**Canonical Quantization**

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# Contents



# 1 Quantization of Free Fields

We first illustrate the quantization of simple cases of fields which satisfy Klein , Dirac or Maxwell equations. These fields are usually referred to as free fields because the solutions to these wave equations are plane wave $e^{ipx}$. The quantization of field is a straightforward generalization of the quantization in the non-relativistic quantum mechanics where we impose the commutaiton relations between generalized coordinates $q_i$ and their conjugate momenta $p_j$,

$$[q_i, p_j] = i\hbar\delta_{ij}$$

where $p_j$ is defined by

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \qquad L : \text{Lagrangian}$$

The Hamiltonian is

$$H = \sum_i p_i \dot{q}_i - L$$

Thus in the field theory we replace $q_i$ by $\phi(x)$ and $L\left(q_i, \dot{q}_j\right)$ by $\mathcal{L}(\phi, \partial_\mu\phi)$.

# 2 Scalar field

Consider a scalar field $\phi$ which satifies the Klein-Gordon equation

$$\left(\partial^\mu\partial_\mu + \mu^2\right)\phi = 0$$

The corresponding Lagrangian density is of the form

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{\mu^2}{2}\phi^2$$

because the Euler-Lagrange equation for this $\mathcal{L}$

$$\partial^\mu \left( \frac{\partial \mathcal{L}}{\partial \left( \partial^\mu \phi \right)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

gives

$$\partial^\mu \partial_\mu \phi + \mu^2 \phi = 0$$

which is exactly the Klein-Gordon equation.

### 2.1 Canonical quantization

First we compute the conjugate momentum

$$\pi \left( \overrightarrow{x}, t \right) = \frac{\partial \mathcal{L}}{\partial \left( \partial_0 \phi \right)} = \left( \partial_0 \phi \right)$$

Impose commutation relations,

$$\left[ \phi \left( \overrightarrow{x}, t \right), \pi \left( \overrightarrow{x}, t \right) \right] = i \delta^3 \left( \overrightarrow{x} - \overrightarrow{y} \right), \qquad \left[ \phi \left( \overrightarrow{x}, t \right), \phi \left( \overrightarrow{y}, t \right) \right] = 0, \qquad \left[ \pi \left( \overrightarrow{x}, t \right), \pi \left( \overrightarrow{y}, t \right) \right] = 0 \tag{1}$$

The Hamiltonian density is

$$\mathcal{H} = \pi \partial_0 \phi - \mathcal{L} = \frac{1}{2} \left[ \left( \partial^0 \phi \right)^2 + \left( \overrightarrow{\nabla} \phi \right)^2 \right] + \frac{1}{2} \mu^2 \phi^2$$

Note that we can compute the commutator

$$\left[ H, \phi(\overrightarrow{x}, t) \right] = \int d^3 y \left[ \mathcal{H}, \phi(\overrightarrow{x}, t) \right]$$

**Mode expansion**

To find the physical consequence of this Hamiltonian, we note that this is very similar to the case of the simple harmonic oscillator where Hamitonian depends quadratically on the coordinates. Recall that the classical solutions to Klein-Gordon equation are of the form,

$$\exp \left( i k_0 t - \overrightarrow{k} \cdot \overrightarrow{x} \right) \quad \text{with} \quad k_0^2 = \overrightarrow{k}^2 + \mu^2$$

We can expand the field operator in terms of classical solutions,

$$\phi \left( \overrightarrow{x}, t \right) = \int \frac{d^3 k}{\sqrt{\left( 2\pi \right)^3 2 w_k}} \left[ a \left( \overrightarrow{k} \right) e^{-ik \cdot x} + a^\dagger \left( \overrightarrow{k} \right) e^{ik \cdot x} \right], \qquad k_0 = \sqrt{\overrightarrow{k}^2 + \mu^2}$$

In this expansion the coefficients $a(k)$ and $a^\dagger(k)$ are operators and are independent of time. In order to make use of the commutation relations in Eq(1),we solve $a(k)$ and $a^\dagger(k)$ in term of $\phi$ and $\partial_0 \phi$,

$$a(k) = i \int d^3 x \frac{e^{ik \cdot x}}{\sqrt{\left( 2\pi \right)^3 2 w_k}} \overleftrightarrow{\partial_0} \phi \left( x \right) \qquad a^\dagger(k) = -i \int d^3 x \frac{e^{-ik \cdot x}}{\sqrt{\left( 2\pi \right)^3 2 w_k}} \overleftrightarrow{\partial_0} \phi \left( x \right)$$

where

$$f \overleftrightarrow{\partial_0} g \equiv f \partial_0 g - \left( \partial_0 f \right) g$$

