Path Integral method

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Path integral formalism was originally developed to have close relationship to classical dynamics. For example, the transition amplitude in coordinate space is expressed in term of action S

$$\langle f|i\rangle = \int [dx]\, e^{iS/\hbar}$$

From this we can see that as $\hbar \to 0$, the trajectory with smallest action dominates, the action principle. This formalism uses the ordinary functions not the operators. Later in the study of non-Abelian gauge theory, the need for removing unphysical degrees of freedom can be more easily accomodated in the path integral formalism by imposing constraints in the integral.

1 Quantum Mechanics in 1-dimension

In the quantum mechanics, the transition matrix element from initial state $|q,t\rangle$ to final state $\langle q',t'|$, can be written as,

$$\langle q't'|qt\rangle = \langle q|'^{-iH(t-t')}|q\rangle$$

where $|q\rangle's$ are eigenstates of the position operator Q in the Schrodinger picture,

$$Q|q\rangle = q|q\rangle$$

and $|q,t\rangle$ denotes the corresponding state in Heisenberg picture,

$$|q,t\rangle = e^{iHt}|q\rangle$$

In the path integral formalism,this transition matrix element can be written as

$$\langle q't'|qt\rangle = N\int [dq] exp\{i\int_t^{t'} d\tau L(q,\dot{q})\}$$

We now explain how this formula come about and what this formula means. First divide the interval $(t\prime,t)$ into n intervals with spacing,

$$\delta t = \frac{t'-t}{n}$$

and write the transition matrix element as,

$$\langle q'|e^{-iH(t'-t)}|q\rangle = \int dq_1...dq_{n-1}\langle q'|e^{-iH\delta t}|q_{n-1}\rangle\langle q_{n-1}|e^{-iH\delta t}|q_{n-2}\rangle...\langle q_1|e^{-iH\delta t}|q\rangle$$

For δt small enough, we can approximate each of matrix elements as

$$\langle q'|e^{-iH\delta t}|q\rangle = \langle q'|(1-iH(P,Q)\delta t)|q\rangle + O\left((\delta t)^2\right) + ...$$

If we take the Hamiltonian in the simple form,

$$H(P,Q) = \frac{p^2}{2m} + V(Q)$$

then

$$\begin{aligned}
\langle q'|H|q\rangle &= \langle q'|\frac{p^2}{2m}|q\rangle + V(\frac{q+q'}{2})\delta(q-q') \\
&= \int \langle q'|\frac{p^2}{2m}|p\rangle\langle p|q\rangle(\frac{dp}{2\pi}) + V(\frac{q+q'}{2})\int \frac{dp}{2\pi}e^{ip(q'-q)} \\
&= \int \frac{dp}{2\pi}e^{ip(q'-q)}[\frac{p^2}{2m} + V(\frac{q+q'}{2})]
\end{aligned}$$

where we have used

$$\langle p|q\rangle = e^{-ipq}$$

which is the momentum eigenfunction in coordinate space. Exponentiation of this infinitesmal result gives

$$\langle q'|e^{-iH\delta t}|q\rangle \simeq \int \frac{dp}{2\pi} e^{ip(q'-q)}\{1 - i\delta t[\frac{p^2}{2m} + V(\frac{q+q'}{2})]\} \simeq \int \frac{dp}{2\pi} \exp\left[ip(q'-q)\right] \exp\left[-i\delta t[\frac{p^2}{2m} + V(\frac{q+q'}{2})]\right]$$

The whole transition matrix element can then be written as

$$\langle q'|e^{-iH(t'-t)}|q\rangle \cong \int (\frac{dp_1}{2\pi})...(\frac{dp_n}{2\pi}) \int dq_1...dq_{n-1} \exp\{i\left[\sum_{i=1}^{n} p_i(q_i - q_{i-1}) - (\delta t)H(p_i, \frac{q_i + q_{i+1}}{2})\right]\}$$

This can be written formally as

$$\langle q'|e^{-iH(t'-t)}|q\rangle = \int [\frac{dpdq}{2\pi}] exp\{i \int_t^{t'} dt[p\dot{q} - H(p,q)]\}$$

$$\equiv \lim_{n\to\infty} \int (\frac{dp_1}{2\pi})...(\frac{dp_n}{2\pi}) \int dq_1...dq_{n-1} exp\{i \sum_{i=1} \delta t[p_i(\frac{q_i - q_{i-1}}{\delta t}) - H(p_i, \frac{q_i + q_{i+1}}{2})]\}$$

