Note 6

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Contents

1 Quantum Electrodynamics

The Lagrangian density for QED is of the form,

$$\mathcal{L} = \overline{\psi}(x)\gamma^\mu(i\partial_\mu - eA_\mu)\psi(x) - m\overline{\psi}(x)\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Equations of motion are

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)\psi(x) &= eA_\mu\gamma^\mu\psi \qquad \text{non-linear coupled equations}\\ \partial_v F^{\mu\nu} &= e\overline{\psi}\gamma^\mu\psi\end{aligned}$$

<u>Quantization</u>
Write $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

$$\begin{aligned}\mathcal{L}_0 &= \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\ \mathcal{L}_{int} &= -e\overline{\psi}\gamma^\mu\psi A_\mu\end{aligned}$$

where $\mathcal{L}_0$, contains only the quadratic part and are the free field Lagrangian we studied before while $\mathcal{L}_{int}$ is the part describing interaction.

Conjugate momenta for the fermion field is

$$\frac{\partial\mathcal{L}}{\partial(\partial_0\psi_\alpha)} = i\psi_\alpha^\dagger(x)$$

For electromagnetic fields, we choose the gauge

$$\overrightarrow{\nabla}\cdot\overrightarrow{A} = 0$$

then

$$\pi^i = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^i)} = -F^{0i} = E^i$$

From equation of motion

$$\partial_\nu F^{0\nu} = e\psi^\dagger\psi \qquad \Longrightarrow \qquad -\nabla^2 A^0 = e\psi^\dagger\psi$$

Thus A^0 is not zero but it is not an independent dynamical variable and can be expressed in terms of other field,

$$A^0 = e\int d^3x' \frac{\psi^\dagger(x',t)\,\psi(x',t)}{4\pi|\overrightarrow{x'}-\overrightarrow{x}|} = e\int \frac{d^3x'\,\rho(x',t)}{|\overrightarrow{x}-\overrightarrow{x'}|}$$

Commutation relation

$$\begin{aligned}\{\psi_\alpha(\overrightarrow{x},t), \psi^\dagger_\beta(\overrightarrow{x'},t)\} &= \delta_{\alpha\beta}\delta^3(\overrightarrow{x}-\overrightarrow{x'}) \qquad \{\psi_\alpha(\overrightarrow{x},t), \psi_\beta(\overrightarrow{x'},t)\} = ... = 0 \\ \left[\dot{A}_i(\overrightarrow{x},t), A_j(\overrightarrow{x'},t)\right] &= i\delta^{tr}_{ij}(\overrightarrow{x}-\overrightarrow{x'})\end{aligned}$$

where

$$\delta^{tr}_{ij}(\vec{x}-\vec{y}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}(\delta_{ij} - \frac{k_ik_j}{k^2})$$

Commutators involving A_0 can be worked out as follows,

$$\left[A_0(\overrightarrow{x},t), \psi_\alpha(\overrightarrow{x'},t)\right] = e\int \frac{d^3x''}{4\pi|\overrightarrow{x}-\overrightarrow{x''}|}\left[\psi^\dagger(\overrightarrow{x''},t)\,\psi(\overrightarrow{x''},t), \psi_\alpha(\overrightarrow{x'},t)\right] = -\frac{e}{4\pi}\frac{\psi_\alpha(\overrightarrow{x'},t)}{|\overrightarrow{x}-\overrightarrow{x'}|}$$

Hamiltonian

$$\begin{aligned}\mathcal{H} &= \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_\alpha)}\dot{\psi}_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_0 A^k)}\dot{A}_k - \mathcal{L} \\ &= \psi^\dagger\left(-i\overrightarrow{\alpha}\cdot\overrightarrow{\nabla} + \beta m\right)\psi + \frac{1}{2}\left(\overrightarrow{E}^2 + \overrightarrow{B}^2\right) + \overrightarrow{E}\cdot\overrightarrow{\nabla}A_0 + e\overline{\psi}\gamma^\mu\psi A_\mu\end{aligned}$$