Essentially a(k) and a†(k) are field operators in momentum space. It is straightforward to compute their commutators to give

$$\left[a\left(\vec{k}\right),\, a^{\dagger}\left(\vec{k'}\right)\right]=\delta^{3}\left(\vec{k}-\vec{k'}\right)\qquad\left[a\left(\vec{k}\right),\, a\left(\vec{k'}\right)\right]=0\qquad\left[a^{\dagger}\left(\vec{k}\right),\, a^{\dagger}\left(\vec{k'}\right)\right]=0$$

These commutators look the same as those in simple harmonic oscillators. The Hamiltonian is

$$\begin{aligned} H &= \int d^{3}x\mathcal{H}=\frac{1}{2}\int d^{3}x\left[\dot{\phi}^{2}+|\vec{\nabla}\phi|^{2}+\mu^{2}\phi^{2}\right] \\ &= \frac{1}{2}\int d^{3}kw_{k}\left[a^{\dagger}\left(\vec{k}\right)a\left(\vec{k}\right)+a\left(\vec{k}\right)a^{\dagger}\left(\vec{k}\right)\right]=\int d^{3}k\mathcal{H}_{k} \end{aligned}$$

with

$$\mathcal{H}_{k}=\frac{w_{k}}{2}\left[a^{\dagger}\left(k\right)a\left(k\right)+a\left(k\right)a^{\dagger}\left(k\right)\right]$$

Thus $H$ is a superposition of many harmonic oscillator with frequency $w_k$.

Recall that from Noether's theorem, the momentum operator is of the form,

$$P_{i}=\int d^{3}xT_{0i}=\int d^{3}x\frac{\partial\mathcal{L}}{\partial\left(\partial_{0}\phi\right)}\partial_{i}\phi=\int d^{3}x\pi\partial_{i}\phi$$

and we have the commutator,

$$\begin{aligned} \left[P_{i},\phi(\vec{x},t)\right] &= \int d^{3}y\left[\pi(\vec{y},t)\partial_{i}\phi(\vec{y},t),\phi(\vec{x},t)\right] \\ &= \int d^{3}y\partial_{i}\phi(\vec{y},t)\left(-i\right)\delta^{3}(\vec{x}-\vec{y})=-i\partial_{i}\phi(\vec{x},t) \end{aligned}$$

In terms of creation and annihilation operators, momentum operator can be written as

$$\vec{P}=\frac{1}{2}\int d^{3}k\,\vec{k}\left[a^{\dagger}\left(k\right)a\left(k\right)+a\left(k\right)a^{\dagger}\left(k\right)\right]=\int d^{3}k\,\vec{p}_{k}$$

with

$$\vec{p}_{k}=\frac{\vec{k}}{2}\left[a^{\dagger}\left(k\right)a\left(k\right)+a\left(k\right)a^{\dagger}\left(k\right)\right]$$

Note that in the usual harmonic oscillator

$$aa^{\dagger}=a^{\dagger}a+1$$

But here

$$a\left(k\right)a^{\dagger}\left(k\right)=a^{\dagger}\left(k\right)a\left(k\right)+\delta^{3}\left(0\right)$$

We can interpret $\delta^{3}\left(0\right)$ as follows. From

$$\delta^{3}\left(\vec{k}\right)=\int\frac{d^{3}x}{\left(2\pi\right)^{3}}e^{i\vec{k}\cdot\vec{x}}$$

we see that as $\vec{k}\rightarrow0$

$$\delta^{3}\left(0\right)=\left(2\pi\right)^{-3}\int d^{3}x=\frac{V}{\left(2\pi\right)^{3}}$$

where $V$ is the total volume of the system. Then

$$H=\int d^{3}kw_{k}\left[a^{\dagger}\left(k\right)a\left(k\right)+\frac{\left(2\pi\right)^{-3}}{2}V\right]=\int d^{3}kw_{k}a^{\dagger}\left(k\right)a\left(k\right)+\frac{1}{2}\int\frac{Vd^{3}k}{\left(2\pi\right)^{3}}w_{k}$$

The last term is just a constant(although infinite) and will be dropped. To achieve this more formally, we introduce the normal ordering device.

## 2.2 Normal ordering

In normal ordering, we move all creation operators a†(k) to the left of annihilation operators a(k) and denote the operation by :f$(a, a^\dagger)$ :. For example,

.

$$
\begin{aligned}
&: \quad a(k)a\dagger(k) := a\dagger(k)a(k) \\
&: \quad a\dagger(k)a(k) := a\dagger(k)a(k)
\end{aligned}
$$

Let the vaccum be defined by

$$a(k)|0\rangle = 0 \qquad \forall \overrightarrow{k} \qquad \Longrightarrow \langle 0|a\dagger(k) = 0$$

Then we will have the general property

$$\langle 0| : f\left(a, a^\dagger\right) : |0\rangle = 0$$

Thus if we define the Hamiltonican by normaling ordering,then we can remove the constant term,

$$H = \frac{1}{2}\int d^3k w_k : [a\dagger(k)a(k) + a(k)a\dagger(k)] := \int d^3k w_k a\dagger(k)a(k)$$