In most case, Hamiltonian depends quadractically on p. We can use the formula

$$\int_{-\infty}^{+\infty} \frac{dx}{2\pi} e^{-ax^2+bx} = \frac{1}{\sqrt{4\pi a}} e^{\frac{b^2}{4a}}$$

to carry out the integration over momentum to get

$$\int \frac{dp_i}{2\pi} \exp[\frac{-i\delta t}{2m} p_i^2 + ip_i(q_i - q_{i-1})] = (\frac{m}{2\pi i \delta t})^{1/2} exp[\frac{im(q_i - q_{i-1})^2}{2\delta t}]$$

Then

$$\langle q'|e^{-iH(t'-t)}|q\rangle = \lim_{n\to\infty} (\frac{m}{2\pi i\delta t})^{n/2} \int \prod_{i=1}^{n-1} dq_i \exp\{i \sum_{i=1}^{n} \delta t[\frac{m}{2}(\frac{q_i - q_{i-1}}{\delta t})^2 - V]\}$$

or

$$\langle q't'|qt\rangle = \langle q'|e^{-iH(t'-t)}|q\rangle = N \int [dq] \exp\{i \int_t^{t'} d\tau[\frac{m}{2}\dot{q}^2 - V(q)]\}$$

This is the path integral representation for the probability amplitude from initial state $|q,t\rangle$ to final state $\langle q',t'|$. The combination in the exponential is just the action for this simple case and we get

$$\langle q't'|qt\rangle == N \int [dq] \exp iS$$

2 Green's functions

In order to generalize this formula to case of field theory where the basic entity is the vacuum expectation value of field operators, we consider the time-ordered product of the coordinate operators in Heisenberg picture between ground state $|0\rangle$,

$$G(t_1,t_2) = \langle 0|T(Q^H(t_1)Q^H(t_2))|0\rangle$$

Inserting complete sets of states, we get

$$G(t_1,t_2) = \int dqdq' \langle 0|q',t'\rangle\langle q',t'|T(Q^H(t_1)Q^H(t_2))|q,t\rangle\langle q,t|0\rangle$$

The matrix element

$$\langle 0|q,t\rangle = \phi_0(q)e^{-iE_0t} = \phi_0(q,t)$$

is the wavefunction for ground state. Consider the case

$$t' > t_1 > t_2 > t,$$

We can write

$$\langle q',t'|T(Q^H(t_1)Q^H(t_2))|q,t\rangle = \langle q'|e^{-iH(t'-t_1)}Q^s e^{-iH(t_1-t_2)}Q^s e^{-iH(t_2-t)}|q\rangle$$

$$= \int \langle q'|e^{-iH(t'-t_1)}|q_1\rangle q_1 \langle q_1|e^{-iH(t_1-t_2)}|q_2\rangle q_2 \langle q_2|e^{-iH(t_2-t)}|q\rangle dq_1 dq_2$$

$$= \int [\frac{dpdq}{2\pi}] q_1(t_1) q_2(t_2) exp\{i \int_t^{t'} d\tau [p\dot{q} - H(p,q)]\}$$

It is not hard to see that for the other time sequence

$$t' > t_2 > t_1 > t,$$

we get the same formula, because the path integral orders the time sequence automatically through the division of time interval into small pieces. The Green's function is then

$$G(t_1,t_2) = \int dqdq' \phi_0(q',t')\phi_0^*(q,t) \int [\frac{dpdq}{2\pi}] q_1(t_1) q_2(t_2) exp\{i \int_t^{t'} d\tau [p\dot{q} - H(p,q)]\} \tag{1}$$

We can remove the ground state wavefunction $\phi_0(q,t)$ by the following procedure. Write

$$\langle q',t'|\theta(t_1,t_2)|q,t\rangle = \int dQdQ' \langle q',t'|Q',T'\rangle\langle Q',T'|\theta(t_1,t_2)|Q,T\rangle\langle Q,t|q,t\rangle$$

where

$$\theta(t_1,t_2) = T(Q^H(t_1)Q^H(t_2))$$

Let $|n>$ be the energy eigenstate with energy E_n and wave function ϕ_n, i.e.,

$$H|n>= E_n|n>, \quad \langle q|n\rangle = \phi_n^*(q)$$

Then

$$\langle q',t'|Q',t'\rangle = \langle q'|e^{-iH(t'-T')}|Q'\rangle = \sum_n \langle q'|n\rangle e^{-iE_n(t'-T')}\langle n|Q'\rangle = \sum_n \phi_n^*(q')\phi_n(Q')e^{-iE_n(t'-T')}$$