But in the Hamiltonian, upon integration by part, the term $\overrightarrow{E}\cdot\overrightarrow{\nabla}A_0$ cancels the time component of the last term, $e\overline{\psi}\gamma^0\psi A_0$

$$H = \int d^3x\mathcal{H} = \int d^3x\{\psi^\dagger\left[\overrightarrow{\alpha}\cdot(-i\overrightarrow{\nabla} - e\overrightarrow{A}) + \beta m\right]\psi + \frac{1}{2}\left(\overrightarrow{E}^2 + \overrightarrow{B}^2\right)\}$$

Thus A_0 does not appear in the interaction. But if we write

$$\overrightarrow{E} = \overrightarrow{E_l} + \overrightarrow{E_t} \qquad \text{where} \quad \overrightarrow{E_l} = -\overrightarrow{\nabla}A_0 \qquad , \overrightarrow{E_t} = -\frac{\partial\overrightarrow{A}}{\partial t}$$

Then

$$\frac{1}{2}\int d^3x\left(\overrightarrow{E}^2 + \overrightarrow{B}^2\right) = \frac{1}{2}\int d^3x\overrightarrow{E_l}^2 + \int d^3x\left(\overrightarrow{E_t}^2 + \overrightarrow{B}^2\right)$$

The longitudinal part is

$$\frac{1}{2}\int d^3x\overrightarrow{E_l}^2 = \frac{e^2}{4\pi}\int d^3xd^3y\frac{\rho(\overrightarrow{x},t)\,\rho(\overrightarrow{y},t)}{|\overrightarrow{x}-\overrightarrow{y}|} \qquad \text{Coulomb interaction}$$

Then we can write the Hamiltonian as

$$H = \int d^3x\{\psi^\dagger\left[\overrightarrow{\alpha}\cdot(-i\overrightarrow{\nabla} - e\overrightarrow{A}) + \beta m\right]\psi + \frac{1}{2}\left(\overrightarrow{E_t}^2 + \overrightarrow{B}^2\right) + \frac{e^2}{4\pi}\int d^3xd^3y\frac{\rho(\overrightarrow{x},t)\,\rho(\overrightarrow{y},t)}{|\overrightarrow{x}-\overrightarrow{y}|}$$

Note that the electromagnetic fields in the second term are transverse as radiation fields should be. The last term is just the static Coulomb interaction.

Even though we can set up the commutators or anti-commutators for quantization, it is difficult, if not impossible, to find the physical consequences.This is because we do not know how to solve the classical equations of motion which is highly non-linear. Without the classical solutions we can not carryout the mode expansion to introduce the creation and annihilation operators and it is difficult to find the eigenvalues and eigenstates of the Hamiltonian. The only approximation we know how to do in field theory is the perturbation theory. We will now set up the framework for the perturbation.

Recall that the free field part $\overrightarrow{A}_0$ satisfy massless Klein-Gordon equation

$$\Box \overrightarrow{A}^{(0)} = 0$$

The general solution is

$$\overrightarrow{A}^{(0)}(\vec{x},t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_{\lambda} \overrightarrow{\epsilon}(k,\lambda)[a(k,\lambda)e^{-ikx} + a^{+}(k,\lambda)e^{ikx}] \qquad w = k_0 = |\overrightarrow{k}|$$

$$\vec{\epsilon}(k,\lambda), \ \ \lambda = 1,2 \quad \text{with } \vec{k}\cdot\vec{\epsilon}(k,\lambda) = 0$$

Standard choice

$$\vec{\epsilon}(k,\lambda)\cdot\vec{\epsilon}(k,\lambda') = \delta_{\lambda\lambda'}, \quad \vec{\epsilon}(-k,1) = -\vec{\epsilon}(k,1), \quad \vec{\epsilon}(-k,2) = \vec{\epsilon}(-k,2)$$