Similarly,

$$\overrightarrow{p} = \frac{1}{2}\int d^3k \overrightarrow{p_k} : [a\dagger(k)a(k) + a(k)a\dagger(k)] := \int d^3k \overrightarrow{p_k} a\dagger(k)a(k)$$

It is then easy to write down the eigenstates and eigenvalues of H and $\overrightarrow{p}$

For example,the state defined by

$$|\overrightarrow{k}\rangle = \sqrt{(2\pi)^3 2w_k} a\dagger(k)|0\rangle$$

will have the property,

$$
\begin{aligned}
H|\overrightarrow{k}\rangle &= w_k|\overrightarrow{k}\rangle \\
\overrightarrow{p}|\overrightarrow{k}\rangle &= \overrightarrow{k}|\overrightarrow{k}\rangle \qquad \text{where } w_k = \sqrt{\overrightarrow{k}^2 + \mu^2}
\end{aligned}
$$

we can interpret this as one-particle state because it has definite energy $w_k$ and definite momentum $\overrightarrow{k}$, with relation

$$w_k^2 - \overrightarrow{k}^2 = \mu^2$$

Similarly,we can define 2 particle satate by

$$|\overrightarrow{k}_1, \overrightarrow{k}_2\rangle = \sqrt{(2\pi)^3 2w_{k_1}}\sqrt{(2\pi)^3 2w_{k_2}} a\dagger(\overrightarrow{k_1})a\dagger(\overrightarrow{k_2})|0\rangle$$

## 2.3 Bose statistics

Any arbitrary state can be expand in terms of states with definite number of particles,

$$|\Phi\rangle = \left[C_0 + \sum_{i=1}^{\infty}\int d^3k_1 d^3k_2 ... d^3k_n C_n(k_1, k_2, ..., k_n) a\dagger(\overrightarrow{k_1})a\dagger(\overrightarrow{k_2})...a\dagger(\overrightarrow{k_n})|0\rangle\right]$$

where $C_n(k_1, k_2, ..., k_n)$ can be interprented as the momentum space wavefunction. Since

$$\left[a^{\dagger}(k_i), a^{\dagger}(k_j)\right] = 0$$

we see that

$$C_n(k_1, ..., k_i, ..., k_j ..., k_n) = C_n(k_1, ..., k_j, ..., k_i ..., k_n)$$

This means the wavefunction $C_n(k_1, k_2, ..., k_n)$ statistics Bose statistics

# 3 Fermion fields

To quantize the fermion field we can proceed as the case for scalar field. Start with Dirac equation for free particles

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0 \qquad \text{or} \qquad \overline{\psi}\left(-i\gamma^{\mu}\overleftarrow{\partial_{\mu}} - m\right) = 0$$

The Lagrangian density for this equation is of the form

$$\mathcal{L} = \overline{\psi_{\alpha}}(i\gamma^{\mu}\partial_{\mu} - m)_{\alpha\beta}\psi_{\beta}$$

Conjugate momentum density is

$$\pi_{\alpha} = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi_{\alpha})} = i\psi_{\alpha}^{\dagger}$$

If we impose the commutation relation like the scalar field, we will get Dirac particles satisfying Bose statistics which is not correct physically.

## 3.1 Anti-commutators

It turns out that we need to impose anticommutation relations in order to satisfy the Fermi-Dirac statistics.

$$\{\pi_{\alpha}(\overrightarrow{x}, t), \psi_{\beta}(\overrightarrow{y}, t)\} = i\delta^3(\overrightarrow{x} - \overrightarrow{y})\,\delta_{\alpha\beta} \qquad \{\psi_{\alpha}(\overrightarrow{x}, t), \psi_{\beta}(\overrightarrow{y}, t)\} = 0 \qquad \{\pi_{\alpha}(\overrightarrow{x}, t), \pi_{\beta}(\overrightarrow{y}, t)\} = 0$$

Hamiltonian density is given by

$$\mathcal{H} = \sum_{\alpha}\pi_{\alpha}\dot{\psi}_{\alpha} - \mathcal{L} = i\psi^{\dagger}\gamma_0\gamma_0\partial_0\psi - \overline{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi = \overline{\psi}\left(i\overrightarrow{\gamma}\cdot\overrightarrow{\nabla} + m\right)\psi$$

**Mode expansion**

The expansion of fermion field in terms of classical solutions to the Dirac equation take the form,

$$\begin{aligned}
\psi(\overrightarrow{x}, t) &= \sum_s \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2E_p}}\left[b(p,s)\,u(p,s)\,e^{-ip\cdot x} + d^{\dagger}(p,s)\,v(p,s)\,e^{ip\cdot x}\right] \\
\psi^{\dagger}(\overrightarrow{x}, t) &= \sum_s \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2E_p}}\left[b^{\dagger}(p,s)\,u^{\dagger}(p,s)\,e^{ip\cdot x} + d(p,s)\,v^{\dagger}(p,s)\,e^{-ip\cdot x}\right]
\end{aligned}$$

