## Contents

1 Interaction Theory ..... 1
1.1 Physical states ..... 3
1.2 In-fields and in-states-asymptotic conditions ..... 3
2 S-matrix ..... 6
3 LSZ reduction ..... 7
3.1 In and Out fields for Fermions ..... 8
4 U matrix ..... 9
4.1 Perturbation Expansion of Vaccum expectation value ..... 11
4.2 Feynman Propagators ..... 13
4.3 Vaccum Amplitude ..... 14
5 Cross section and Decay rate ..... 17
5.1 Decay rates ..... 18
5.2 Cross section ..... 18
6 Feynman Rules ..... 19
6.1 Example in $\lambda \phi^{3}$ theory ..... 20

# Note 4 Interaction Theory and Feynman Rule 

Ling fong Li

## 1 Interaction Theory

As an illustration we discuss the electromagnetic interaction. From the principle of minimum substitution, the Lagrangian density is of the form,

$$
\mathcal{L}=\bar{\psi}(x) \gamma^{\mu}\left(i \partial_{\mu}-e A_{\mu}\right) \psi(x)-m \bar{\psi}(x) \psi(x)-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}
$$

Equations of motion are

$$
\begin{aligned}
\left(i \gamma^{\mu} \partial_{\mu}-m\right) \psi(x) & =e A_{\mu} \gamma^{\mu} \psi \quad \text { non-linear coupled equations } \\
\partial_{v} F^{\mu \nu} & =e \bar{\psi} \gamma^{\mu} \psi
\end{aligned}
$$

## Quantization

Write $\mathcal{L}=\mathcal{L}_{0}+\mathcal{L}_{\text {int }}$

$$
\begin{aligned}
\mathcal{L}_{0} & =\bar{\psi}\left(i \gamma^{\mu} \partial_{\mu}-m\right) \psi-\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \\
\mathcal{L}_{i n t} & =-e \bar{\psi} \gamma^{\mu} \psi A_{\mu}
\end{aligned}
$$

where $\mathcal{L}_{0}$, contains only the quadratic part and are the free field Lagrangian we studied before while $\mathcal{L}_{\text {int }}$ is the part describing interaction.

Conjugate momenta for the fermion field is

$$
\frac{\partial \mathcal{L}}{\partial\left(\partial_{0} \psi_{\alpha}\right)}=i \psi_{\alpha}^{\dagger}(x)
$$

For electromagnetic fields, we choose the gauge

$$
\vec{\nabla} \cdot \vec{A}=0
$$

then

$$
\pi^{i}=\frac{\partial \mathcal{L}}{\partial\left(\partial_{0} A^{i}\right)}=-F^{0 i}=E^{i}
$$

From equation of motion

$$
\partial_{\nu} F^{0 \nu}=e \psi^{\dagger} \psi \quad \Longrightarrow \quad-\nabla^{2} A^{0}=e \psi^{\dagger} \psi
$$

Thus $A^{0}$ is not zero but it is not an independent dynamical variable and can be expressed in terms of other field,

$$
A^{0}=e \int d^{3} x^{\prime} \frac{\psi^{\dagger}\left(x^{\prime}, t\right) \psi\left(x^{\prime}, t\right)}{4 \pi\left|\overrightarrow{x^{\prime}}-\vec{x}\right|}=e \int \frac{d^{3} x^{\prime} \rho\left(x^{\prime}, t\right)}{\left|\vec{x}-\overrightarrow{x^{\prime}}\right|}
$$

Commutation relation

$$
\begin{aligned}
\left\{\psi_{\alpha}(\vec{x}, t), \psi_{\beta}^{\dagger}\left(\overrightarrow{x^{\prime}}, t\right)\right\} & =\delta_{\alpha \beta} \delta^{3}\left(\vec{x}-\overrightarrow{x^{\prime}}\right) \quad\left\{\psi_{\alpha}(\vec{x}, t), \psi_{\beta}\left(\overrightarrow{x^{\prime}}, t\right)\right\}=\ldots=0 \\
{\left[\dot{A}_{i}(\vec{x}, t), A_{j}\left(\overrightarrow{x^{\prime}}, t\right)\right] } & =i \delta_{i j}^{t r}\left(\vec{x}-\overrightarrow{x^{\prime}}\right)
\end{aligned}
$$

Commutators involving $A_{0}$ can be worked out as follows,

$$
\left[A_{0}(\vec{x}, t), \psi_{\alpha}\left(\overrightarrow{x^{\prime}}, t\right)\right]=e \int \frac{d^{3} x^{\prime \prime}}{4 \pi\left|\vec{x}-\overrightarrow{x^{\prime \prime}}\right|}\left[\psi^{\dagger}\left(\overrightarrow{x^{\prime \prime}}, t\right) \psi\left(\overrightarrow{x^{\prime \prime}}, t\right), \psi_{\alpha}\left(\overrightarrow{x^{\prime}}, t\right)\right]=-\frac{e}{4 \pi} \frac{\psi_{\alpha}\left(\overrightarrow{x^{\prime}}, t\right)}{\left|\vec{x}-\overrightarrow{x^{\prime}}\right|}
$$

Hamiltonian

$$
\begin{aligned}
\mathcal{H} & =\frac{\partial \mathcal{L}}{\partial\left(\partial_{0} \psi_{\alpha}\right)} \dot{\psi}_{\alpha}+\frac{\partial \mathcal{L}}{\partial\left(\partial_{0} A^{k}\right)} \dot{A}_{k}-\mathcal{L} \\
& =\psi^{\dagger}(-i \vec{\alpha} \cdot \vec{\nabla}+\beta m) \psi+\frac{1}{2}\left(\vec{E}^{2}+\vec{B}^{2}\right)+\vec{E} \cdot \vec{\nabla} A_{0}+e \bar{\psi} \gamma^{\mu} \psi A_{\mu}
\end{aligned}
$$

and

$$
H=\int d^{3} x \mathcal{H}=\int d^{3} x\left\{\psi^{\dagger}[\vec{\alpha} \cdot(-i \vec{\nabla}-e \vec{A})+\beta m] \psi+\frac{1}{2}\left(\vec{E}^{2}+\vec{B}^{2}\right)\right\}
$$

$A_{0}$ does not appear in the interaction,
But if we write

$$
\vec{E}=\vec{E}_{l}+\vec{E}_{t} \quad \text { where } \quad \vec{E}_{l}=-\vec{\nabla} A_{0} \quad, \overrightarrow{E_{t}}=-\frac{\partial \vec{A}}{\partial t}
$$

Then

$$
\frac{1}{2} \int d^{3} x\left(\vec{E}^{2}+\vec{B}^{2}\right)=\frac{1}{2} \int d^{3} x \vec{E}_{l}^{2}+\int d^{3} x\left({\overrightarrow{E_{t}}}^{2}+\vec{B}^{2}\right)
$$

The longitudinal part is

$$
\frac{1}{2} \int d^{3} x \vec{E}_{l}^{2}=\frac{e}{4 \pi} \int d^{3} x d^{3} y \frac{\rho(\vec{x}, t) \rho(\vec{y}, t)}{|\vec{x}-\vec{y}|} \quad \text { Coulomb interaction }
$$

Even though we can set up the commutators or anti-commutators for quantization, it is difficult, if not impossible, to find the physical consequences.This is because we do not know how to solve the classical equations of motion which is highly non-linear. Without the classical solutions we can not carryout the mode expansion to introduce the creation and annihilation operators and it is difficult to find the eigenvalues and eigenstates of the Hamiltonian. The only approximation we know how to do in field theory is the perturbation theory. We will now set up the framework for the perturbation.

### 1.1 Physical states

In high energy physics, we study the interactions by scattering processes.We assume that the interactions of interest are all short-range in nature. Then far away from the interaction region, particles propagate as free particles.

Choose the physical states to be eigensates of energy momentum operators,

$$
P_{\mu}|\Psi\rangle=p_{\mu}|\Psi\rangle
$$

They are required to satisfy following reasonable requirements;
(a) The eigenvalues $p_{\mu}$ all lie within forward light cone,

$$
p^{2}=p_{\mu} p^{\mu} \geqslant 0, \quad p_{0} \geqslant 0
$$

(b) There exists a non-degenerate Lorentz invariant ground state $\mid 0>$ with lowest energy taken to be the zero point ,

$$
p^{0}|0\rangle=0
$$

which implies

$$
\vec{p}|0\rangle=0
$$

(c) There exists stable single particle states $\left|\overrightarrow{p_{i}}\right\rangle$ with $p_{i}^{2}=m_{i}^{2}$ for each stable particle.
(d) The vaccum and one particle states form discrete spectrum in $p^{\mu}$

We associate a field $\phi(x)$ for each discrete state appearing in the spectrum of $p^{\mu}$ and assume that interactions do not violently the spectrum of states. This means that there is no room in this formalism to describe bound states which are not there to begin with.

### 1.2 In-fields and in-states-asymptotic conditions

For simplicity, consider

$$
\mathcal{L}=\frac{1}{2}\left(\partial_{\mu} \phi\right)^{2}-\frac{\mu_{0}^{2}}{2} \phi^{2}+\frac{\lambda}{4!} \phi^{4}
$$

Equation of motion

$$
\left(\square+\mu_{0}^{2}\right) \phi=j(x)=\frac{\lambda}{3!} \phi^{3}
$$

conjugate momenta

$$
\pi(x)=\frac{\partial \mathcal{L}}{\partial_{0} \phi}=\partial_{0} \phi
$$

## Commutation relations

$$
[\pi(x, t), \phi(y, t)]=-i \delta^{3}(x-y) \quad[\pi(x, t), \pi(y, t)]=[\phi(x, t), \phi(y, t)]=0
$$

In scattering problem, at $t=-\infty$, particles progate freely. Let $\phi_{i n}(x)$ be the operator which creates free particle propagating with physical mass $\mu$.

$$
\left(\square+\mu^{2}\right) \phi_{i n}(x)=0
$$

Here we allow the particle's physical mass $\mu$ to be different from $\mu_{0}$ that appear in the Largrangian. This has to do with the renormalization effect which will be described later. We will assume that $\phi_{i n}(x)$ transforms under coordinate displacements and Lorentz transformation in the same way as $\phi(x)$. In particular,

$$
\left[p_{\mu}, \phi_{i n}(x)\right]=-i \partial_{\mu} \phi_{i n}(x)
$$

This implies that $\phi_{\text {in }}(x)$ creates one particle state from vacuum. To see this, consider states with definite momentum,

$$
P^{\mu}|n\rangle=p_{n}^{\mu}|n\rangle
$$

Then

$$
-i \partial_{\mu}\langle n| \phi_{i n}(x)|0\rangle=\langle n|\left[p_{\mu}, \phi_{i n}(x)\right]|0\rangle=p_{n}^{\mu}\langle n| \phi_{i n}(x)|0\rangle
$$

From this we get

$$
\left(\square+\mu^{2}\right)\langle n| \phi_{i n}(x)|0\rangle=\left(\mu^{2}-p_{n}^{2}\right)\langle n| \phi_{i n}(x)|0\rangle=0, \quad \Longrightarrow p_{n}^{2}=\mu^{2}
$$

This means $\langle n| \phi_{i n}(x)|0\rangle$ is non-zero only for state $\langle n|$ with $p_{n}^{2}=\mu^{2}$ which is an one-particle state. Thus the state $\phi_{\text {in }}(x)|0\rangle$ is an one-particle state with mass $\mu^{2}$.