It is convienent to write the mode expansion as,

$$A_\mu(\vec{x},t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_{\lambda} \epsilon_\mu(k,\lambda)[a(k,\lambda)e^{-ikx} + a^{+}(k,\lambda)e^{ikx}]$$

where

$$\epsilon_\mu(k,\lambda) = (0, \vec{\epsilon}(k,\lambda))$$

1.1 Photon Propagator

As the scalar field, the Feynman propagatpr for the photon is defined as

$$\begin{aligned} iD_{\mu\nu}(x,x') &= \langle 0 | T(A_\mu(x)A_\nu(x')) | 0 \rangle \\ &= \theta(t-t')\langle 0|A_\mu(x)A_\nu(x')|0\rangle + \theta(t'-t)\langle 0|A_\nu(x')A_\mu(x)|0\rangle \end{aligned}$$

Using the mode expansion, we see that

$$D_{\mu\nu}(x,x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x'-x)}}{k^2 + i\varepsilon} \sum_{\lambda=1}^{2} \epsilon_\nu(k,\lambda)\epsilon_\mu(k,\lambda)$$

The polarization vectors $\epsilon_\mu(k,\lambda)$, $\lambda = 1,2$ are perpendicular to each other. We need 2 more unit vectors to form a complete set of vectors in Minkowski space. It is not are hard to see that they are

$$\eta^\mu = (1,0,0,0), \qquad \hat{k}^\mu = \frac{k^\mu - (k\cdot\eta)\eta^\mu}{\sqrt{(k\cdot\eta)^2 - k^2}}$$

and the completeness relation is of the form,

$$\begin{aligned} \sum_{\lambda=1}^{2} \epsilon_\nu(k,\lambda)\epsilon_\mu(k,\lambda) &= -g_{\mu\nu} - \eta_\mu\eta_\nu - \hat{k}_\mu\hat{k}_\nu \\ &= -g_{\mu\nu} - \frac{k_\mu k_\nu}{(k\cdot\eta)^2 - k^2} + \frac{(k\cdot\eta)(k_\mu\eta_\nu + \eta_\mu k_\nu)}{(k\cdot\eta)^2 - k^2} - \frac{k^2\eta_\mu\eta_\nu}{(k\cdot\eta)^2 - k^2} \end{aligned}$$

If we define propagator in momentum space as

$$D_{\mu\nu}\left(x,x'\right)=\int\frac{d^4k}{\left(2\pi\right)^4}e^{-ik\cdot\left(x'-x\right)}D_{\mu\nu}\left(k\right)$$

then

$$D_{\mu\nu}\left(k\right)=\frac{1}{k^2+i\varepsilon}\left[-g_{\mu\nu}-\frac{k_\mu k_\nu}{\left(k\cdot\eta\right)^2-k^2}+\frac{\left(k\cdot\eta\right)\left(k_\mu\eta_\nu+\eta_\mu k_\nu\right)}{\left(k\cdot\eta\right)^2-k^2}-\frac{k^2\eta_\mu\eta_\nu}{\left(k\cdot\eta\right)^2-k^2}\right]$$

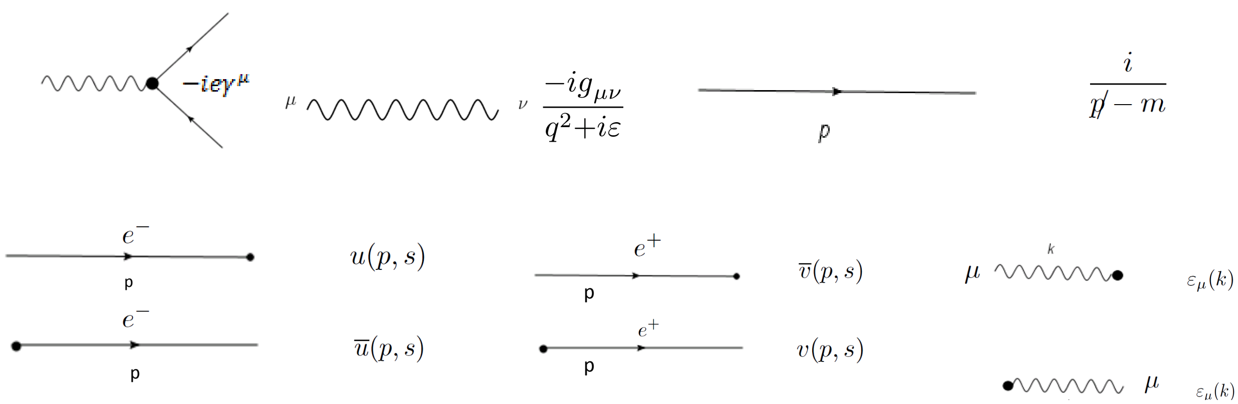
It turns out that the terms proportional to k_μ will not contribute to any physical processes becuse of current conservation and the last term is of the form $\delta_{\mu0}\,\delta_{\nu0}$ will be cancelled by the Coulom interaction..