To isolate the ground state wavefunction, we take an "unusual limit",

$$\lim_{t'\to -i\infty} \langle q',t'|Q',T'\rangle = \phi_0^*(q')\phi_0(Q')e^{-E_0|t'|}e^{iE_0T'}$$

Similarity,

$$\lim_{t\to i\infty} \langle Q,T|q,t\rangle = \phi_0(q)\phi_0^*(Q)e^{-E_0|t|}e^{-iE_0T}$$

With these we can write

$$\mathop{lim}_{\substack{t'\to -i\infty \\ t\to i\infty}} \langle q',t'|\theta(t_1,t_2)|q,t\rangle = \int dQdQ' \phi_0^*(q')\phi_0(Q')\langle Q',T'|\theta(t_1,t_2)|Q,T\rangle\phi_0^*(Q)\phi_0(q)e^{-E_0|t'|}e^{iE_0T'}e^{-iE_0T}e^{-E_0|t|}$$

$$= \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|}G(t_1,t_2)$$

It is easy to see that

$$\mathop{lim}_{\substack{t'\to -i\infty \\ t\to i\infty}} \langle q',t'|q,t\rangle = \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|}$$

Finally, the Green function can be written as,

$$G(t_1,t_2) = \mathop{lim}_{\substack{t'\to -i\infty \\ t\to i\infty}} \left[\frac{\langle q',t'|T(Q^H(t_1)Q^H(t_2))|q,t\rangle}{\langle q',t'|q,t\rangle}\right]$$

$$= \mathop{lim}_{\substack{t'\to -i\infty \\ t\to i\infty}} \frac{1}{\langle q',t'|q,t\rangle} \int \left[\frac{dpdq}{2\pi}\right] q(t_1)q(t_2)\exp\{i\int_t^{t'} d\tau[p\dot{q} - H(p,q)]\}$$

This can generalized to n-point Green's function with the result,

$$G(t_1,t_2,...,t_n) = \langle 0|T(q(t_1)q(t_2)...q(t_n))|0\rangle$$

$$= \mathop{lim}_{\substack{t'\to -i\infty \\ t\to i\infty}} \frac{1}{\langle q',t'|q,t\rangle} \int \left[\frac{dpdq}{2\pi}\right] q(t_1)q(t_2)...q(t_n)\exp\{i\int_t^{t'} d\tau[p\dot{q} - H(p,q)]\}$$

It is very useful to introduce generating functional for these n-point functions

$$W[J] = \lim_{\substack{t' \to -i\infty \\ t \to i\infty}} \frac{1}{\langle q', t' | q, t \rangle} \int \left[\frac{dpdq}{2\pi} \right] \exp\{ i \int_t^{t'} d\tau [p\dot{q} - H(p,q) + J(\tau) q(\tau)] \}$$

Then

$$G(t_1, t_2, ..., t_n) = (-i)^n \left. \frac{\delta^n}{\delta J(t_1) ... \delta J(t_n)} \right|_{J=0}$$

The unphysical limit, $t' \to -i\infty, t \to i\infty$, should be interpreted in term of Eudidean Green's functions defined by

$$S^{(n)}(\tau_1, \tau_2, ..., \tau_n) = i^n G^{(n)}(-i\tau_1, -i\tau_2, ..., -i\tau_n)$$

Generating functional for $S^{(n)}$ is then

$$W_E[J] = \lim_{\substack{\tau' \to \infty \\ \tau \to -\infty}} \int [dq] \frac{1}{\langle q', t' | q, t \rangle} \exp\{ \int_{\tau}^{\tau'} d\tau" [-\frac{m}{2} \left(\frac{dq}{d\tau"} \right)^2 - V(q) + J(\tau")q(\tau")] \}$$

Since we can adjust the zero point of $V(q)$ such that

$$\frac{m}{2} \left(\frac{dq}{d\tau} \right)^2 + V(q) > 0$$

which provides the damping to give a converging Gaussian integral. In this form, we can see that any constant in the path integral which is independent of q will be canceled out in the generation functional.

