu\partial_\mu + m)\psi = 0$$

This is usually referred to as Dirac equation in covariant form. Note that the anti-commutations are now in a simpler form,

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

2.1 Probability interpretation

We can now show that Dirac equation give a correct form for the peobability. From the Dirac equation in the hermitian form we get

$$-i\frac{\partial\psi^\dagger}{\partial t} = (\{-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi\})^\dagger$$

and

$$i(\frac{\partial\psi^\dagger}{\partial t}\psi + \psi^\dagger\frac{\partial\psi}{\partial t}) = \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi - \{(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi\}^\dagger\psi$$

Integrate over space, we get

$$\begin{aligned} i\frac{d}{dt}\int d^3x(\psi^\dagger\psi) &= \int\{-i\psi^\dagger(\vec{\alpha}\cdot\vec{\nabla})\psi - i\{(\vec{\alpha}\cdot\vec{\nabla})\psi\}^\dagger\psi\}d^3x \\ &= -i\int\vec{\nabla}\psi^\dagger(\vec{\alpha}\cdot\vec{\nabla})\psi d^3x = 0 \end{aligned}$$

The probability $\int d^3x(\psi^\dagger\psi)$ is conserved and positive.

2.2 Solution to Dirac equation

We look for solution in the plane wave form,

$$\psi(x) = e^{-ipx}\begin{pmatrix} u \\ l \end{pmatrix}$$

where u and l are 2 components column vector. Then Dirac equation becomes

$$(\not{p} - m)\begin{pmatrix} u \\ l \end{pmatrix} = 0 \qquad \text{where} \qquad \not{p} = \gamma^\mu p_\mu$$

Using the representation given in Eq(8), we get

$$\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} u \\ l \end{pmatrix} = p_0\begin{pmatrix} u \\ l \end{pmatrix}$$

Or

$$\begin{cases} (p_0 - m)u - (\vec{\sigma}\cdot\vec{p})l = 0 \\ -(\vec{\sigma}\cdot\vec{p})u + (p_0 + m)l = 0 \end{cases} \tag{9}$$

These are homogeneous linear equations of u and l. Non-trivial solution exists if

$$\begin{vmatrix} p_0 - m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & p_0 + m \end{vmatrix} = 0$$

It is easy to see that this determinantal condition gives

$$p_0^2 = \vec{p}^2 + m^2 \qquad \text{or} \qquad p_0 = \pm\sqrt{\vec{p}^2 + m^2}$$

(a) Positive energy solution $p_0 = E = \sqrt{\vec{p}^2 + m^2}$,
Substitute this into Eqs(9) we get,

$$l = \frac{\vec{\sigma} \cdot \vec{p}}{E+m} u$$

We can write the solution in the form,

$$\psi = e^{-ipx} \begin{pmatrix} u \\ l \end{pmatrix} = e^{-ipx} N \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \chi$$

Here χ is an arbitrary 2 components vector and N is normalization constant to be determined later.

(b) Negative energy solution $p_0 = -E = -\sqrt{\vec{p}^2 + m^2}$,

Similarly, the solution can be written as,

$$\psi = e^{-ipx} N \begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix} \chi$$

The standard notation for these 4-component column vector, *spinors* are,

$$u(p.s) = N \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \chi_s \quad v(p,s) = e^{-ipx} N \begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix} \chi_s \quad N = \sqrt{E+m}$$

Dirac conjugate

One of the unual features of Dirac equation in momentum space

$$(\not{p} - m)\psi(p) = 0$$

is not hermitian. This is because in the Hermitian conjugate

$$\psi^\dagger(p)(\not{p}^\dagger - m) = 0$$

$\gamma_\mu' s$ are not hermitian,

$$\gamma_0^\dagger = \gamma_0 \quad \gamma_i^\dagger = -\gamma_i$$

But we can write

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0$$

Then

$$\psi^\dagger(p)(\gamma_0\gamma_\mu\gamma_0 p^\mu - m) = 0 \qquad \text{or} \qquad \psi^\dagger(p)\gamma_0(\gamma_\mu p^\mu - m) = 0$$

Or

$$\bar{\psi}(\not{p} - m) = 0 \quad \text{where} \quad \bar{\psi} = \psi^\dagger \gamma_0 \quad \text{Dirac conjugate}$$

2.3 Dirac equation under Lorentz transformation

Unlike the Klein-Gordon equation which is invariant under Lorentz transformation, Dirac equation is not. We now study how Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

behaves under Lorentz transformation

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$$

In the new coordinate system, the Dirac equation is of the form

$$(i\gamma^{\mu}\partial'_{\mu} - m)\psi'(x') = 0 \tag{10}$$

Note that we have used the same γ matrices (In general, different sets of γ-matrices are related by similarity transformation - Pauli's theorem). Assume that $\psi'(x')$ and $\psi(x)$ are related by a linear transformation,

$$\psi'(x') = S\psi(x)$$

We need to find the operator S. Invert the Lorentz transformation

$$x^{\gamma} = \Lambda^{\gamma}_{\mu} x'^{\mu} \qquad \Longrightarrow \qquad \frac{\partial}{\partial x'^{\mu}} = \frac{\partial}{\partial x^{\gamma}}\frac{\partial x^{\gamma}}{\partial x'^{\mu}} = \Lambda^{\gamma}_{\mu}\frac{\partial}{\partial x^{\gamma}}$$