1.2 Feynman rule in QED

The interaction Hamiltonian is of the form,

$$H_{int}=e\int d^3x\bar{\psi}\gamma^\mu\psi A_\mu$$

The Feynman propagators, vertices and external wave functions are given below.

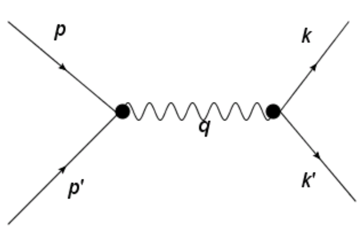


2 $e^+e^-\rightarrow\mu^+\mu^-$

2.1 Total Cross Section

First we label the momenta for this reaction as

$$e^+\left(p'\right)+e^-\left(p\right)\rightarrow\mu^+\left(k'\right)+\mu^-\left(k\right)$$



From the Feynman rule we can write the matrix element as

$$M(e^+e^- \rightarrow \mu^+\mu^-) = \bar{v}(p',s')\left(-ie\gamma^\mu\right)u\left(p,s\right)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(k',r')\left(-ie\gamma^\nu\right)v\left(k,r\right)$$
$$= \frac{ie^2}{q^2}\bar{v}(p',s')\gamma^\mu u\left(p,s\right)\bar{u}(k',r')\gamma_\mu v\left(k,r\right)$$

where $q = p + p'$. Notet that the electron vertex will have the property,

$$q_\mu \bar{v}(p')\gamma^\mu u\left(p\right) = \left(p+p'\right)_\mu \bar{v}(p')\gamma^\mu u\left(p\right) = \bar{v}(p')\left(\not{p}+\not{p}'\right)u\left(p\right) = 0$$

This illustrates the assertion that the term proportional to photon momentum will not contribute in the physical processes. To compute the cross section, we need to write out M^* which contains the factor $\left(\bar{v}\gamma^\mu u\right)^*$ and can be simplified as

$$\left(\bar{v}\gamma^\mu u\right)^* = u^\dagger\left(\gamma^\mu\right)^\dagger\left(\gamma_0\right)^\dagger v = u^\dagger\gamma_0\gamma^\mu v = \bar{u}\gamma^\mu v$$

More generally we write

$$\left(\bar{v}\Gamma u\right)^* = \bar{u}\bar{\Gamma}v, \qquad \text{with } \bar{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0$$

It is easy to see that

$$\bar{\gamma}_\mu = \gamma_\mu$$
$$\overline{\gamma_\mu\gamma_5} = -\gamma_\mu\gamma_5$$
$$\overline{\not{a}\not{b}\cdots\not{p}} = \not{p}\cdots\not{b}\not{a}$$

Usually we are interested in the unpolarized cross section which requires the spin sum,

$$\sum_s u_\alpha\left(p,s\right)\bar{u}_\beta\left(p,s\right) = \left(\not{p}+m\right)_{\alpha\beta}$$

$$\sum_s v_\alpha\left(p,s\right)\bar{v}_\beta\left(p,s\right) = \left(\not{p}-m\right)_{\alpha\beta}$$