Since $\phi_{i n}(x)$ satisfies free field equation, we can expand $\phi_{i n}(x)$ in terms of free solution of Klein-Gordon equation,

$$
\phi_{i n}(x)=\int d^{3} k\left[a_{i n}(k) f_{k}(x)+a_{i n}^{\dagger}(k) f_{k}^{*}(x)\right] \quad f_{k}(x)=\frac{1}{\sqrt{(2 \pi)^{3} 2 w_{k}}} e^{-i k \cdot x}
$$

Invert this expansion

$$
a_{i n}(k)=i \int d^{3} x f_{k}^{*}(x) \overleftrightarrow{\partial_{0}} \phi_{i n}(x)
$$

We also have

$$
\left[p^{\mu}, a_{i n}(k)\right]=-k^{\mu} a_{i n}(k), \quad\left[p^{\mu}, a_{i n}^{\dagger}(k)\right]=k^{\mu} a_{i n}^{\dagger}(k)
$$

States are defined by

$$
\begin{aligned}
\left|k_{1}, i n\right\rangle & =\sqrt{(2 \pi)^{3} 2 w_{k}} a_{i n}^{\dagger}(k)|0\rangle \\
\left.\left|k_{1}, k_{2}, \ldots k_{n}\right| i n\right\rangle & =\left[\prod_{i} \sqrt{(2 \pi)^{3} 2 w_{k_{i}}} a_{i n}^{\dagger}\left(k_{i}\right)\right]|0\rangle
\end{aligned}
$$

With normalization

$$
\left\langle k_{2}, i n \mid k_{1}, i n\right\rangle=(2 \pi)^{3} 2 w_{1} \delta^{3}\left(\overrightarrow{k_{1}}-\overrightarrow{k_{2}}\right)
$$

$$
\left\langle p_{1}, p_{2}, \ldots, p_{m}, i n\right| k_{1}, k_{2}, \ldots k_{n}|i n\rangle=0
$$

unless $\mathrm{m}=\mathrm{n}$ and $\left(p_{1}, p_{2}, \ldots, p_{m}\right)$ conicides with $\left(k_{1}, k_{2}, \ldots k_{n}\right)$
Relation between $\phi_{\text {in }}(x)$ and $\phi(x)$
We now want to find the relation between interacting field $\phi(x)$ and the $\phi_{i n}(x)$ which is a free field in order to set up the perturbation expansion. The field equations for these fields can be written as

$$
\begin{gathered}
\left(\square+\mu_{0}^{2}\right) \phi(x)=j(x) \quad \text { or } \quad\left(\square+\mu^{2}\right) \phi(x)=j(x)+\delta \mu^{2} \phi(x)=\widetilde{j(x)} \quad \delta \mu^{2}=\mu^{2}-\mu_{0}^{2} \\
\left(\square+\mu^{2}\right) \phi_{\text {in }}(x)=0
\end{gathered}
$$

Formally we can relate the solution to the inhomogeneous equation $\phi(x)$ to the solution to homogeneous equation $\phi_{i n}(x)$ as,

$$
\sqrt{z} \phi_{i n}(x)=\phi(x)-\int d^{4} y \triangle_{r e t}\left(x-y, \mu^{2}\right) \widetilde{j(y)}
$$

where

$$
\left(\square_{x}+\mu^{2}\right) \triangle_{r e t}\left(x-y, \mu^{2}\right)=\delta^{4}(x-y) \quad \text { and } \quad \triangle_{r e t}\left(x-y, \mu^{2}\right)=0 \quad \text { for } \quad x_{0}<y_{0}
$$

is the usual retarded Green's function. This suggests that as $x_{0} \rightarrow-\infty, \phi(x) \rightarrow \sqrt{z} \phi_{i n}(x)$. This relation which relates 2 operators can be viewed as strong convergence relation.It turns out that this leads to contradiction. The argument is quite technical and will not be discussed here.

Correct asymptotic condition (Lehmann, Symanzik, and Zimmermann)
Let $|\alpha\rangle,|\beta\rangle$ be any two normalizable states, $\phi^{f}(t)$ is defined by smearing $\phi(x)$ over space-like region

$$
\phi^{f}(t) \equiv i \int d^{3} x f_{k}^{*}(\vec{x}, t) \overleftrightarrow{\partial_{0}} \phi(\vec{x}, t) \quad \text { with } \quad\left(\square+\mu^{2}\right) f=0
$$

where $f_{k}(\vec{x}, t)$ is an arbitrary normalizable solution to Klein-Gordon equation. Then the correct asymptotic condition is

$$
\lim _{x_{0} \rightarrow-\infty}\langle\alpha| \phi^{f}(t)|\beta\rangle=\sqrt{z}\langle\alpha| \phi_{i n}^{f}(t)|\beta\rangle
$$

where

$$
\phi_{i n}^{f}(t)=i \int d^{3} x f^{*}(\vec{x}, t) \overleftrightarrow{\partial_{0}} \phi_{i n}(\vec{x}, t) \quad \text { time-independent }
$$

This is known as weak convergence relation.

## Out fields and out states

Just like the case of in-field and in states, we can also reduce the dynamics to that of free particles for $t \rightarrow \infty$ by defining

$$
\begin{array}{r}
\left(\square+\mu_{0}^{2}\right) \phi_{\text {out }}(x)=0 \\
\phi_{\text {out }}(x)=\int d^{3} k\left[a_{\text {out }}(k) f_{k}(x)+a_{\text {out }}^{\dagger}(k) f_{k}^{*}(x)\right], \quad\left[p^{\mu}, a_{\text {out }}^{\dagger}(k)\right]=-k^{\mu} a_{\text {out }}^{\dagger}(k)
\end{array}
$$

Asymptotic condition

$$
\lim _{t \rightarrow \infty}\langle\alpha| \phi^{f}(t)|\beta\rangle=\sqrt{z}\langle\alpha| \phi_{\text {out }}^{f}(t)|\beta\rangle
$$

## 2 S-matrix

Description of scattering processes: start with state with $n$ non-interacting particles.They interact when they are close to each other. After interaction, $m$ particles seperate and again propagate freely.

Denote the initial state by

$$
\left|p_{1}, p_{2}, \ldots, p_{n}, i n\right\rangle=|\alpha, i n\rangle
$$

and final state by

$$
\left.\left.\mid p_{1}^{\prime}, p_{2}^{\prime}, \ldots, p_{m}^{\prime}, \text { out }\right\rangle=\mid \beta, \text { out }\right\rangle
$$

The S-matrix for the transistion from initial state $\mid \alpha$, in $\rangle$ to final state $\mid \beta$, out $\rangle$ is

$$
\left.S_{\beta \alpha} \equiv\langle\beta, \text { out }| \alpha, \text { in }\right\rangle
$$

We can introduce S-operator which will take an in - state and turn it into out - state,

$$
\langle\beta, \text { out }| \equiv\langle\beta, \text { in }| S \quad\langle\beta, \text { out }| S^{-1}=\langle\beta, \text { in }|
$$

then the $S$ - matrix element can be written as a matrix element of $S$-operator between 2 in - states,

$$
\left.\left.S_{\beta \alpha}=\langle\beta, \text { out }| \alpha, \text { in }\right\rangle=\langle\beta, \text { in }| S \mid \alpha, \text { in }\right\rangle
$$

## Properties of S-matrix

(a) From the stability of vacuum $\left|S_{00}\right|=1$

$$
\langle 0, \text { in }| S=\langle 0, \text { out }|=e^{-i \varphi}\langle 0, \text { in }|
$$

(b) Stability of the one-particle state requires

$$
\langle p, \text { in }| S \mid p, \text { in }\rangle=\langle p, \text { out }| p, \text { in }\rangle=1 \quad \because \mid p, \text { in }\rangle=\mid p, \text { out }\rangle
$$

(c)

$$
\phi_{\text {in }}(x)=S \phi_{\text {out }}(x) S^{-1}
$$

To prove this, consider

$$
\left.\left.\langle\beta, \text { out }| \phi_{\text {out }}(x) \mid \alpha, \text { in }\right\rangle=\langle\beta, \text { in }| S \phi_{\text {out }}(x) \mid \alpha, \text { in }\right\rangle
$$

Since $\langle\beta$,out $| \phi_{\text {out }}(x)$ is an out state

$$
\langle\beta, \text { out }| \phi_{\text {out }}(x)=\langle\beta, \text { in }| \phi_{\text {in }}(x) S
$$

then

$$
\langle\beta, i n| S \phi_{\text {out }}(x)|\alpha, i n\rangle=\langle\beta, i n| \phi_{\text {in }}(x) S|\alpha, i n\rangle
$$

Or

$$
S \phi_{\text {out }}=\phi_{\text {in }}(x) S \quad \text { or } \phi_{\text {in }}(x)=S \phi_{\text {out }}(x) S^{-1}
$$

(d) Unitarity

Since

$$
\left.\left.\langle\alpha, \text { in }| S=\langle\alpha, \text { out }|, \quad \Longrightarrow S^{\dagger} \mid \alpha, \text { in }\right\rangle=\mid \alpha, \text { out }\right\rangle
$$

Combinnng these two relations gives

$$
\left.\left.\langle\beta, \text { in }| S S^{\dagger} \mid \alpha, \text { in }\right\rangle=\langle\beta, \text { out }| \alpha, \text { out }\right\rangle=\delta_{\beta \alpha}
$$

As operator we see that $S S^{\dagger}=1$,similar argument $\Longrightarrow S^{\dagger} S=1$
(e) S is translational and Lorentz invariance

Under Lorentz transformation

$$
x^{\mu} \rightarrow x^{\prime \mu}=\Lambda_{\nu}^{\mu} x^{\nu}+b^{\mu}
$$

then

$$
U(\Lambda, b) S U^{-1}(\Lambda, b)=S
$$

where

$$
U(\Lambda, b) \phi(x) U^{-1}(\Lambda, b)=\phi(\Lambda x+b)
$$

Proof:

$$
\begin{aligned}
\phi_{\text {in }}(\Lambda x+b) & =U(\Lambda, b) \phi_{\text {in }}(x) U^{-1}(\Lambda, b)=U S \phi_{\text {out }}(x) S^{-1} U^{-1} \\
& =\left(U S U^{-1}\right) \phi_{\text {out }}(\Lambda x+b)\left(U S^{-1} U^{-1}\right)
\end{aligned}
$$

But

$$
\phi_{\text {in }}(\Lambda x+b)=S \phi_{\text {out }}(\Lambda x+b) S^{-1}
$$

This implies that

$$
U(\Lambda, b) S U^{-1}(\Lambda, b)=S
$$

which completes the proof.