Invert these relations

$$
\begin{aligned}
b(p,s) &= \int \frac{d^3x e^{ip\cdot x}}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}} u^\dagger(p,s)\psi(\overrightarrow{x},t) \qquad b^\dagger(p,s) = \int \frac{d^3x e^{-ip\cdot x}}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}} \psi^\dagger(\overrightarrow{x},t) u(p,s) \\
d^\dagger(p,s) &= \int \frac{d^3x e^{-ip\cdot x}}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}} \upsilon^\dagger(p,s)\psi(\overrightarrow{x},t) \qquad d(p,s) = \int \frac{d^3x e^{ip\cdot x}}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}} \psi^\dagger(\overrightarrow{x},t)\upsilon(p,s)
\end{aligned}
$$

From these we can compute the anti-commutation relations,

$$
\{b(p,s), b^\dagger(p',s')\} = \delta_{ss'}\delta^3(\overrightarrow{p} - \overrightarrow{p}'), \qquad \{d(p,s), d^\dagger(p',s')\} = \delta_{ss'}\delta^3\left(\overrightarrow{p} - \overrightarrow{p}'\right)
$$

and all other anticommutators vanish.

The Hamiltonian is of the form,

$$
\begin{aligned}
H &= \int d^3x\mathcal{H} = \int \overline{\psi}\left(i\overrightarrow{\gamma}\cdot\overrightarrow{\nabla} + m\right)\psi d^3x = i\int \psi^\dagger \partial_0 \psi d^3x \\
&= \sum_s \int d^3p E_p \left[b^\dagger(p,s) b(p,s) - d(p,s) d^\dagger(p,s)\right] = \sum_s \int d^3p \mathcal{H}_{ps} \\
\text{where} \quad \mathcal{H}_{ps} &= E_p\left[b^\dagger(p,s) b(p,s) - d(p,s) d^\dagger(p,s)\right]
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\overrightarrow{p} &= \sum_s d^3p \overrightarrow{p}_p \\
\overrightarrow{p}_p &= \overrightarrow{p}\left[b^\dagger(p,s) b(p,s) - d(p,s) d^\dagger(p,s)\right]
\end{aligned}
$$

We can compute the commutators of $H$ with $b^\dagger(p,s)$ to get

$$
\begin{aligned}
\left[H, b^\dagger(p,s)\right] &= \sum_{s'} d^3p' \left[b^\dagger(p',s') b(p',s'), b^\dagger(p,s)\right] E_p \\
&= \int d^3p' E_p \sum_{s'} \left[\left(b^\dagger(p',s')\{b(p',s'), b^\dagger(p,s)\} - \{b^\dagger(p',s'), b^\dagger(p,s)\} b(p',s')\right)\right] \\
&= b^\dagger(p,s) E_p
\end{aligned}
$$

This means $b^\dagger(p,s)$ is an operator which increases the energy eigenvalue by $E_p$, hence creation operator. Similarly,

$$
\left[\overrightarrow{p}, b^\dagger(p,s)\right] = \overrightarrow{p} b^\dagger(p,s)
$$

and $b^\dagger(p,s)$ increases the monetum eigenvalue by $\overrightarrow{p}$ .Combine these two, we can say that $b^\dagger(p,s)$ creats a particle with $E_p$ and $\overrightarrow{p}$ with relation $E_p = \sqrt{\overrightarrow{p}^2 + m^2}$.

In the same way,we can show that, $d^\dagger(p,s)$ also creates a particle with same mass and will be shown later to have opposite charge as those created by $b^\dagger(p,s)$.

### 3.2 Symmetry

The Lagrangian given by

$$
\mathcal{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi
$$

is invariant under the phase transformation,

$$\psi(x) \rightarrow e^{i\alpha}\psi(x) \Longrightarrow \psi^{\dagger}(x) \rightarrow \psi^{\dagger}(x)\, e^{-i\alpha} \qquad \alpha : \text{some real constant}$$

From Noether's theorem,the conserved current for this symmetry is

$$j_{\mu} = \overline{\psi}\gamma_{\mu}\psi$$

The corresponding conserved charge is

$$Q = \int j_0(x)\, d^3x = \sum_s \int d^3p \left[N^{+}(p,s) - N^{-}(p,s)\right]$$

where

$$N_{ps}^{+} = b^{\dagger}(p,s)\, b(p,s) \qquad N_{ps}^{-} = d^{\dagger}(p,s)\, d(p,s)$$

are the number operators. Thus particle and anti-particle have opposite "charge".