A typical calculation is shown below,

$$\sum_{s,s'}\bar{v}_\alpha(p',s')\left(\gamma^\mu\right)_{\alpha\beta}u_\beta\left(p,s\right)\bar{u}_\rho(p,s)\left(\gamma^\nu\right)_{\rho\sigma}v_\sigma\left(p,s\right)$$
$$= \sum_{s'}\bar{v}_\alpha(p',s')\left(\gamma^\mu\right)_{\alpha\beta}\left(\not{p}+m\right)_{\beta\rho}\left(\gamma^\nu\right)_{\rho\sigma}v_\sigma\left(p,s\right)$$
$$= \left(\gamma^\mu\right)_{\alpha\beta}\left(\not{p}+m\right)_{\beta\rho}\left(\gamma^\nu\right)_{\rho\sigma}\left(\not{p}-m\right)_{\sigma\alpha}$$
$$= Tr\left[\gamma^\mu\left(\not{p}+m\right)\gamma^\nu\left(\not{p}-m\right)\right]$$

Now we need to compute the trace of product of γ matrices. The calculations are straightforward and we will just quote the results,

$$Tr\left(\gamma^\mu\right) = 0$$
$$Tr\left(\gamma^\mu\gamma^\nu\right) = 4g^{\mu\nu}$$
$$Tr\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = 4\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}\right)$$

$$Tr\left(\not{a}_1\not{a}_2\cdots\not{a}_n\right) = \left(a_1\cdot a_2\right)Tr\left(\not{a}_3\cdots\not{a}_n\right) - \left(a_1\cdot a_3\right)Tr\left(\not{a}_2\cdots\not{a}_n\right) + \cdots + \left(a_1\cdot a_n\right)Tr\left(\not{a}_2\not{a}_3\cdots\not{a}_{n-1}\right), \qquad n \text{ even}$$
$$= 0 \qquad n \text{ odd}$$

With these tools we get

$$\frac{1}{4}\sum_{spin}\left|M(e^+e^-\to\mu^+\mu^-)\right|^2=\frac{e^4}{q^4}Tr\left[\left(\not{p}'-m_e\right)\gamma^\mu\left(\not{p}+m_e\right)\gamma^\nu\right]Tr\left[\left(\not{k}'+m_\mu\right)\gamma_\mu\left(\not{k}+m_\mu\right)\gamma^\nu\right]$$

$$\begin{aligned}Tr\left[\left(\not{p}'-m_e\right)\gamma^\mu\left(\not{p}+m_e\right)\gamma^\nu\right] &= Tr\left[\not{p}'\gamma^\mu\not{p}\gamma^\nu\right]-m^2Tr\left[\gamma^\mu\gamma^\nu\right]\\ &= 4\left[p'^\mu p^\nu-g^{\mu\nu}\left(p\cdot p'\right)+p^\mu p'^\nu\right]-4m_e^2g^{\mu\nu}\end{aligned}$$

$$\begin{aligned}Tr\left[\left(\not{k}'+m_\mu\right)\gamma_\mu\left(\not{k}-m_\mu\right)\gamma^\nu\right] &= Tr\left[\not{k}'\gamma_\mu\not{k}\gamma^\nu\right]-m_\mu^2Tr\left[\gamma_\mu\gamma^\nu\right]\\ &= 4\left[k'^\mu k^\nu-g^{\mu\nu}\left(k\cdot k'\right)+k^\mu k'^\nu\right]-4m_\mu^2g^{\mu\nu}\end{aligned}$$

Usually, we are interested in this reaction for energies much higher than m_μ. Then

$$\frac{1}{4}\sum_{spin'}\left|M(e^+e^-\to\mu^+\mu^-)\right|^2=8\frac{e^4}{q^4}\left[(p\cdot k)\left(p'\cdot k'\right)+(p'\cdot k)\left(p\cdot k'\right)\right]$$

In the center of mass frame, we write the momenta as

$$p_\mu=(E,0,0,E)\,,\qquad p'_\mu=(E,0,0,-E)$$

$$k_\mu=\left(E,\vec{k}\right),\qquad k'_\mu=\left(E,-\vec{k}\right),\qquad \text{with }\vec{k}\cdot\hat{z}=\left|\vec{k}\right|\cos\theta$$