## 3 LSZ reduction

We now work to set up the framework to compute the transistion matrix element $S_{\beta \alpha}$.
Consider

$$
\left.S_{\beta, \alpha p}=\langle\beta, \text { out }| \alpha, p, \text { in }\right\rangle
$$

Using the creation operator for the in - state, we get

$$
\begin{aligned}
S_{\beta, \alpha p} & \left.=\langle\beta, \text { out }| \alpha, p, \text { in }\rangle=\sqrt{(2 \pi)^{3} 2 w_{p}}\langle\beta, \text { out }| a_{\text {in }}^{\dagger}(p) \mid \alpha, \text { in }\right\rangle \\
& \left.\left.\left.=\sqrt{(2 \pi)^{3} 2 w_{p}}\langle\beta, \text { out }| a_{\text {out }}^{\dagger}(p) \mid \alpha, \text { in }\right\rangle+\langle\beta, \text { out }|\left[a_{\text {in }}^{\dagger}(p)-a_{\text {out }}^{\dagger}(p)\right] \mid \alpha, \text { in }\right\rangle\right] \\
& \left.\left.=\sqrt{(2 \pi)^{3} 2 w_{p}}[\langle\beta-p, \text { out }| \alpha, \text { in }\rangle-i\langle\beta, \text { out }| \int d^{3} x f_{p}(x) \overleftrightarrow{\partial_{0}}\left[\phi_{\text {in }}(x)-\phi_{\text {out }}(x)\right] \alpha, \text { in }\right\rangle\right]
\end{aligned}
$$

Here $\langle\beta-p$, out $|$ is the state obtained from $\langle\beta$, out $|$ by removing a particle with momentum $\vec{p}$. Use the symptotic conditions

$$
\langle\alpha| \phi_{\text {in }}(x)|\beta\rangle=\frac{1}{\sqrt{z}} \lim _{t \rightarrow-\infty}\langle\alpha| \phi(x)|\beta\rangle, \quad\langle\alpha| \phi_{\text {out }}(x)|\beta\rangle=\frac{1}{\sqrt{z}} \lim _{t \rightarrow \infty}\langle\alpha| \phi(x)|\beta\rangle
$$

and the identity

$$
\begin{aligned}
\left(\lim _{x_{0} \rightarrow \infty}-\lim _{x_{0} \rightarrow-\infty}\right) \int d^{3} x g_{1}(x) \overleftrightarrow{\partial_{0}} g_{2}(x) & =\int_{-\infty}^{\infty} d^{4} x \partial_{0}\left(g_{1}(x) \overleftrightarrow{\partial_{0}} g_{2}(x)\right) \\
& =\int_{-\infty}^{\infty} d^{4} x\left[g_{1}(x) \partial_{0}^{2} g_{2}(x)-\partial_{0}^{2} g_{1}(x) g_{2}(x)\right]
\end{aligned}
$$

we get
$\int d^{3} x f_{p}(x) \overleftrightarrow{\partial_{0}}\left[\phi_{\text {in }}(x)-\phi_{\text {out }}(x)\right]=\int d^{4} x\left[\partial_{0}^{2} f_{p}(x) \phi(x)-f_{p}(x) \partial_{0}^{2} \phi(x)\right]=-\int d^{4} x f_{p}(x) \quad\left(\square+\mu^{2}\right) \phi(x)$
where we have used $\partial_{0}^{2} f_{p}(x)=\left(\partial_{i}^{2}-\mu^{2}\right) f_{p}(x)$ and carry out the integration by parts. Put these together, we get the reduction formula,

$$
\left.\left.\langle\beta, \text { out }| \alpha, p, \text { in }\rangle=\sqrt{(2 \pi)^{3} 2 w_{p}}\langle\beta-p, \text { out }| \alpha, \text { in }\right\rangle \left.+\frac{i}{\sqrt{z}} \int e^{-i p \cdot x} d^{4} x\left(\square+\mu^{2}\right)\langle\beta, \text { out }| \phi(x) \right\rvert\, \alpha, \text { in }\right\rangle
$$

To remove a particle with momentum $p^{\prime}$ from $\beta$ in the matrix element $\langle\beta$,out $| \phi(x) \mid \alpha$, in $\rangle$, write $\beta=\gamma p^{\prime}$ and use the annililation operator,

$$
\begin{gathered}
\left.\left.\langle\beta, \text { out }| \phi(x) \mid \alpha, \text { in }\rangle=\left\langle\gamma p^{\prime}, \text { out }\right| \phi(x) \mid \alpha, \text { in }\right\rangle=\sqrt{(2 \pi)^{3} 2 w_{p^{\prime}}}\langle\gamma, \text { out }| a_{\text {out }}\left(p^{\prime}\right) \phi(x) \mid \alpha, \text { in }\right\rangle \\
\left.=\sqrt{(2 \pi)^{3} 2 w_{p^{\prime}}}\left[\langle\gamma, \text { out }| \phi(x) a_{\text {in }}\left(p^{\prime}\right) \mid \alpha, \text { in }\right\rangle-\langle\gamma, \text { out }|\left(a_{\text {out }}\left(p^{\prime}\right) \phi(x)-\phi(x) a_{\text {in }}\left(p^{\prime}\right) \mid \alpha, \text { in }\right\rangle\right] \\
\left.\left.\left.=\sqrt{(2 \pi)^{3} 2 w_{p^{\prime}}[ }\langle\gamma, \text { out }| \phi(x) \mid \alpha-p^{\prime}, \text { in }\right\rangle-i \int d^{3} y\langle\gamma, \text { out }|\left(\phi_{\text {out }}(y) \phi(x)-\phi(x) \phi_{\text {in }}(y)\right) \mid \alpha, \text { in }\right\rangle \overleftrightarrow{\partial_{0}} f_{p^{\prime}}^{*}(y)\right] \\
\left.\left.\left.=\sqrt{(2 \pi)^{3} 2 w_{p^{\prime}}}\left[\langle\gamma, \text { out }| \phi(x) \mid \alpha-p^{\prime}, \text { in }\right\rangle-\frac{i}{\sqrt{z}} \int d^{3} y\left(\lim _{y 0 \rightarrow \infty}-\lim _{y_{0} \rightarrow-\infty}\right)\langle\gamma, \text { out }|(T(\phi(y) \phi(x))) \right\rvert\, \alpha, \text { in }\right\rangle \overleftrightarrow{\partial_{0}} f_{p^{\prime}}^{*}(y)\right]
\end{gathered}
$$

Following the same procedure as before, we can get

$$
\left.\left.\langle\beta, \text { out }| \phi(x) \mid \alpha, \text { in }\rangle=\sqrt{(2 \pi)^{3} 2 w_{p^{\prime}}}\left\{\langle\gamma, \text { out }| \phi(x) \mid \alpha-p^{\prime}, \text { in }\right\rangle\right\} \left.+\frac{i}{\sqrt{z}} \int d^{4} y\langle\gamma, \text { out }| T(\phi(y) \phi(x)) \right\rvert\, \alpha, \text { in }\right\rangle\left(\overleftarrow{\square}+\mu^{2}\right) e^{i p \cdot x}
$$

It is clear how to remove all particles from "in" and "out" state

$$
\begin{aligned}
\left.\left\langle p_{1}, \ldots, p_{n}, \text { out }\right| q_{1}, \ldots, q_{m}, \text { in }\right\rangle= & \left(\frac{i}{\sqrt{z}}\right)^{m+n} \prod_{i=1}^{m} \prod_{j=1}^{n} \int d^{4} x_{i} d^{4} y_{j} e^{-i q_{i} x_{i}}\left(\overrightarrow{\square_{x}}+\mu^{2}\right) \\
& \langle 0| T\left(\phi\left(y_{1}\right) \ldots \phi\left(y_{m}\right) \phi\left(x_{1}\right) \ldots \phi\left(x_{m}\right)\right)|0\rangle\left(\overleftarrow{\square_{y_{j}}}+\mu^{2}\right) e^{i p_{j} \cdot x_{j}}
\end{aligned}
$$

for all $p_{j} \neq q_{i}$

### 3.1 In and Out fields for Fermions

It is now clear how to generalize the discussion to the case of fermions. The in-field can be written as

$$
\psi_{i n}(x)=\int d^{3} p \sum_{s}\left[b_{i n}(p, s) U_{p, s}(x)-d_{i n}^{\dagger}(p, s) V_{p, s}(x)\right]
$$

where

$$
U_{p, s}(x)=\frac{1}{\sqrt{(2 \pi)^{3} 2 E_{p}}} u(p, s) e^{-i p \cdot x} \quad V_{p, s}(x)=\frac{1}{\sqrt{(2 \pi)^{3} 2 E_{p}}} v(p, s) e^{i p \cdot x}
$$