3 Field Theory

We can extend the treatment for quantum mechanics to field theory of a scalar field $\phi(x)$ with following replacements,

$$\prod_{i=1}^{\infty} [dq_i dp_i] \longrightarrow [d\phi(x) d\pi(x)]$$

$$L(q, \dot{q}) \longrightarrow \int \mathcal{L}(\phi, \partial_\mu \phi) d^3x \qquad H(p,q) \longrightarrow \int \mathcal{H}(\phi, \pi) d^3x$$

For example, the generating functional for scalar field is of the form

$$W[J] \backsim \int [d\phi][d\pi] \exp\{ i \int d^4x [\pi(x) \partial_0 \phi - \mathcal{H}(\pi, \phi) + J(x)\phi(x)] \}$$

$$\backsim \int [d\phi] \exp\{ i \int d^4x [\mathcal{L}(\phi, \partial_\mu \phi) + J(x)\phi(x)] \}$$

Note that the functional derivative is defined by

$$\frac{\delta F[\phi(x)]}{\delta \phi(y)} = \lim_{\varepsilon \to 0} \frac{F[\phi(x) + \varepsilon \delta(x-y)] - F[\phi(x)]}{\varepsilon}$$

Then we see that

$$\frac{\delta W[J]}{\delta J(y)} = i \int [d\phi] \phi(y) \exp\{ i \int d^4x [\mathcal{L}(\phi, \partial_\mu \phi) + J(x)\phi(x)] \} \tag{2}$$

and

$$\frac{\delta^2 W[J]}{\delta J(y_1) \delta J(y_2)} = (i)^2 \int [d\phi] \phi(y_1) \phi(y_2) \exp\{ i \int d^4x [\mathcal{L}(\phi, \partial_\mu \phi) + J(x)\phi(x)] \}$$

Consider the example of $\lambda\phi^4$ theory

$$\mathcal{L}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_1(\phi)$$

$$\mathcal{L}_0(\phi) = \frac{1}{2}(\partial_\lambda \phi)^2 - \frac{\mu^2}{2}\phi^2, \qquad \mathcal{L}_1(\phi) = -\frac{\lambda}{4!}\phi^4$$

For conveience we use Euclidean time to carry the computations. The generating functional

$$W\left[J\right]=\int\left[d\phi\right]\exp\{-\int d^{4}x[\frac{1}{2}(\frac{\partial\phi}{\partial\tau})^{2}+\frac{1}{2}(\overrightarrow{\bigtriangledown}\phi)^{2}+\frac{\mu^{2}}{2}\phi^{2}+\frac{\lambda}{4!}\phi^{4}-J\phi]\}$$

can be written as

$$W\left[J\right]=\left[\exp\int d^{4}x\mathcal{L}_{I}\left(\frac{\delta}{\delta J\left(x\right)}\right)\right]W_{0}\left[J\right]$$

where

$$W_{0}\left[J\right]=\int\left[d\phi\right]\exp[-\frac{1}{2}\int d^{4}xd^{4}y\phi(x)K(x,y)\phi(y)+\int d^{4}zJ(z)\phi(z)]$$

and

$$K(x,y)=\delta^{4}(x-y)\left(-\frac{\partial^{2}}{\partial\tau^{2}}-\overrightarrow{\bigtriangledown}^{2}+\mu^{2}\right)$$

We have used Eq(2) to write the interaction term in terms of function derivative with repect to the source $J\left(x\right)$. The Gaussian integral for many variables is

$$\int d\phi_{1}d\phi_{2}...d\phi_{n}\exp\left[-\frac{1}{2}\sum_{i,j}\phi_{i}K_{ij}\phi_{j}+\sum_{k}J_{k}\phi_{k}\right]\backsim\frac{1}{\sqrt{detK}}\exp\left[\frac{1}{2}\sum_{i,j}J_{i}(K^{-1})_{ij}J_{j}\right]$$

Apply this to the case of scalar fields,

$$W_{0}\left[J\right]=\exp\left[\frac{1}{2}\int d^{4}xd^{4}yJ(x)\bigtriangleup(x,y)J(y)\right]$$

where

$$\int d^{4}yK(x,y)\bigtriangleup(y,z)=\delta^{4}\left(x-z\right)$$

It is not difficult to see that

$$\bigtriangleup(x,y)=\int\frac{d^{4}k_{E}}{\left(2\pi\right)^{4}}\frac{e^{ik_{E}(x-y)}}{k_{E}^{2}+\mu^{2}}$$

where $k_{E}=(ik_{0},\overrightarrow{k})$,the Euclidean momentum

Perturbative expansion in power of λ gives

$$W\left[J\right]=W_{0}\left[J\right]\{1+\lambda w_{1}\left[J\right]+\lambda^{2}w_{2}\left[J\right]+...\}$$

where

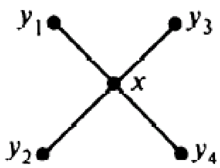
$$w_{1}=-\frac{1}{4!}W_{0}^{-1}\left[J\right]\{\int d^{4}x\left[\frac{\delta}{\delta J(x)}\right]^{4}\}W_{0}\left[J\right]$$