In the approximation $m_\mu=0$, we have $E=\left|\vec{k}\right|$ and

$$q^2=\left(p+p'\right)^2=4E^2,\qquad p\cdot k=p'\cdot k'=E^2\left(1-\cos\theta\right),$$

$$p'\cdot k=p\cdot k'=E^2\left(1+\cos\theta\right)$$

Then

$$\begin{aligned}\frac{1}{4}\sum_{spin'}|M|^2 &= \frac{8e^4}{16E^4}\left[E^4\left(1-\cos\theta\right)^2+E^4\left(1+\cos\theta\right)^2\right]\\ &= e^4\left(1+\cos^2\theta\right)\end{aligned}$$

Note that under the parity $\theta\to\pi-\theta$. We see that this matrix element conserves the parity. Recall that the cross section for this processes is related to the amplitude as

$$d\sigma=\frac{1}{I}\frac{1}{2E}\frac{1}{2E}(2\pi)^4\delta^4(p+p'-k-k')\frac{1}{4}\sum_{spin'}|M|^2\frac{d^3k}{(2\pi)^32\omega}\frac{d^3k'}{(2\pi)^32\omega'}$$

We can the $\delta-$function to carry out some of the integrations as follows. For convience we introduce the quantity ρ, called the phase space, given by

$$\begin{aligned}\rho &= \int(2\pi)^4\delta^4(p+p'-k-k')\frac{d^3k}{(2\pi)^32\omega}\frac{d^3k'}{(2\pi)^32\omega'}\\ &= \frac{1}{4\pi^2}\int\delta\left(2E-\omega-\omega'\right)\frac{d^3k}{4\omega\omega'}=\frac{1}{32\pi^2}\int\delta\left(E-\omega\right)\frac{k^2dkd\Omega}{\omega^2}=\frac{d\Omega}{32\pi^2}\end{aligned}$$

The flux factor is

$$I=\frac{1}{E_1E_2}\sqrt{\left(p_1\cdot p_2\right)^2-m_1^2m_2^2}=\frac{1}{E^2}2E^2=2$$

The differential crossection is then

$$d\sigma = \frac{1}{2}\frac{1}{4E^2}\left(\frac{1}{4}\sum_{spin'}|M|^2\right)\frac{d\Omega}{32\pi^2}$$

Or

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\left(1+\cos^2\theta\right)$$

where $\alpha = \dfrac{e^2}{4\pi}$ is the fine structure constant. The total cross section is

$$\sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{\alpha^2\pi}{3E^2}$$

Or

$$\sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{4\alpha^2\pi}{3s} \qquad \text{with} \qquad s = (p_1+p_2)^2 = 4E^2$$

2.2 $e^+e^- \to hadrons$

One of the interesting processess in e^+e^- collider is the reaction

$$e^+e^- \to hadrons$$

As we will discuss later according to QCD, the theory of strong interaciton, this processes will go through

$$e^+e^- \to q\bar{q}$$

and then $q\bar{q}$ will trun into hadrons. Since the coupling of photon to $q\bar{q}$ differs from the coupling to $\mu^+\mu^-$ only in their charges we can write down the cross section for $q\bar{q}$ as

$$\sigma\left(e^+e^- \to q\bar{q}\right) = 3\left(Q_q^2\right)\frac{4\alpha^2\pi}{3s} = 3\left(Q_q^2\right)\sigma\left(e^+e^- \to \mu^+\mu^-\right)$$