Inversion

$$
\begin{array}{ll}
b_{i n}(p, s)=\int d^{3} x U_{p, s}^{\dagger}(x) \psi_{i n}(x) & d_{i n}(p, s)=\int d^{3} x \psi_{i n}^{\dagger}(x) V_{p, s}(x) \\
b_{i n}^{\dagger}(p, s)=\int d^{3} x \psi_{i n}^{\dagger}(x) U_{p, s}(x) & d_{i n}^{\dagger}(p, s)=\int d^{3} x V_{p, s}^{\dagger}(x) \psi_{i n}(x)
\end{array}
$$

## Reduction formula for fermions

(a) remove electron from the in-state

$$
\left.\langle\beta, \text { out }| \alpha ; \text { ps, in }\rangle \left.=-\frac{i}{\sqrt{z_{2}}} \int d^{4} x\langle\beta, \text { out }| \overline{\psi_{\alpha}}(x) \right\rvert\, \alpha, \text { in }\right\rangle \overleftarrow{\left(-i \gamma^{\mu} \partial_{\mu}-m\right)_{\alpha \beta}} u(p, s) e^{-i p \cdot x}
$$

(b) remove positron(anti-particle) from the in-state

$$
\left.\langle\beta, \text { out }| \alpha ; \overline{p s}, \text { in }\rangle \left.=\frac{i}{\sqrt{z_{2}}} \int d^{4} x e^{-i \bar{p} \cdot x} \overline{\nu_{\alpha}}(\bar{p}, \bar{s}) \overrightarrow{\left(i \gamma^{\mu} \partial_{\mu}-m\right)_{\alpha \beta}}\langle\beta, \text { out }| \psi_{\beta}(x) \right\rvert\, \alpha, \text { in }\right\rangle
$$

(c) remove electron from out-state

$$
\left.\left.\left\langle\beta ; p^{\prime} s^{\prime}, \text { out }\right| \alpha, \text { in }\right\rangle \left.=-\frac{i}{\sqrt{z_{2}}} \int d^{4} x \overline{u_{\alpha}}\left(p^{\prime}, s^{\prime}\right) e^{i p^{\prime} \cdot x} \overrightarrow{\left(i \gamma^{\mu} \partial_{\mu}-m\right)_{\alpha \beta}}\langle\beta, \text { out }| \psi_{\beta}(x) \right\rvert\, \alpha, \text { in }\right\rangle
$$

(d) remove positron from out-state

$$
\left.\left.\left\langle\beta ; \bar{p}^{\prime} \bar{s}^{\prime}, \text { out }\right| \alpha, \text { in }\right\rangle \left.=\frac{i}{\sqrt{z_{2}}} \int d^{4} x\langle\beta, \text { out }| \overline{\psi_{\alpha}}(x) \right\rvert\, \alpha, \text { in }\right\rangle \overleftarrow{\left(-i \gamma^{\mu} \partial_{\mu}-m\right)_{\alpha \beta}} v\left(\bar{p}^{\prime} \bar{s}^{\prime}\right) e^{-i \bar{p}^{\prime} \cdot x}
$$

## 4 U matrix

For the purpose of perturbation theory we want to find the relation between interacting fields $\phi(x), \pi(x)$ and the free fields $\phi_{i n}(x), \pi_{i n}(x)$. Assum they are related by unitary U matrix,

$$
\phi(\vec{x}, t)=U^{-1}(t) \phi_{i n}(\vec{x}, t) U(t), \quad \pi(\vec{x}, t)=U^{-1}(t) \pi_{i n}(\vec{x}, t) U(t)
$$

The in-fields satisfy the free field equation of motion,

$$
\begin{equation*}
\partial_{0} \phi_{i n}(x)=i\left[H_{i n}\left(\phi_{i n}, \pi_{i n}\right), \phi_{i n}\right], \quad \partial_{0} \pi_{i n}(x)=i\left[H_{i n}\left(\phi_{i n}, \pi_{i n}\right), \pi_{i n}\right] \tag{1}
\end{equation*}
$$

where $H_{i n}\left(\phi_{i n}, \pi_{i n}\right)$ is the free field Hamiltonian with mass $\mu$. On the other hand, the time evolution of interacting field is governed by full Hamiltonian,

$$
\partial_{0} \phi(x)=i[H(\phi, \pi), \phi], \quad \partial_{0} \pi(x)=i[H(\phi, \pi), \pi]
$$

Then from equation for $\phi_{i n}$, we get

$$
\begin{aligned}
\partial_{0} \phi_{i n} & =\frac{\partial}{\partial t}\left[U \phi U^{-1}\right]=\frac{\partial U}{\partial t} \phi U^{-1}+U \frac{\partial \phi}{\partial t} U^{-1}+U \phi \frac{\partial U^{-1}}{\partial t} \\
& =\frac{\partial U}{\partial t}\left(U^{-1} \phi_{i n} U\right) U^{-1}+U(i[H(\phi, \pi), \phi]) U^{-1}-U \phi U^{-1} \frac{\partial U}{\partial t} U^{-1} \\
& =\left(\frac{\partial U}{\partial t} U^{-1}\right) \phi_{i n}+i\left[H\left(\phi_{i n}, \pi_{i n}\right), \phi_{i n}\right]-\phi_{i n} \frac{\partial U}{\partial t} U^{-1}
\end{aligned}
$$

Using $\mathrm{Eq}(1)$, we simplify

$$
\left[\frac{\partial U}{\partial t} U^{-1}+i H_{I}\left(\phi_{i n}, \pi_{i n}\right), \phi_{i n}\right]=0
$$

where $H_{I}\left(\phi_{i n}, \pi_{i n}\right)=H\left(\phi_{i n}, \pi_{i n}\right)-H_{i n}\left(\phi_{i n}, \pi_{i n}\right)$ contains all the interaction. Similarly, we can show

$$
\left[\frac{\partial U}{\partial t} U^{-1}+i H_{I}\left(\phi_{i n}, \pi_{i n}\right), \pi_{i n}\right]=0
$$

This means the combination $\frac{\partial U}{\partial t} U^{-1}+i H_{I}$ commutes with all the operators, we can take this to be a c-number. For simlicity,we can take this c-number to be zero.Thus

$$
\begin{equation*}
\mathrm{i} \frac{\partial U(t)}{\partial t}=H_{I}(t) U(t) \tag{2}
\end{equation*}
$$

For convenience, we define

$$
U\left(t, t^{\prime}\right) \equiv U(t) U^{-1}\left(t^{\prime}\right) \quad \text { time evolution operator }
$$

then $\mathrm{Eq}(2)$ becomes,

$$
i \frac{\partial U\left(t, t^{\prime}\right)}{\partial t}=H_{I}(t) U\left(t, t^{\prime}\right) \quad \text { with } \quad U(t, t)=1
$$

We can convert this to integral equation

$$
U\left(t, t^{\prime}\right)=1-i \int_{t^{\prime}}^{t} d t_{1} H_{I}\left(t_{1}\right) U\left(t_{1}, t^{\prime}\right)
$$

which includes the initial condition. Iterate this equation assuming $H_{I}$ is "small",

$$
\begin{aligned}
U\left(t, t^{\prime}\right)= & 1-i \int_{t^{\prime}}^{t} d t_{1} H_{I}\left(t_{1}\right)+(-i)^{2} \int_{t^{\prime}}^{t} d t_{1} H_{I}\left(t_{1}\right) \int_{t^{\prime}}^{t_{1}} d t_{2} H_{I}\left(t_{2}\right)+\ldots \\
& +(-i)^{n} \int_{t^{\prime}}^{t} d t_{1} \int_{t^{\prime}}^{t_{1}} d t_{2} \ldots \int_{t^{\prime}}^{t_{n-1}} d t_{n} H_{I}\left(t_{1}\right) H_{I}\left(t_{2}\right) \ldots H_{I}\left(t_{n}\right)+\ldots
\end{aligned}
$$

The second term can be written as

$$
\begin{aligned}
U^{(2)} & =(-i)^{2} \int_{t^{\prime}}^{t} d t_{1} \int_{t^{\prime}}^{t_{1}} d t_{2} H_{I}\left(t_{1}\right) H_{I}\left(t_{2}\right)=(-i)^{2} \int_{t^{\prime}}^{t} d t_{2} \int_{t_{2}}^{t} d t_{1} H_{I}\left(t_{1}\right) H_{I}\left(t_{2}\right) \\
& =(-i)^{2} \int_{t^{\prime}}^{t} d t_{1} \int_{t_{2}}^{t} d t_{2} H_{I}\left(t_{2}\right) H_{I}\left(t_{1}\right)
\end{aligned}
$$

where we have interchange the order of integration. Renaming $t_{1}$ and $t_{2}$, we get

$$
U^{(2)}=(-i)^{2} \int_{t^{\prime}}^{t} d t_{1} \int_{t_{2}}^{t} d t_{2} H_{I}\left(t_{2}\right) H_{I}\left(t_{1}\right)
$$

We can use time-ordered product to combine these two equivalent expression so that the $t_{2}$ integration goes from $t^{\prime}$ to $t$

$$
U^{(2)}=\frac{(-i)^{2}}{2} \int_{t^{\prime}}^{t} d t_{1} \int_{t^{\prime}}^{t} d t_{2} T\left(H_{I}\left(t_{2}\right) H_{I}\left(t_{1}\right)\right)
$$

We can generalize these steps to higher terms in $U$ so that

$$
\begin{aligned}
U\left(t, t^{\prime}\right) & =1+\sum_{n=1}^{\infty} \frac{(-i)^{n}}{n!} \int_{t^{\prime}}^{t} d t_{1} \int_{t^{\prime}}^{t} d t_{2} \ldots \int_{t^{\prime}}^{t} d t_{n} T\left(H_{I}\left(t_{1}\right) H_{I}\left(t_{2}\right) \ldots H_{I}\left(t_{n}\right)\right) \\
& =T\left(\exp \left[-i \int_{t^{\prime}}^{t} d^{4} x \mathcal{H}_{I}\left(\phi_{i n}, \pi_{i n}\right)\right]\right)
\end{aligned}
$$

### 4.1 Perturbation Expansion of Vaccum expectation value

From LSZ reduction formula we see that the scattering matrix element, $S$-matrix can be written in terms of vacuum expectation value of the form,

$$
\tau\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\langle 0| T\left(\phi\left(x_{1}\right) \phi\left(x_{2}\right) \ldots \phi\left(x_{n}\right)\right)|0\rangle
$$