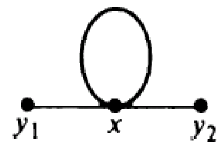
$$w_{2}=-\frac{1}{2\left(4!\right)^{2}}W_{0}^{-1}\left[J\right]\{\int d^{4}x\left[\frac{\delta}{\delta J(x)}\right]^{4}\}^{2}W_{0}\left[J\right]$$

Use the explicit form for $W_0[J]$,

$$\begin{aligned}W_{0}\left[J\right] &= 1+\frac{1}{2}\int d^{4}xd^{4}yJ(x)\bigtriangleup(x,y)J(y)+\\ &\left(\frac{1}{2}\right)^{2}\frac{1}{2!}\int d^{4}y_{1}d^{4}y_{2}d^{4}y_{3}d^{4}y_{4}\left[J(y_{1})\bigtriangleup(y_{1},y_{2})J(y_{2})J(y_{3})\bigtriangleup(y_{3},y_{4})J(y_{4})\right]+...\end{aligned}$$

we can compute w_1 as follows





$$w_1 = -\frac{1}{4!}\left[\int \triangle(x,y_1)\triangle(x,y_2)\triangle(x,y_3)\triangle(x,y_4)J(y_1)J(y_2)J(y_3)J(y_4) + 3!\triangle(x,y_1)\triangle(x,y_2)J(y_1)J(y_2)\triangle(x,x)\right]$$

where we have dropped all J independent terms, and all arguments (x_i, y_i) are integrated over. In this computation we have used the identity,

$$\frac{\delta}{\delta J(x)}\int d^4y_1 J(y_1) f(y_1) = \int \delta^4(x-y_1)\, d^4y_1 f(y_1) = f(x)$$

graphical representation for w_1

The connected Green's function is

$$G^{(n)}(x_1, x_2, ...x_n) = \frac{\delta^n \ln W[J]}{\delta J(x_1)\delta J(x_2)...\delta J(x_n)}|_{J=0}$$

Thus replacing y_i by external x_i, we get contributions for 4-point,2-point functions,

4 Grassmann algebra

For the quantization of fermion fields, using path integral, we need to integrate over anti-commuting c-number functions.This can be realized as elements of Grassmann algebra. We now give a simple introduction to this anti-commuting algebra.

In an n-dimensional Grassmann algebra,the n generators θ_1, θ_2, θ_3, ..., θ_n satisfy the anti-commutation relations,

$$\{\theta_i, \theta_j\} = 0 \qquad i, j = 1, 2, ..., n$$

and every element can be expanded in a finite series,

$$P(\theta) = P_0 + P^{(1)}_{i_1}\theta_{i_1} + P^{(2)}_{i_1 i_2}\theta_{i_1}\theta_{i_2} + ... + P^{(n)}_{i_1...i_n}\theta_{i_1}...\theta_{i_n}$$

Simplest case:n=1

$$\{\theta, \theta\} = 0 \qquad \text{or} \qquad \theta^2 = 0 \qquad P(\theta) = P_0 + \theta P_1$$

We can define the "differentiation" and "integration" as follows,

$$\frac{d}{d\theta}\theta = \theta\frac{\overleftarrow{d}}{d\theta} = 1 \quad \Longrightarrow \frac{d}{d\theta}P(\theta) = P_1$$

Integration is defined by translational invariant,

$$\int d\theta P(\theta) = \int d\theta P(\theta + \alpha)$$

where α is another Grassmann variable. This implies

$$\int d\theta = 0$$

Normalize the integral such that

$$\int d\theta\theta = 1$$

Then

$$\int d\theta P(\theta) = P_1 = \frac{d}{d\theta} P(\theta)$$

Consider a change of variable

$$\theta \to \widetilde{\theta} = a + b\theta$$

Since

$$\int d\widetilde{\theta} P\left(\widetilde{\theta}\right) = \frac{d}{d\widetilde{\theta}} P\left(\widetilde{\theta}\right) = P_1$$

$$\int d\theta P\left(\widetilde{\theta}\right) = \int d\theta \left[P_0 + \widetilde{\theta} P_1\right] = \int d\theta \left[P_0 + (a + b\theta) P_1\right] = bP_1$$

we get

$$\int d\widetilde{\theta} P\left(\widetilde{\theta}\right) = \int d\theta \left(\frac{d\widetilde{\theta}}{d\theta}\right)^{-1} P\left(\widetilde{\theta}(\theta)\right)$$

Thus the "Jacobian" is the inverse of that for c-number integration.