## 4 Electromagnetic fields

Start with free Maxwell's equations,

$$\nabla \cdot \overrightarrow{B} = 0, \quad \nabla \times \overrightarrow{E} + \frac{\partial \overrightarrow{B}}{\partial t} = 0, \tag{2}$$

$$\nabla \cdot \vec{E} = 0, \qquad \frac{1}{\mu_0}\nabla \times \overrightarrow{B} - \epsilon_0 \frac{\partial \overrightarrow{E}}{\partial t} = 0 \tag{3}$$

Introduce vector and scalar potentials $\overrightarrow{A}, \phi$ by

$$\overrightarrow{B} = \nabla \times \overrightarrow{A}, \qquad \overrightarrow{E} = -\nabla\phi - \frac{\partial \overrightarrow{A}}{\partial t} \tag{4}$$

These solve those Maxwell's equations given in Eq(2). It is convenient to write the relations in Eq(4) as

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} \quad \text{with} \quad F^{0i} = \partial^{0}A^{i} - \partial^{i}A^{0} = -E^{i}, \quad F^{ij} = \partial^{i}A^{j} - \partial^{j}A^{i} = -\epsilon_{ijk}B_k$$

The other two equations in Eq(3) can be written as

$$\partial_{\nu}F^{\mu\nu} = 0, \quad \mu = 0,1,2,3$$

For example

$$\mu = 0, \quad \partial_i F^{0i} = 0 \quad \Rightarrow \quad \nabla \cdot \overrightarrow{E} = 0$$

$$\mu = i, \quad \partial_{\nu}F^{i\nu} = 0 \quad \Rightarrow \quad \nabla \times \overrightarrow{B} - \frac{\partial \overrightarrow{E}}{\partial t} = 0$$

Note that in natural unit $c^2 = \dfrac{1}{\mu_0\epsilon_0} = 1$.

### 4.1 Gauge Invariance

It is easy to see that $F^{\mu\nu}$ which is an antisymmetric derivative of $A_\mu$ is invariant under the transformation,

$$A^\mu \longrightarrow A^\mu + \partial^\mu \alpha \qquad \alpha = \alpha(x)$$

$\alpha(x)$ is an arbitrary function.This is usually referred to as gauge transformation. It means that given a set of $\overrightarrow{B}$ and $\overrightarrow{E}$ fields, their solutions in terms of potentials $\overrightarrow{A}$,and $\phi$ are not unique. One can change them by gauge function $\alpha(x)$ and still get the same $\overrightarrow{B}$ and $\overrightarrow{E}$ fieldst. This property is usually called the **gauge invariance**.

It turns out that the Lagrangian density given by,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(\overrightarrow{E}^2 - \overrightarrow{B}^2)$$

will give Maxwell equations as consequence of Euler-Lagrange equations. Conjugate momenta

$$\pi_0 = \frac{\partial L}{\partial(\partial_0 A_0)} = 0, \qquad \pi^i(x) = \frac{\partial L}{\partial(\partial_0 A_i)} = -F^{0i} = E^i$$

Here we see that the time component $A_0$ does not have conjugate momenta and is therefore not a dynamical degree of freedom.
Hamiltanian density is of the form,

$$\mathcal{H} = \pi^k \dot{A}_k - \mathcal{L} = (\partial^k A^0 - \partial^0 A^k)\partial_0 A_k + \frac{1}{2}\partial_\mu A_\nu(\partial^\mu A\nu) - \partial^\nu A\mu) = \frac{1}{2}(\overrightarrow{E}^2 + \overrightarrow{B}^2) + (\overrightarrow{E}\cdot\nabla)A_0$$

Using $\overrightarrow{\nabla}\cdot\overrightarrow{E} = 0$, which is part of the equation of motion, we can write Hamiltonian as,

$$H = \int d^3x \mathcal{H} = \frac{1}{2}\int d^3x(\overrightarrow{E}^2 + \overrightarrow{B}^2)$$

### 4.2 Quantization

For the quantization, we impose the commutation relation,

$$[\pi^i(\overrightarrow{x}, t), A^j(\overrightarrow{y}, t)] = -i\delta_{ij}\delta^3(\overrightarrow{x} - \overrightarrow{y}), \quad ...$$

But this is not consistent with $\overrightarrow{\nabla}\cdot\vec{E} = 0$ because

$$[\nabla\cdot E(x,t), A_j(x,t)] = -i\partial_j\delta^3(x-y) \neq 0$$

Note that the $\delta-$function can be written in momentum space as

$$\partial_j\delta^3(\overrightarrow{x} - \overrightarrow{y}) = i\int \frac{d^3k}{(2\pi)^3} e^{i\overrightarrow{k}\cdot(\overrightarrow{x}-\overrightarrow{y})} k_j$$

In order to get zero for the commutator of $\nabla\cdot E$, we can do the following replacement,

$$\delta_{ij}\delta^3(\vec{x} - \vec{y}) \rightarrow \delta^{tr}_{ij}(\vec{x} - \vec{y}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}(\delta_{ij} - \frac{k_i k_j}{k^2})$$

then

$$\partial_i \delta^{tr}_{ij} \delta^3(\vec{x}-\vec{y}) = i \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}(\cdot\vec{x}-\vec{y})} k_i (\delta_{ij} - \frac{k_i k_j}{k^2}) = 0$$