Using U matrix we can write this in terms of $\phi_{i n}$

$$
\begin{aligned}
\tau & =\langle 0| T\left(U^{-1}\left(t_{1}\right) \phi_{i n}\left(x_{1}\right) U\left(t_{1}, t_{2}\right) \phi_{i n}\left(x_{2}\right) U\left(t_{2}, t_{3}\right) \ldots U\left(t_{n-1}, t_{n}\right) \phi_{i n}\left(x_{n}\right) U\left(t_{n}\right)\right)|0\rangle \\
& =\langle 0| T\left(U^{-1}(t) U\left(t, t_{1}\right) \phi_{i n}\left(x_{1}\right) \ldots \phi_{i n}\left(x_{n}\right) U\left(t_{n}, t^{\prime}\right) U\left(t^{\prime}\right)\right)|0\rangle
\end{aligned}
$$

Let $t>t_{1} \ldots t_{n}>t^{\prime}$,then we can pull $U^{-1}(t)$ and $U\left(t^{\prime}\right)$ out of the time-ordered product, and combine $U^{\prime} s$ and $\phi_{i n}$

$$
\begin{aligned}
\tau & \left.=\langle 0| U^{-1}(t) T U\left(t, t_{1}\right) \phi_{i n}\left(x_{1}\right) \ldots \phi_{i n}\left(x_{n}\right) U\left(t_{n}, t^{\prime}\right)\right) U\left(t^{\prime}\right)|0\rangle \\
& =\langle 0| U^{-1}(t) T\left(\phi_{i n}\left(x_{1}\right) \ldots \phi_{i n}\left(x_{n}\right) \exp \left[-i \int_{t^{\prime}}^{t} H_{I}\left(t^{\prime \prime}\right) d t^{\prime \prime}\right]\right) U\left(t^{\prime}\right)|0\rangle
\end{aligned}
$$

We need to take care of the factor $U^{-1}(t)$ and $U\left(t^{\prime}\right)$. First we show the following theorem.
Theorem: $|0\rangle$ is an eigenstate of $U(-t)$ as $t \rightarrow \infty$
Proof: Consider a matrix element of the type $\langle p, \alpha, i n| U(-t)|0\rangle$. Use the method the same as reduction formula we can write

$$
\begin{aligned}
\langle p, \alpha, i n| U(-t)|0\rangle & =\sqrt{(2 \pi)^{3} 2 w_{p}}\langle\alpha, \text { in }| a_{i n}(p) U(-t)|0\rangle \\
& =i \sqrt{(2 \pi)^{3} 2 w_{p}} \int d^{3} x f_{p}^{*}\left(\vec{x},-t^{\prime}\right) \overleftrightarrow{\partial_{0}^{\prime}}\langle\alpha, i n| \phi_{i n}\left(\vec{x},-t^{\prime}\right) U(-t)|0\rangle \\
& =i \sqrt{(2 \pi)^{3} 2 w_{p}} \int d^{3} x f_{p}^{*}\left(\vec{x},-t^{\prime}\right) \overleftrightarrow{\partial_{0}^{\prime}}\langle\alpha, i n| U\left(-t^{\prime}\right) \phi\left(\vec{x},-t^{\prime}\right) U\left(-t^{\prime}\right) U(-t)|0\rangle
\end{aligned}
$$

Carrying out the operation $\overleftrightarrow{\partial_{0}^{\prime}}$ and taking the limit $t=t^{\prime}$,, we get

$$
\begin{aligned}
& f_{p}^{*}\left(\vec{x},-t^{\prime}\right) \overleftrightarrow{\partial_{0}^{\prime}} U\left(-t^{\prime}\right) \phi\left(\vec{x},-t^{\prime}\right) U\left(-t^{\prime}\right) U(-t) \\
= & \partial_{0}^{\prime} f_{p}^{*}\left(\vec{x},-t^{\prime}\right) U(-t) \phi(-t)--f_{p}^{*}\left(\vec{x},-t^{\prime}\right)\left[\dot{U}(-t) \phi(-t)+U(-t) \dot{\phi}(-t)+U(-t) \phi(-t) \dot{U}^{-1}(-t) U^{-1}(-t)\right]
\end{aligned}
$$

Then

$$
\langle p, \alpha, i n| U(-t)|0\rangle=\sqrt{(2 \pi)^{3} 2 w_{p}}\left\{\langle\alpha, i n| U(-t) a_{i n}(p)|0\rangle+i \int d^{3} x f_{p}^{*}\left(\vec{x},-t^{\prime}\right)\langle\alpha, i n| \dot{U} \phi+U \phi \dot{U}^{-1} U|0\rangle\right\}
$$

In the last term

$$
\begin{aligned}
\dot{U} \phi+U \phi \dot{U}^{-1} U & =\dot{U}\left(U^{-1} \phi_{i n} U\right)+U\left(U^{-1} \phi_{i n} U\right)\left(-U^{-1} \dot{U} U^{-1}\right) U \\
& =\dot{U} U^{-1} \phi_{i n} U-\phi_{i n} \dot{U} U^{-1} U=\left[\dot{U} U^{-1}, \phi_{i n}\right] U \\
& =-i\left[H_{I}\left(\phi_{i n}, \pi_{i n}\right), \phi_{i n}\right] U=0
\end{aligned}
$$

if there is no derivative coupling. Then we get the result $\langle p, \alpha, i n| U(-t)|0\rangle=0$ as $t \rightarrow \infty$ for all instates.This means

$$
U(-t)|0\rangle=\lambda_{-}|0\rangle \quad \lambda_{-} \quad \text { some phase as } t \rightarrow \infty
$$

This completes the proof.
Similarly we can show that

$$
U(t)|0\rangle=\lambda_{+}|0\rangle \quad \lambda_{+} \quad \text { some phase as } t \rightarrow \infty
$$

These phases can be written as

$$
\begin{aligned}
\lambda_{-} \lambda_{+}^{*} & =\langle 0| U(-t)|0\rangle\langle 0| U^{-1}(t)|0\rangle=\langle 0| U(-t) U^{-1}(t)|0\rangle \\
& =\langle 0| T\left(\exp \left[i \int_{-t}^{t} H_{I}\left(t^{\prime}\right) d t^{\prime}\right]\right)|0\rangle \\
& =\left[\langle 0| T \exp \left(-i \int_{-t}^{t} H_{I}\left(t^{\prime}\right) d t^{\prime}\right)|0\rangle\right]^{-1}
\end{aligned}
$$

Now we have express the vacuum expectation value $\tau\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ completely in terms of $\phi_{i n}$,

$$
\begin{aligned}
\tau\left(x_{1}, x_{2}, \ldots, x_{n}\right) & =\langle 0| U^{-1}(-t) T\left(\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right) \ldots \phi_{\text {in }}\left(x_{n}\right) \exp \left(-i \int_{-t}^{t} H_{I}\left(t^{\prime}\right) d t^{\prime}\right)\right) U(t)|0\rangle \\
& =\lambda_{-} \lambda_{+}^{*}\langle 0| T\left(\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right) \ldots \phi_{\text {in }}\left(x_{n}\right) \exp \left(-i \int_{-t}^{t} H_{I}\left(t^{\prime}\right) d t^{\prime}\right)\right)|0\rangle
\end{aligned}
$$

or

$$
\tau\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\frac{\langle 0| T\left(\phi_{i n}\left(x_{1}\right) \phi_{i n}\left(x_{2}\right) \ldots \phi_{i n}\left(x_{n}\right) \exp \left(-i \int_{-\infty}^{\infty} H_{I}\left(t^{\prime}\right) d t^{\prime}\right)\right)|0\rangle}{\langle 0| T\left(\exp \left(-i \int_{-\infty}^{\infty} H_{I}\left(t^{\prime}\right) d t^{\prime}\right)\right)|0\rangle}
$$

To do any computation we need to expand the exponential of $H_{I}$, to write

$$
\tau\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\frac{\sum_{m=0}^{\infty} \frac{(-i)^{m}}{m!} \int_{-\infty}^{\infty} d y_{1} \ldots d y_{m}\langle 0| T\left(\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right) \ldots \phi_{\text {in }}\left(x_{n}\right) \mathcal{H}_{I}\left(y_{1}\right) \mathcal{H}_{I}\left(y_{2}\right) \ldots \mathcal{H}_{I}\left(y_{m}\right)\right)|0\rangle}{\sum_{m=0}^{\infty} \frac{(-i)^{m}}{m!} \int_{-\infty}^{\infty} d y_{1} \ldots d y_{m}\langle 0| T\left(\mathcal{H}_{I}\left(y_{1}\right) \mathcal{H}_{I}\left(y_{2}\right) \ldots \mathcal{H}_{I}\left(y_{m}\right)\right)|0\rangle}
$$

## Wick's theorem

To compute the matrix elements of time-ordered products of free fields $\phi_{i n}$ between vacuum, the procedure is straightforward but tedious. The strategy is to convert time-ordered product to normal ordering whose vacuum expectation values are trivial to do. The results are summarized in the form of Wick's theorem which is stated below;

$$
\begin{aligned}
& T\left(\phi_{\text {in }}\left(x_{1}\right) \ldots \phi_{\text {in }}\left(x_{n}\right)\right)=: \phi_{\text {in }}\left(x_{1}\right) \ldots \phi_{\text {in }}\left(x_{n}\right):+\left[\langle 0| \phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right)|0\rangle: \phi_{\text {in }}\left(x_{3}\right) \phi_{\text {in }}\left(x_{4}\right) \ldots \phi_{\text {in }}\left(x_{n}\right):+ \text { permutations }\right] \\
& \quad+\left[\langle 0| \phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right)|0\rangle\langle 0| \phi_{\text {in }}\left(x_{3}\right) \phi_{\text {in }}\left(x_{4}\right)|0\rangle: \phi_{\text {in }}\left(x_{5}\right) \ldots \phi_{\text {in }}\left(x_{n}\right):+ \text { permutations }\right] \ldots \\
& \quad+\left\{\begin{array}{c}
{\left[\langle 0| \phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right)|0\rangle\langle 0| \phi_{\text {in }}\left(x_{3}\right) \phi_{\text {in }}\left(x_{4}\right)|0\rangle \ldots\langle 0| \phi_{\text {in }}\left(x_{n-1}\right) \phi_{\text {in }}\left(x_{n}\right)|0\rangle+\text { permutations }\right] \text { neven }} \\
{\left[\langle 0| \phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right)|0\rangle \ldots\langle 0| \phi_{\text {in }}\left(x_{n-2}\right) \phi_{\text {in }}\left(x_{n-1}\right)|0\rangle \phi_{\text {in }}\left(x_{n}\right)+\text { permutations }\right] \mathrm{n} \text { odd }}
\end{array}\right.
\end{aligned}
$$