Then Eq(10) becomes

$$(i\gamma^{\mu}\Lambda^{\alpha}_{\mu}\partial_{\alpha} - m)S\psi(x) = 0 \qquad \text{or} \qquad (i(S^{-1}\gamma^{\mu}S)\Lambda^{\alpha}_{\mu}\partial_{\alpha} - m)\psi(x) = 0$$

In order for this equation to be equivalent to the original Dirac equation, we require

$$(S^{-1}\gamma^{\mu}S)\Lambda^{\alpha}_{\mu} = \gamma^{\alpha} \quad \text{or} \quad (S^{-1}\gamma^{\mu}S) = \Lambda^{\mu}_{\alpha}\gamma^{\alpha} \tag{11}$$

To construct S, we consider infinitesimal transformation

$$\Lambda^{\mu}_{\nu} = g^{\mu}_{\nu} + \epsilon^{\mu}_{\nu} + O(\epsilon^2) \qquad \text{with} \quad |\epsilon^{\mu}_{\nu}| << 1$$

Pseudo-othogonality implies

$$g_{\mu\nu}(g^{\mu}_{\alpha} + \epsilon^{\mu}_{\alpha})(g^{\nu}_{\beta} + \epsilon^{\nu}_{\beta}) = g_{\alpha\beta}$$

Or

$$\epsilon_{\alpha\beta} + \epsilon_{\beta\alpha} = 0, \qquad \Longrightarrow \qquad \epsilon_{\alpha\beta} \text{ antisymmetric}$$

Write S as $S = 1 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu} + O(\epsilon^2)$ then $S^{-1} = 1 + \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}$ $\quad \sigma_{\mu\nu}$: 4×4 matrices. Then Eq(11) yields,

$$(1 + \frac{i}{4}\sigma_{\alpha\beta}\epsilon^{\alpha\beta})\gamma^{\mu}(1 - \frac{i}{4}\sigma_{\alpha\beta}\epsilon^{\alpha\beta}) = (g^{\mu}_{\alpha} + \epsilon^{\mu}_{\alpha})\gamma^{\alpha}$$

Or

$$\epsilon^{\alpha\beta}\frac{i}{4}[\sigma_{\alpha\beta}, \gamma^{\mu}] = \epsilon^{\mu}_{\alpha}\gamma^{\alpha} = \frac{1}{2}\epsilon^{\alpha\beta}(g^{\mu}_{\alpha}\gamma_{\beta} - g^{\mu}_{\beta}\gamma_{\alpha})$$

Identifying coefficient of $\varepsilon^{\alpha\beta}$, we get

$$[\sigma_{\alpha\beta}, \gamma_{\mu}] = 2i(g_{\beta\mu}\gamma_{\alpha} - g_{\alpha\mu}\gamma_{\beta}) \tag{12}$$

It is straightforward to check that $\sigma_{\alpha\beta}$ given by

$$\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_{\alpha}, \gamma_{\beta}]$$

satisfy Eq(12). It is not hard to see that for the finite Lorentz transformation, we have

$$\psi'(x') = S\psi(x), \qquad \text{with} \quad S = exp[-\frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}] \tag{13}$$

Note that

$$\sigma^{\dagger}_{\mu\nu} = \gamma_0\sigma_{\mu\nu}\gamma_0 \qquad \text{and} \qquad S^{\dagger} = \gamma^0 S^{-1}\gamma^0$$

Thus S is not unitary. From $\psi^{'}(x^{'}) = S\psi$ we get

$$\psi^{\dagger'}(x^{'}) = \psi^{\dagger}S^{\dagger} = \psi^{\dagger}\gamma^{0}S^{-1}\gamma^{0}, \qquad \text{or} \qquad \bar{\psi}'(x^{'}) = \bar{\psi}(x)S^{-1}$$

This shows that $\bar{\psi}$ has simple transformation property.

Fermion bilinears

Even though the Dirac wave function ψ has rather complicate transformation under the Lorentz transformation as shown in Eq(13). The fermion bi-linears $\bar{\psi}_{\alpha}(x)\psi_{\beta}(x)$ has rather simple transformation. For example,

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)S^{-1}S\psi(x) = \bar{\psi}(x)\psi(x)$$

This means that the combination $\bar{\psi}(x)\psi(x)$ is Lorentz invariant. Similarly, we can work out the other combination to get the following results.

$\bar{\psi}\gamma_{\mu}\psi$ 4-vector
$\bar{\psi}\gamma_{\mu}\gamma_{5}\psi$ axial vector
$\bar{\psi}\sigma_{\mu\nu}\psi$ 2nd rank antisymmetric ensor
$\bar{\psi}\gamma_{5}\psi$ pseudo scalar

where $\gamma_{5} = i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}$

Hole Theory (Dirac 19)

To solve the problem with negative energy states, Dirac proposed that the vaccum is the one in which $E < 0$ states are all filled and $E > 0$ states are empty. Then Pauli exclusion principle will prevent an electron from moving into E<0 states. In this picture hole in the negative sea, i.e. absence of an electron with charge $-|e|$ with negative energy $-|E|$ is equivalent to a presence of a particle with energy $|E|$ and charge $+|e|$. This new particle is called "positron" and sometime also called $anti-particle$. This correspondence of particle and anti-particle is called *charge conjugation*.