It is easy to generalize to the case of n-dimensional Grassmann algebra,

$$\frac{d}{d\theta_i}(\theta_1, \theta_2, \theta_3, ..., \theta_n) = \delta_{i_1}\theta_2...\theta_n - \delta_{i_2}\theta_1\theta_3...\theta_n + ... + (-1)^{n-1} \delta_{in}\theta_1\theta_2...\theta_{n-1}$$

$$\{d\theta_i, d\theta_j\} = 0$$

$$\int d\theta_i = 0 \qquad \int d\theta_i \theta_j = \delta_{ij}$$

For a change of variables of the form

$$\widetilde{\theta_i} = b_{ij}\theta_j$$

we have

$$\int d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1 P\left(\widetilde{\theta}\right) = \int d\theta_n...d\theta_1 \left[det\frac{d\widetilde{\theta}}{d\theta}\right]^{-1} P\left(\widetilde{\theta}(\theta)\right)$$

Proof:

$$\widetilde{\theta_1}\widetilde{\theta_2}...\widetilde{\theta_n} = b_{1i_1}b_{2i_2}...b_{ni_n}\theta_{i_1}...\theta_{i_n}$$

RHS is non-zero only if $i_1, i_2..., i_n$ are all different and we can write

$$\begin{aligned}
\widetilde{\theta}_1\widetilde{\theta}_2...\widetilde{\theta}_n &= b_{1i_1}b_{2i_2}...b_{ni_n}\epsilon_{i_1,i_2...,i_n}\theta_{i_1}...\theta_{i_n} \\
&= (\det b)\,\theta_1\theta_2\theta_3...\theta_n
\end{aligned}$$

From the normalization condition,

$$1 = \int d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1 \left(\widetilde{\theta}_1\widetilde{\theta}_2...\widetilde{\theta}_n\right) = (\det b) \int d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1 (\theta_1\theta_2\theta_3...\theta_n)$$

we see that

$$d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1 = (\det b)^{-1} d\theta_1...d\theta_n$$

In field theory,we need to make use of Gaussian integral,

$$G(A) \equiv \int d\theta_n...d\theta_1 \exp\left(\frac{1}{2}(\theta, A\theta)\right) \qquad \text{where } (\theta, A\theta) = \theta_i A_{ij}\theta_j$$

First consider the simple case of n=2, where

$$A = \left(\begin{array}{cc} 0 & A_{12} \\ -A_{12} & 0 \end{array} \right)$$

Then

$$G(A) = \int d\theta_2 d\theta_1 \exp(\theta_1\theta_2 A_{12}) \simeq \int d\theta_2 d\theta_1 (1 + \theta_1\theta_2 A_{12}) = A_{12} = \sqrt{\det A}$$

The generalization to arbitrary n is

$$G(A) = \int d\theta_n ... d\theta_1 \exp\left(\frac{1}{2}(\theta, A\theta)\right) = \sqrt{\det A} \qquad \text{n even}$$

and for "complex" Grassmann variables

$$\int d\theta_n d\overline{\theta_n} d\theta_{n-1} d\overline{\theta}_{n-1} ... d\theta_1 d\overline{\theta_1} exp\left(\overline{\theta}, A\theta\right) = \det A$$

For the Fermion fields, the generating functional is of the form,

$$W[\eta, \overline{\eta}] = \int [d\psi(x)] \left[d\overline{\psi}(x)\right] \exp\{i \int d^4x \left[\mathcal{L}\left(\psi, \overline{\psi}\right) + \overline{\psi}\eta + \overline{\eta}\psi\right]\}$$

It is not hard to see that if $\mathcal{L}$ depends on $\psi, \overline{\psi}$ quadratically

$$\mathcal{L} = \left(\overline{\psi}, A\psi\right)$$

then we have

$$W = \int [d\psi(x)] \left[d\overline{\psi}(x)\right] exp\{\int d^4x \overline{\psi} A\psi\} = \det A$$