This theorem can be proved by induction. We will illustrate this for the simple case of $n=2$. It is clear that the difference between time-ordered product and normal ordering is a c-number,

$$
T\left(\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right)\right)=: \phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right):+(c-\text { number })
$$

To compute this c-number we can take matrix element between vacuum state,

$$
\langle 0| T\left(\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right)\right)|0\rangle=(c-\text { number })
$$

Then we get

$$
T\left(\phi_{i n}\left(x_{1}\right) \phi_{i n}\left(x_{2}\right)\right)=: \phi_{i n}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right):+\langle 0| T\left(\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right)\right)|0\rangle
$$

Most useful application of Wick's theorem

$$
\begin{array}{ll}
\langle 0| T\left(\phi_{\text {in }}\left(x_{1}\right) \ldots \phi_{\text {in }}\left(x_{n}\right)\right)|0\rangle & =0 \quad \mathrm{n} \text { odd } \\
\langle 0| T\left(\phi_{\text {in }}\left(x_{1}\right) \ldots \phi_{\text {in }}\left(x_{n}\right)\right)|0\rangle & =\sum_{\text {permutation }}\left[\langle 0| T\left(\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right)\right)|0\rangle\langle 0| T\left(\phi_{\text {in }}\left(x_{3}\right) \phi_{\text {in }}\left(x_{4}\right)\right)|0\rangle \ldots\right] \quad \mathrm{n} \text { even }
\end{array}
$$

## Notation

$$
\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right)=\langle 0| T\left(\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right)\right)|0\rangle \quad \text { Contraction }
$$

Example:

$$
\begin{aligned}
\langle 0| T\left(\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right):\right. & \left.\phi_{\text {in }}^{3}\left(y_{1}\right):: \phi_{\text {in }}^{3}\left(y_{2}\right):\right)|0\rangle \\
= & \langle 0| T\left(\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right): \phi_{\text {in }}\left(y_{1}\right) \phi_{\text {in }}\left(y_{1}\right) \phi_{\text {in }}\left(y_{1}\right):: \phi_{\text {in }}\left(y_{2}\right) \phi_{\text {in }}\left(y_{2}\right) \phi_{\text {in }}\left(y_{2}\right):\right)|0\rangle \\
& +\ldots
\end{aligned}
$$

### 4.2 Feynman Propagators

From Wick's theorem we see that the most important quantity in the computation of matrix element is the vacuum expecation of two free fields, called Feynman propagator. It is strightforward to work out these propagator for various type of fields. The result for real scalar field is,

$$
\begin{aligned}
\langle 0| T\left(\phi_{\text {in }}(x) \phi_{\text {in }}(y)\right)|0\rangle & =i \triangle_{F}\left(x-y, \mu^{2}\right)=i \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{e^{-i k \cdot(x-y)}}{k^{2}-\mu^{2}+i \varepsilon}=i \int \frac{d^{4} k}{(2 \pi)^{4}} e^{-i k \cdot(x-y)} i \triangle_{F}(k) \\
\text { with } i \triangle_{F}(k) & =\frac{i}{k^{2}-\mu^{2}+i \varepsilon}
\end{aligned}
$$

For complex scalar field

$$
\langle 0| T\left(\phi_{i n}(x) \phi_{i n}^{*}(y)\right)|0\rangle=i \triangle_{F}\left(x-y, \mu^{2}\right)=i \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{e^{-i k \cdot(x-y)}}{k^{2}-\mu^{2}+i \varepsilon}
$$

Fermion field
$\langle 0| T\left(\psi_{\alpha}^{i n}(x) \bar{\psi}_{\beta}^{i n}(y)\right)|0\rangle=i S_{F}(x-y, m)_{\alpha \beta}=i \int \frac{d^{4} p}{(2 \pi)^{4}} e^{-i p \cdot(x-y)} \frac{\left(\gamma^{\mu} p_{\mu}+m\right)_{\alpha \beta}}{p^{2}-m^{2}+i \varepsilon}=\int \frac{d^{4} p}{(2 \pi)^{4}} e^{-i p \cdot(x-y)} i_{F}(p)_{\alpha \beta}$
photon field

$$
\begin{aligned}
\langle 0| T\left(A_{\mu}^{i n}(x) A_{v}^{i n}(y)\right)|0\rangle= & i D_{F}^{t r}(x-y)=i \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{e^{-i k \cdot(x-y)}}{k^{2}+i \varepsilon} \times \\
& {\left[-g_{\mu \nu}-\frac{k_{\mu} k_{\nu}}{(k \cdot \eta)^{2}-k^{2}}+\frac{(k \cdot \eta)\left(k_{\mu} \eta_{\nu}+k_{\nu} \eta_{\mu}\right)}{(k \cdot \eta)^{2}-k^{2}}-\frac{k^{2} \eta_{\mu} \eta_{\nu}}{(k \cdot \eta)^{2}-k^{2}}\right] }
\end{aligned}
$$

where $\eta_{\mu}=(1,0,0,0)$
It can be shown that in QED only term contributes is " $-g_{\mu \nu}$ " as a consequence of the gauge invariance.

## Graphical representation

$$
\begin{array}{cc}
{ }_{y}^{\circ} \cdots \cdots{ }_{x}^{\circ} & i \triangle_{F}\left(x-y, \mu^{2}\right) \\
{ }_{y}^{\beta} \longrightarrow>{ }_{x}^{\alpha} & i S_{F}(x-y, m)_{\alpha \beta} \\
\sim^{\nu} \sim \sim \sim \sim \sim \sim^{\mu} & i D_{F}^{t r}(x-y)
\end{array}
$$

Each line (propagator) represents a contraction in Wick's expansion e.q.

1. $\phi_{\text {in }}\left(x_{1}\right) \phi_{i n}\left(x_{2}\right): \phi_{\text {in }}\left(y_{1}\right) \phi_{\text {in }}\left(y_{1}\right) \phi_{\text {in }}\left(y_{1}\right):: \phi_{\text {in }}\left(y_{2}\right) \phi_{i n}\left(y_{2}\right) \phi_{\text {in }}\left(y_{2}\right):$

$$
x_{1} \ldots \ldots x_{2}
$$

$$
y_{1}^{\prime}=-\cdots=-\cdots y_{2}
$$

2. $\phi_{i n}\left(x_{1}\right) \phi_{i n}\left(x_{2}\right): \phi_{i n}\left(y_{1}\right) \phi_{i n}(\underbrace{}_{1}) \phi_{i n}\left(y_{1}\right):: \phi_{i n}\left(y_{2}\right) \phi_{i n}\left(y_{2}\right) \phi_{i n}\left(y_{2}\right):$


### 4.3 Vaccum Amplitude

In the denominator of $\tau$-function, there are no external lines

$$
\sum_{m=0}^{\infty} \frac{(-i)^{m}}{m!} \int_{-\infty}^{\infty} d^{4} y_{1} \ldots d^{4} y_{m}\langle 0| T\left(\mathcal{H}_{I}\left(\phi_{\text {in }}\left(y_{1}\right)\right) \ldots \mathcal{H}_{I}\left(\phi_{\text {in }}\left(y_{m}\right)\right)\right)|0\rangle
$$

e.q. 2 nd order term for the case $\mathcal{H}_{I}=\frac{\lambda}{3!}: \phi_{\text {in }}^{3}$ :

$$
\begin{aligned}
\langle 0| T\left(\mathcal{H}_{I}\left(\phi_{i n}\left(y_{1}\right)\right) \mathcal{H}_{I}\left(\phi_{\text {in }}\left(y_{m}\right)\right)\right)|0\rangle & =\left(\frac{\lambda}{3!}\right)^{2}\langle 0| T\left(: \phi_{i n}^{3}\left(y_{1}\right):: \phi_{i n}^{3}\left(y_{2}\right):\right)|0\rangle \\
& =\left(\frac{\lambda}{3!}\right)^{2}: \phi_{i n}\left(y_{1}\right) \phi_{i n}\left(y_{1}\right) \phi_{\text {in }}\left(y_{1}\right):: \phi_{i n}\left(y_{2}\right) \phi_{i n}\left(y_{2}\right) \phi_{i n}\left(y_{2}\right): 3 \times 2
\end{aligned}
$$

closed loop diagram :graphs with no external lines(lines with open end)
disconnected diagram :a subgraph not connected to any external lines connected diagram :graph not disconnected