So the non-zero commutator is modified to be of the form,

$$[E^i(x,t), A_j(y,t)] = -i\delta^{tr}_{ij}(\vec{x}-\vec{y})$$

As a consequence, we also have

$$[E^i(x,t), \overrightarrow{\nabla} \cdot \vec{A}(y,t)] = 0$$

Now that $A_0$ and $\overrightarrow{\nabla} \cdot \vec{A}$ commute with all operators, they must be C-number. In other words, Ao and longitudinal part of A are not dynamical degree of freedom.
We can choose a gauge such that $A_0 = 0 \; and \; \nabla \cdot \vec{A} = 0$ (radiation gauge) In this gauge

$$\pi^i = \partial^i A^0 - \partial^0 A^i = -\partial^0 A^i$$

$$[\partial_0 A^i(\vec{x},t), A^j(\vec{y},t)] = i\delta^{tr}_{ij}(\vec{x}-\vec{y})$$

Mode expansion
Equation of motion $\quad \partial_\nu F^{\mu\nu} = 0$ can be written as

$$\partial_\nu \left(\partial^\nu A^\mu - \partial^\mu A^\nu\right) = \Box A^\mu - \partial^\mu \left(\partial_\upsilon A^\upsilon\right) = 0$$

In the gauge we have chosen,

$$A_0 = 0, \overrightarrow{\nabla} \cdot \overrightarrow{A} = 0$$

we have $\partial_\upsilon A^\upsilon = 0$. Then the wave equation become

$$\Box \overrightarrow{A} = 0 \qquad \text{massless Klein-Gordon equation}$$

The general solution is

$$A(\vec{x},t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_\lambda \epsilon(\vec{k},\lambda)[a(k,\lambda)e^{-ikx} + a^+(k,\lambda)e^{ikx}] \qquad w = k_0 = |\overrightarrow{k}|$$

In the gauge $\nabla \cdot A = 0$, there are only two independent degree of freedom

$$\vec{\epsilon}(k,\lambda), \lambda = 1,2 \quad \text{with } \vec{k} \cdot \vec{\epsilon}(k,\lambda) = 0$$

Standard choice

$$\vec{\epsilon}(k,\lambda) \cdot \vec{\epsilon}(k,\lambda') = \delta_{\lambda\lambda'}, \quad \vec{\epsilon}(-k,1) = -\vec{\epsilon}(k,1), \quad \vec{\epsilon}(-k,2) = \vec{\epsilon}(-k,2)$$

We can solve for $a(k,\lambda)$ and $a^+(k,\lambda)$ to get

$$a(k,\lambda) = i \int \frac{d^3x}{\sqrt{(2\pi)^3 2\omega}} [e^{ik\cdot x} \overleftrightarrow{\partial_0} \vec{\epsilon}(k,\lambda) \cdot \vec{A}(x)] \tag{5}$$

$$a^+(k,\lambda) = -i \int \frac{d^3x}{\sqrt{(2\pi)^3 2\omega}} [e^{-ik\cdot x} \overleftrightarrow{\partial_0} \vec{\epsilon}(k,\lambda) \cdot \vec{A}(x)] \tag{6}$$

The commutation relations are of the form,

$$[a(k,\lambda), a^+(k',\lambda')] = \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}') \quad [a(k,\lambda), a(k',\lambda')] = 0, \; [a^+(k,\lambda), a^+(k',\lambda')] = 0$$

The normal ordered form for the Hamiltonian and momentum operators are

$$H = \frac{1}{2}\int d^3x : (E^2 + B^2) := \int d^3k\omega \sum_\lambda a^+(k,\lambda)a(k,\lambda) \tag{7}$$

$$\vec{P} = \int d^3x : E \times B := \int d^3k\vec{k} \sum_\lambda a^+(k,\lambda)a(k,\lambda) \tag{8}$$

The vaccum is defined by

$$a(\vec{k},\lambda)|0>= 0 \quad \forall \vec{k},\lambda$$

# 5 Lorentz group

In the derivation of Dirac equation it is not clear what is the meaning of the Dirac $\gamma$ matrices. It turns out that they are related to representations of Lorentz group. The Lorentz group is a collection of linear transformations of space-time coordinates

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$$

which leaves the proper time

$$\tau^2 = (x^o)^2 - (\overrightarrow{x})^2 = x^\mu x^\nu g_{\mu\nu} = x^2$$

invariant. This requires the transformation matrix $\Lambda^\mu\nu$ satisfies the pseudo-orthogonality relation

$$\Lambda^\mu_\alpha \Lambda^\nu_\beta g_{\mu\nu} = g_{\alpha\beta}$$

## 5.1 Generators

For infinitesmal transformation, we write

$$\Lambda^\mu_\alpha = g^\mu_\alpha + \epsilon^\mu_\alpha \qquad \text{with} \quad |\varepsilon^\mu_\alpha| \ll 1$$