All graphs appearing in the numerator of the $\tau$-function can be seperated uniquely into connected and disconnected parts

$$
\begin{aligned}
& \sum_{m=0}^{\infty} \frac{(-i)^{m}}{m!} \int_{-\infty}^{\infty} d^{4} y_{1} \ldots d^{4} y_{m}\langle 0| T\left(\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right) \ldots \phi_{\text {in }}\left(x_{n}\right) \mathcal{H}_{I}\left(\phi_{\text {in }}\left(y_{1}\right)\right) \ldots \mathcal{H}_{I}\left(\phi_{\text {in }}\left(y_{m}\right)\right)\right)|0\rangle \\
= & \sum_{m=0}^{\infty} \frac{(-i)^{m}}{m!} \int_{-\infty}^{\infty} d^{4} y_{1} \ldots d^{4} y_{m}\langle 0| T\left(\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right) \ldots \phi_{\text {in }}\left(x_{n}\right) \mathcal{H}_{I}\left(\phi_{\text {in }}\left(y_{1}\right)\right) \ldots \mathcal{H}_{I}\left(\phi_{\text {in }}\left(y_{s}\right)\right)\right)|0\rangle_{C} \\
& \times \frac{m!}{s!(m-s)!}\langle 0| T\left(\mathcal{H}_{I}\left(\phi_{\text {in }}\left(y_{s+1}\right)\right) \ldots \mathcal{H}_{I}\left(\phi_{\text {in }}\left(y_{m}\right)\right)\right)|0\rangle \\
= & \sum_{s} \frac{(-i)^{s}}{s!} \int_{-\infty}^{\infty} d^{4} y_{1} \ldots d^{4} y_{m}\langle 0| T\left(\phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right) \ldots \phi_{\text {in }}\left(x_{n}\right) \mathcal{H}_{I}\left(\phi_{\text {in }}\left(y_{1}\right)\right) \ldots \mathcal{H}_{I}\left(\phi_{\text {in }}\left(y_{s}\right)\right)\right)|0\rangle \\
& \times \sum_{s} \frac{(-i)^{r}}{r!} \int_{-\infty}^{\infty} d^{4} z_{1} \ldots d^{4} z_{r}\langle 0| T\left(\mathcal{H}_{I}\left(\phi_{\text {in }}\left(z_{1}\right)\right) \ldots \mathcal{H}_{I}\left(\phi_{\text {in }}\left(z_{r}\right)\right)\right)|0\rangle
\end{aligned}
$$

where $r=m-s$
It is not hard to see that

$$
\begin{aligned}
\tau\left(x_{1}, x_{2}, \ldots, x_{n}\right) & =\frac{\sum_{i} G_{i}\left(x_{1}, x_{2}, \ldots, x_{n}\right)}{\sum_{k} D_{k}}=\frac{\sum_{i} G_{i}^{C}\left(x_{1}, x_{2}, \ldots, x_{n}\right) \sum_{k} D_{k}}{\sum_{k} D_{k}} \\
& =\sum_{i} G_{i}^{C}\left(x_{1}, x_{2}, \ldots, x_{n}\right)
\end{aligned}
$$

where $G_{i}^{C}\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ :connected diagrams with n external lines
$D_{k}$ :closed loop diagrams
Hence in calculating $\tau$-function with n external lines, we can ignore all disconnected graphs.
Example $: \mathcal{H}_{I}=\frac{\lambda}{3!} \phi_{\text {in }}^{3}$

$$
\phi\left(q_{1}\right)+\phi\left(q_{2}\right) \longrightarrow \phi\left(p_{1}\right)+\phi\left(p_{2}\right)
$$

$$
\begin{aligned}
S_{\beta \alpha}= & \left.\langle\beta, \text { out }| \alpha, \text { in }\rangle=\left\langle p_{1}, p_{2}, \text { out }\right| q_{1}, q_{2}, \text { in }\right\rangle \\
= & \left(\frac{-i}{\sqrt{z}}\right)^{4} \int d^{4} x_{1} d^{4} x_{2} d^{4} y_{1} d^{4} y_{2} e^{i p_{1} y_{1}} e^{i p_{2} y_{2}}\left(\square_{y_{1}}+\mu^{2}\right)\left(\square_{y_{2}}+\mu^{2}\right) \\
& \times\langle 0| T\left(\phi_{\text {in }}\left(y_{1}\right) \phi_{\text {in }}\left(y_{2}\right) \phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right)\right)|0\rangle\left(\overleftarrow{\square_{x_{1}}}+\mu^{2}\right)\left(\overleftarrow{\square_{x 2}}+\mu^{2}\right) e^{-i q_{1} x_{1}} e^{-i q_{2} x_{2}} \\
= & \left(\frac{-i}{\sqrt{z}}\right)^{4} \int d^{4} x_{1} d^{4} x_{2} d^{4} y_{1} d^{4} y_{2}\left(\mu^{2}-p_{1}^{2}\right)\left(\mu^{2}-p_{2}^{2}\right)\left(\mu^{2}-q_{1}^{2}\right)\left(\mu^{2}-q_{2}^{2}\right) \\
& \times \tau\left(y_{1}, y_{2}, x_{1}, x_{2}\right) e^{i\left(p_{1} y_{1}+p_{2} y_{2}\right)} e^{-i\left(q_{1} x_{1}+q_{2} x_{2}\right)}
\end{aligned}
$$

Perturbation expansion of $\tau$-function
$\tau\left(y_{1}, y_{2}, x_{1}, x_{2}\right)=\sum_{n} \frac{(-i)^{n}}{n!} \int_{-\infty}^{\infty} d^{4} z_{1} \ldots d^{4} z_{n}\langle 0| T\left(\phi_{\text {in }}\left(y_{1}\right) \phi_{\text {in }}\left(y_{2}\right) \phi_{\text {in }}\left(x_{1}\right) \phi_{\text {in }}\left(x_{2}\right) \mathcal{H}_{I}\left(\phi_{i n}\left(z_{1}\right)\right) \ldots \mathcal{H}_{I}\left(\phi_{\text {in }}\left(z_{n}\right)\right)\right)|0\rangle$
Lowest order contribution
$\tau^{(2)}\left(y_{1}, y_{2}, x_{1}, x_{2}\right)=\frac{(-i)^{2}}{2!} \int_{-\infty}^{\infty} d^{4} z_{1} d^{4} z_{2}\langle 0| T\left(\phi_{i n}\left(y_{1}\right) \phi_{i n}\left(y_{2}\right) \phi_{i n}\left(x_{1}\right) \phi_{i n}\left(x_{2}\right)\left(\frac{\lambda}{3!} \phi_{i n}^{3}\left(z_{1}\right)\right)\left(\frac{\lambda}{3!} \phi_{i n}^{3}\left(z_{2}\right)\right)\right)|0\rangle$
Using Wick's theorem, we get for the connected diagrams.



Their contribution to $\tau\left(y_{1}, y_{2}, x_{1}, x_{2}\right)$ is

$$
\begin{aligned}
\tau^{(2)}\left(y_{1}, y_{2}, x_{1}, x_{2}\right)= & \frac{(-i \lambda)^{2}}{2!} \int_{-\infty}^{\infty} d^{4} z_{1} d^{4} z_{2} i \triangle_{F}\left(y_{1}-z_{1}\right) i \triangle_{F}\left(y_{2}-z_{1}\right) \\
& i \triangle_{F}\left(z_{2}-x_{1}\right) i \triangle_{F}\left(z_{2}-x_{2}\right) i \triangle_{F}\left(z_{1}-z_{2}\right)+\ldots
\end{aligned}
$$

use the propagator in momentum space

$$
i \triangle_{F}(x)=\int \frac{d^{4} k}{(2 \pi)^{4}} \frac{i}{k^{2}-\mu^{2}+i \varepsilon} e^{-i k \cdot x}
$$

Then

$$
\begin{aligned}
\tau^{(2)}\left(y_{1}, y_{2}, x_{1}, x_{2}\right)= & \frac{(-i \lambda)^{2}}{2!} \int_{-\infty}^{\infty} d^{4} z_{1} d^{4} z_{2} \int \frac{d^{4} k_{1}}{(2 \pi)^{4}} \int \frac{d^{4} k_{2}}{(2 \pi)^{4}} \ldots \int \frac{d^{4} k_{5}}{(2 \pi)^{4}} e^{-i k_{1} \cdot\left(y_{1}-z_{1}\right)} i \Delta_{F}\left(k_{1}\right) e^{-i k_{2} \cdot\left(z_{1}-x_{1}\right)} i \triangle_{F}\left(k_{2}\right) \\
& e^{-i k_{3} \cdot\left(z_{1}-z_{2}\right)} i \triangle_{F}\left(k_{3}\right) e^{-i k_{4} \cdot\left(y_{2}-z_{2}\right)} i \triangle_{F}\left(k_{4}\right) e^{-i k_{5} \cdot\left(z_{2}-x_{2}\right)} i \triangle_{F}\left(k_{5}\right)
\end{aligned}
$$

$$
\begin{aligned}
& z_{1} \text { integration } \int d^{4} z_{1} e^{i\left(k_{1}-k_{2}-k_{3}\right) \cdot z_{1}}=(2 \pi)^{4} \delta^{4}\left(k_{1}-k_{2}-k_{3}\right) \\
& z_{2} \text { integration } \int d^{4} z_{2} e^{i\left(k_{3}+k_{4}-k_{5}\right) \cdot z_{2}}=(2 \pi)^{4} \delta^{4}\left(k_{3}+k_{4}-k_{5}\right)
\end{aligned}
$$

energy-momentum conservation at each vertex
Then

$$
\begin{aligned}
\tau^{(2)}\left(y_{1}, y_{2}, x_{1}, x_{2}\right)= & \frac{(-i \lambda)^{2}}{2!} \int \frac{d^{4} k_{1}}{(2 \pi)^{4}} \cdots \frac{d^{4} k_{4}}{(2 \pi)^{4}}(2 \pi)^{4} \delta^{4}\left(k_{1}-k_{2}+k_{4}-k_{5}\right) \\
& i \triangle_{F}\left(k_{1}\right) i \triangle_{F}\left(k_{2}\right) i \triangle_{F}\left(k_{4}\right) i \triangle_{F}\left(k_{5}\right) i \triangle_{F}\left(k_{1}-k_{2}\right) e^{-i k_{1} \cdot y_{1}} e^{i k_{2} \cdot x_{1}} e^{-i k_{4} \cdot y_{2}} e^{i k_{5} \cdot x_{2}} \\
& \int \tau^{(2)}\left(y_{1}, y_{2}, x_{1}, x_{2}\right) d^{4} x_{1} d^{4} x_{2} d^{4} y_{1} d^{4} y_{2} e^{i\left(p_{1} y_{1}+p_{2} y_{2}\right)} e^{-i\left(q_{1} x_{1}+q_{2} x_{2}\right)} \\
\Longrightarrow & k_{2}=q_{1} \quad k_{5}=q_{2} \quad p_{1}=k_{1} \quad p_{2}=k_{4}
\end{aligned}
$$

We see that the external line propagators cancell out and

$$
S_{\beta \alpha}=\frac{(-i \lambda)^{2}}{2!}\left(\frac{1}{\sqrt{z}}\right)^{4}(2 \pi)^{4} \delta^{4}\left(p_{1}+p_{2}-q_{1}-q_{2}\right)+\ldots
$$

This is rather simple answer in momentum space.