As before, the pseudo-orthogonality relation, implies, $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$. Consider $f(x^\mu)$, an arbitrary function of $x^\mu$. Under the infinitesimal Lorentz transformation, the change in $f$ is

$$\begin{aligned} f(x^\mu) \quad &\to \quad f(x'^\mu) = f(x^\mu + \varepsilon^\mu_\alpha x^\alpha) \approx f(x^\mu) + \varepsilon_{\alpha\beta} x^\beta \partial_\alpha f \\ &= \quad f(x^\mu) + \frac{1}{2}\varepsilon_{\alpha\beta}[x^\beta\partial^\alpha - x^\alpha\partial^\beta]f(x) + \cdots \end{aligned}$$

Introduce an operator $M_{\mu\nu}$ to represent this change,

$$f(x') = f(x) - \frac{i}{2}\varepsilon_{\alpha\beta} M^{\alpha\beta} f(x) + \cdots$$

then

$$M^{\alpha\beta} = -i(x^\alpha\partial^\beta - x^\beta\partial^\alpha) \tag{9}$$

generators $M_{\mu\nu}$ are called the generators of Lorentz group. Note that for $\alpha,\beta = 1,2,3$ these are just the angular momentum operator.

Using the generators given in Eq(9) it is straightforward to work out commutators of these generators,

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i\{g_{\beta\gamma}M_{\alpha\delta} - g_{\alpha\gamma}M_{\beta\delta} - g_{\beta\delta}M_{\alpha\gamma} + g_{\alpha\delta}M_{\beta\gamma}\}$$

Define

$$M_{ij} = \epsilon_{ijk} J_k, \quad M_{oi} = K_i$$

We can solve for $J_i$ to get

$$J_i = \frac{1}{2}\epsilon_{ijk} M_{jk}$$

We can compute the commutator of $J_i's$,

$$[J_i, J_j] = \left(\frac{1}{2}\right)^2 \varepsilon_{ikl}\varepsilon_{jmn}[M_{kl}, M_{mn}] = (-i)\,(\frac{1}{2})^2 \varepsilon_{ikl}\varepsilon_{jmn}(g_{lm}M_{kn} - g_{km}M_{ln} - g_{ln}M_{km} + g_{kn}M_{lm})$$

$$= (\frac{1}{2})^2\,(-i)\,[-\epsilon_{ikl}\varepsilon_{jln}M_{kn} + \epsilon_{ikl}\varepsilon_{jkn}M_{ln} + \epsilon_{ikl}\varepsilon_{jml}M_{km} - \epsilon_{ikl}\varepsilon_{jmk}M_{lm}]$$

Using identity

$$\epsilon_{abc}\varepsilon_{alm} = (\delta_{bl}\delta_{cm} - \delta_{bm}\delta_{cl})$$

we get

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{10}$$

Thus we can identify $J_i$ as the angular momentum operator.

Similarly, we can derive

$$[K_i, K_j] = -i\epsilon_{ijk} J_k \qquad [J_i, K_j] = i\epsilon_{ijk} K_k \tag{11}$$

Eqs(10,11) are called the Lorentz algebra.

To simplify the Lorentz algebra, we define the combinations

$$A_i = \frac{1}{2}(J_i + iK_i) \quad , B_i = \frac{1}{2}(J_i - iK_i)$$

Then it is straightforward to derive the following commutation relations,

$$[A_i, A_j] = i\epsilon_{ijk} A_k, \qquad [B_i, B_j] = i\epsilon_{ijk} B_k, \qquad [A_i, B_j] = 0$$

This means that the algebra of Lorentz generators factorizes into 2 independent SU(2) algebra. The representations are just the tensor products of the representation of SU(2) algebra. Thus we label the irreducible representation by $(j_1, j_2)$ which transforms as $(2j_1 + 1)$-dim representation under $A_i$ algebra and $(2j_2 + 1)$-dim representation under $B_i$ algebra.

## 5.2 Simple representations

(a) $(\frac{1}{2}, 0)$ representation $\chi_a$

This 2-component object has the following properties,

$$A_i \chi_a = (\frac{\sigma_i}{2})_{ab}\chi_b \qquad \Longrightarrow \qquad \frac{1}{2}(J_i + iK_i)\chi_a = (\frac{\sigma_i}{2})_{ab}\chi_b$$

$$B_i \chi_a = 0 \qquad \Longrightarrow \qquad \frac{1}{2}(J_i - iK_i)\chi_a = 0$$

Combining these realtions we get

$$\vec{J}\chi = (\frac{\vec{\sigma}}{2})\chi, \qquad \vec{K}\chi = -i(\frac{\vec{\sigma}}{2})\chi$$

(b) $(0,\frac{1}{2})$ representation $\eta_a$

Similarly, we can get

$$A_i\eta_a = 0 \qquad \Rightarrow \qquad \frac{1}{2}(J_i + iK_i)\eta_a = 0$$

$$B_i\eta_a = (\frac{\sigma_i}{2})_{ab} \qquad \Longrightarrow \quad \frac{1}{2}(J_i - iK_i)\eta_a = (\frac{\sigma_i}{2})_{ab}\eta_b$$

$$\vec{J}\eta = (\frac{\vec{\sigma}}{2})\eta, \qquad \vec{K}\eta = i(\frac{\vec{\sigma}}{2})\eta$$

If we define a 4-component $\psi$ by putting togather these 2 representations,

$$\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$$

Then the action of the Lorentz generators are

$$\vec{J}\psi = \begin{pmatrix} \frac{\vec{\sigma}}{2} & 0 \\ 0 & \frac{\vec{\sigma}}{2} \end{pmatrix}\psi, \qquad \vec{K}\psi = \begin{pmatrix} -i\frac{\vec{\sigma}}{2} & 0 \\ 0 & i\frac{\vec{\sigma}}{2} \end{pmatrix}\psi \tag{12}$$

$\psi$ are related to the 4-component Dirac field we studied before, but with different representation for the $\gamma$ matrices. This can be seen as follows.
Consider Dirac matrices in the following form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \text{where} \quad \sigma^\mu = (1, \vec{\sigma}) \ \ , \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

More explicitly,

$$\gamma^o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

It is straightforward to check that in this case.

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This means that in 4-component field $\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$, $\chi$ is right-handed and $\eta$ is left-handed. In this representation, it is easy to check that

$$\sigma_{0i} = i\gamma_0\gamma_1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix}$$

$$\sigma_{ij} = i\gamma_i\gamma_j = i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

In the Lorentz transformation of Dirac field,

$$\psi'(x') = S\psi = exp\{-\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}\} = exp\{-\frac{i}{4}(2\sigma_{0i}\varepsilon^{0i} + \sigma_{ij}\varepsilon^{ij})\}$$

Write $\varepsilon^{0i} = \beta^i, \quad \varepsilon^{ij} = \varepsilon^{ijk}\theta^k$

$$\sigma_{ij}\epsilon^{ij} = \varepsilon^{ijk}\theta^k\epsilon_{ijl}\begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix} = 2\begin{pmatrix} \vec{\sigma}\cdot\vec{\theta} & 0 \\ 0 & \vec{\sigma}\cdot\vec{\theta} \end{pmatrix}$$

$$\sigma_0 i\varepsilon^0 i = \begin{pmatrix} -i\vec{\sigma}\cdot\vec{\beta} & 0 \\ 0 & i\vec{\sigma}\cdot\vec{\beta} \end{pmatrix}$$

$\Rightarrow$

$$-\frac{i}{4}(2\sigma_{0i}\varepsilon^{0i} + \sigma_{ij}\varepsilon^{ij}) = \frac{-i}{2}\begin{pmatrix} \vec{\sigma}\cdot\overrightarrow{\theta} - i\vec{\sigma}\cdot\vec{\beta} & 0 \\ 0 & \vec{\sigma}\cdot\overrightarrow{\theta} + i\vec{\sigma}\cdot\vec{\beta} \end{pmatrix}$$

More precisely,

$$\psi'(x') = S\psi = exp\{-\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}\}\psi = \exp\left[\frac{-i}{2}\begin{pmatrix} \vec{\sigma}\cdot\overrightarrow{\theta} - i\vec{\sigma}\cdot\vec{\beta} & 0 \\ 0 & \vec{\sigma}\cdot\overrightarrow{\theta} + i\vec{\sigma}\cdot\vec{\beta} \end{pmatrix}\right]\psi \tag{13}$$

If we write the Lorentz transformations in terms of generators,

$$L = \exp(-iM_{\mu\nu}\varepsilon^{\mu\nu})$$

then in terms of the generators $\overrightarrow{J}, \overrightarrow{K}$

$$L = \exp\left[(-i)\left(\overrightarrow{J}\cdot\overrightarrow{\theta} + \overrightarrow{K}\cdot\overrightarrow{\beta}\right)\right]$$

We then see from Eq(13) that for this $\psi$, $\overrightarrow{J}, \overrightarrow{K}$ are of the form,

$$\overrightarrow{J} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \qquad \overrightarrow{K} = \frac{1}{2}\begin{pmatrix} -i\vec{\sigma} & 0 \\ 0 & i\vec{\sigma} \end{pmatrix}$$

These are the same as those in Eq(12). This demonstrate that the wavefunction which satisfies Dirac equation is just the representation $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ under the Lorentz group. Futhermore, the right-handed components transform as $\left(\frac{1}{2}, 0\right)$ represenation while left-handed components transform as $\left(0, \frac{1}{2}\right)$ representation.