## 5 Cross section and Decay rate

Write the S-matrix elements as

$$
S_{f i}=\delta_{f i}+i(2 \pi)^{4} \delta^{4}\left(p_{f}-p_{i}\right) T_{f i} \quad T_{f i}: \text { invariant amplitude for } i \rightarrow f
$$

For $i \neq f$, the transistion probability is

$$
\left|S_{f i}\right|^{2}=(2 \pi)^{4} \delta^{4}(0)\left[(2 \pi)^{4} \delta^{4}\left(p_{f}-p_{i}\right)\left|T_{f i}\right|^{2}\right]
$$

To interprete $\delta^{4}(0)$, we write

$$
(2 \pi)^{4} \delta^{4}\left(p_{f}-p_{i}\right)=\int d^{4} x e^{-i\left(p_{f}-p_{i}\right) x}
$$

The integration is over some large but finite volume V and time interval T .
Then we can interprete $\delta^{4}(0)$ as

$$
(2 \pi)^{4} \delta^{4}(0)=V T
$$

and write

$$
\left|S_{f i}\right|^{2}=V T\left[(2 \pi)^{4} \delta^{4}\left(p_{f}-p_{i}\right)\left|T_{f i}\right|^{2}\right.
$$

The transistion rate (transistion probability per unit time) is then

$$
\omega_{f i}=(2 \pi)^{4} \delta^{4}\left(p_{f}-p_{i}\right)\left|T_{f i}\right|^{2} V
$$

### 5.1 Decay rates

For a general decay processes with kinematics,

$$
a(p) \rightarrow c_{1}\left(k_{1}\right)+c_{2}\left(k_{2}\right)+\ldots .+c_{n}\left(k_{n}\right) \quad p_{f}=\sum_{l=1}^{n} k_{i} \quad p_{i}=p
$$

The number of states in the volume elements $d^{3} k_{1} d^{3} k_{2} \ldots d^{3} k_{n}$ in momentum space is

$$
\prod_{l=1}^{n} \frac{d^{3} k_{l}}{(2 \pi)^{3} 2 \omega_{k l}}
$$

The transition rate, summing over final states is

$$
d \omega^{\prime}=(2 \pi)^{4} \delta^{4}\left(p-\Sigma_{j=1}^{n} k_{j}\right)\left|T_{f i}\right|^{2} V \prod_{l=1}^{n} \frac{d^{3} k_{l}}{(2 \pi)^{3} 2 \omega_{k l}}
$$

For the invariant normalization of the physical states we've been using

$$
<p\left|p^{\prime}>=(2 \pi)^{3} \delta^{3}\left(\vec{p}-\vec{p}^{\prime}\right) 2 \omega_{p} \quad \Rightarrow \quad<p\right| p>=(2 \pi)^{3} \delta^{3}(0) 2 \omega_{p}=2 V \omega_{p}
$$

which is the number of particle in the initial state.
The decay rate per particle is then

$$
d \omega=\frac{d \omega^{\prime}}{2 V \omega_{p}}=(2 \pi)^{4} \delta^{4}\left(p-\Sigma_{j=1}^{n} k_{j}\right)\left|T_{f i}\right|^{2} \frac{1}{2 \omega_{p}} \prod_{l=1}^{n} \frac{d^{3} k_{l}}{(2 \pi)^{3} 2 \omega_{k l}}
$$

If there are " $m$ " identical particles in the final state, divide this by m !

$$
d \omega=\frac{1}{2 \omega_{p}}\left|T_{f i}\right|^{2} \frac{d^{3} k_{1}}{(2 \pi)^{3} 2 \omega_{1}} \cdots \frac{d^{3} k_{n}}{(2 \pi)^{3} 2 \omega_{n}}(2 \pi)^{4} \delta^{4}\left(p-\Sigma_{j=1}^{n} k_{j}\right) S \quad S=\prod_{j} \frac{1}{\left(m_{j}\right)!}
$$

### 5.2 Cross section

For a scattering processes of the form,

$$
a\left(p_{1}\right)+b\left(p_{2}\right) \rightarrow c_{1}\left(k_{1}\right)+c_{2}\left(k_{2}\right)+\ldots+c_{n}\left(k_{n}\right)
$$

the transition rate is given by, after summing over final states,

$$
d \omega^{\prime}=(2 \pi)^{4} \delta^{4}\left(p_{1}+p_{2}-\Sigma_{j=1}^{n} k_{j}\right)\left|T_{f i}\right|^{2} V \prod_{l=1}^{n} \frac{d^{3} k_{l}}{(2 \pi)^{3} 2 \omega_{k l}}
$$

We normalize this to one particle in the beam and one particle in the target and divide this by the flux $\sim$ relative velocity divided by the volume, to get differential cross section

$$
d \sigma=\frac{1}{2 \omega_{p_{1}} V} \frac{1}{2 \omega_{p_{2}} V}(2 \pi)^{4} \delta^{4}\left(p_{1}+p_{2}-\sum_{j=1}^{n} k_{j}\right)\left|T_{f i}\right|^{2} V \prod_{l=1}^{n} \frac{d^{3} k_{l}}{(2 \pi)^{3} 2 \omega_{k l}} \frac{V}{\left|\overrightarrow{v_{1}}-\overrightarrow{v_{2}}\right|}
$$

Velocity factor can be written as

$$
I=\left|\overrightarrow{v_{1}}-\overrightarrow{v_{2}}\right|=\left|\frac{\overrightarrow{p_{1}}}{E_{1}}-\frac{\overrightarrow{p_{2}}}{E_{2}}\right|
$$

In the C.M. frame $\overrightarrow{p_{1}}=-\overrightarrow{p_{2}}=\vec{p} \quad p_{1}=\left(E_{1}, \vec{p}\right), p_{2}=\left(E_{2},-\vec{p}\right)$

$$
\begin{gathered}
I=\frac{|\vec{p}|}{E_{1} E_{2}}\left(E_{1}+E_{2}\right) \\
\left(p_{1} \cdot p_{2}\right)^{2}=\left(E_{1} E_{2}+\vec{p}^{2}\right)^{2}=E_{1}^{2} E_{2}^{2}+2 E_{1} E_{2} \vec{p}^{2}+\vec{p}^{4} \\
\left(p_{1} \cdot p_{2}\right)^{2}-m_{1}^{2} m_{2}^{2}=\left(\vec{p}^{2}+m_{1}^{2}\right)\left(\vec{p}^{2}+m_{2}^{2}\right)+2 E_{1} E_{2} \vec{p}^{2}+\vec{p}^{4}-m_{1}^{2} m_{2}^{2} \\
=\vec{p}^{2}\left[2 \vec{p}^{2}+\left(m_{1}^{2}+m_{2}^{2}\right)+2 E_{1} E_{2}\right] \\
=\vec{p}^{2}\left(E_{1}+E_{2}\right)^{2} \\
\Rightarrow I=\frac{1}{E_{1} E_{2}} \sqrt{\left(p_{1} \cdot p_{2}\right)^{2}-m_{1}^{2} m_{2}^{2}} \\
d \sigma=\frac{1}{I} \frac{1}{2 \omega_{p_{1}}} \frac{1}{2 \omega_{p_{2}}}(2 \pi)^{4} \delta^{4}\left(p_{1}+p_{2}-\Sigma_{j=1}^{n} k_{j}\right)\left|T_{f i}\right|^{2} \prod_{l=1}^{n} \frac{d^{3} k_{l}}{(2 \pi)^{3} 2 \omega_{l}}
\end{gathered}
$$

## 6 Feynman Rules

Since the final forms for transition matrix elements $T_{f i}$ are quite simple, we can use simple rules to sidestep all those tedious intermediate steps.

Draw all connected Feynman graphs with appropriate external lines.Label each with momenta and impose momentum conservation for each vertex.
1.For each internal fermion line with momentum p,enter the propagator

$$
i S_{F}(p)=\frac{i}{p^{\mu} \gamma_{\mu}-m+i \varepsilon}
$$

2.For each internal boson line of spin 0 , with momentum q , enter the propagator

$$
i \triangle_{F}(q)=\frac{i}{q^{2}-\mu^{2}+i \varepsilon}
$$

3.For each internal photon line with momentum k , enter the propagator

$$
i D_{F}(k)_{\mu v}=\frac{-i g_{\mu \nu}}{k^{2}+i \varepsilon}
$$

4.For each internal momentum 1 not fixed by momentum conservation,enter

$$
\int \frac{d^{4} l}{(2 \pi)^{4}}
$$

5.For each closed fermion loop, enter ( -1 ) .Also they should be factor of $(-1)$ between graphs which differ only by an interchange of two external identical fermion lines.

At each vertex, the factors depend on the explicit form of interactions.
(a) $\frac{1}{3!} \lambda \phi^{3} \quad(-i \lambda)$
(b) $\frac{1}{4!} \lambda \phi^{4} \quad(-i \lambda)$
(c) $e \frac{4!}{\bar{\psi}} \gamma_{\mu} \psi A^{\mu} \quad\left(-i e \gamma_{\mu}\right)$
(d) $f \bar{\psi} \gamma_{5} \psi \phi \quad\left(-i f \gamma_{5}\right)$

### 6.1 Example in $\lambda \phi^{3}$ theory

In $\lambda \phi^{3}$ theory, consider scattering processes $\phi\left(k_{1}\right)+\phi\left(k_{2}\right) \longrightarrow \phi\left(k_{3}\right)+\phi\left(k_{4}\right)$
To second order in $\lambda$, we have following 3 Feynman diagrams for this reaction
We can write down the matrix element for each graph,

(a)

(b)

(c)
$T^{(a)}=(-i \lambda)^{2} \frac{i}{\left(k_{1}-k_{3}\right)^{2}-\mu^{2}} \quad T^{(b)}=(-i \lambda)^{2} \frac{i}{\left(k_{1}+k_{2}\right)^{2}-\mu^{2}} \quad T^{(c)}=(-i \lambda)^{2} \frac{i}{\left(k_{1}-k_{4}\right)^{2}-\mu^{2}}$
Total amplitude $T=T^{(a)}+T^{(b)}+T^{(c)}$
Mandelstam variables

$$
\begin{array}{ccc}
s=\left(k_{1}+k_{2}\right)^{2} & \text { total energy in c.m. frame } \\
t=\left(k_{1}-k_{3}\right)^{2} & \text { momentum transfer(scattering angle) } \\
u=\left(k_{1}-k_{4}\right)^{2} & \\
& & \\
& & \\
& &
\end{array}
$$