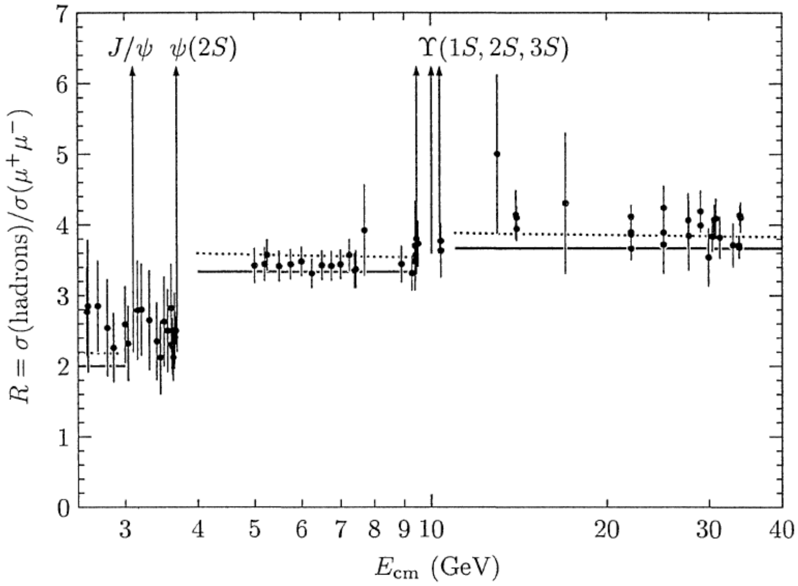
where Q_q is the electric charge of quark q. Here we have multiply the cross section by a factor of 3 to account for the fact that each quark has 3 colors. Then for the cross section to produce hadrons is

$$\frac{\sigma\left(e^+e^- \to hadrons\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = 3\left(\sum_i Q_i^2\right)$$

Here the summation is over quarks which are allowed by the avaliable energies. For example, for energy below the the charm quark only u, d, and s quarks should be included,

$$\frac{\sigma\left(e^+e^- \to hadrons\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = 3\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] = 2$$

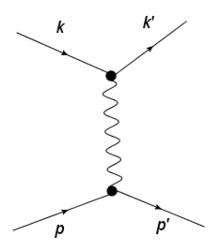
which is not far from the reality.



3 $ep \to ep$

We now consider the electron proton elastic scattering,

$$e(k) + p(p) \longrightarrow e(k') + p(p')$$



In general proton is a hadron and strong interaction needed to be included. But for simplicity, we will discuss the case where proton has no strong interaction and will dicuss the inclusion of the strong interaction later. The lowest order contribution is coming from the photo exchange and the amplitude is of the form,

$$\begin{aligned} M(e+p \quad \to \quad e+p) &= \bar{u}(p',s')\left(-ie\gamma^{\mu}\right)u(p,s)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(k',r')\left(-ie\gamma^{\nu}\right)u(k,r) \\ &= \frac{ie^2}{q^2}\bar{u}(p',s')\gamma^{\mu}u(p,s)\,\bar{u}(k',r')\gamma_{\mu}u(k,r) \end{aligned}$$

where $q = k - k'$. For the unploarized cross section, we sum over the spins of final states and average over the spin of the inital states,

$$\frac{1}{4}\sum_{spin}|M(e+p\to e+p)|^2 = \frac{e^4}{q^4}Tr\left[\left(\not{p}'+M\right)\gamma^{\mu}\left(\not{p}+M\right)\gamma^{\nu}\right]Tr\left[\left(\not{k}'+m_e\right)\gamma_{\mu}\left(\not{k}+m_e\right)\gamma^{\nu}\right]$$

Again we will neglec the mass of electron. It is straight to compute the traces to get

$$Tr\left[\not{k}'\gamma_{\mu}\not{k}\gamma^{\nu}\right] = 4\left[k'^{\mu}k^{\nu} - g^{\mu\nu}\left(k\cdot k'\right) + k^{\mu}k'^{\nu}\right]$$

$$Tr\left[\left(\not p' + M\right)\gamma^{\mu}\left(\not p + M\right)\gamma^{\nu}\right] = 4\left[p'^{\mu}p^{\nu} - g^{\mu\nu}\left(p\cdot p'\right) + p^{\mu}p'^{\nu}\right] + 4M^2 g^{\mu\nu}$$

Then

$$\frac{1}{4}\sum_{spin}\left|M(e^+e^- \to \mu^+\mu^-)\right|^2 = \frac{e^4}{q^4}\left\{8\left[(p\cdot k)\left(p'\cdot k'\right) + (p'\cdot k)\left(p\cdot k'\right)\right] - 8M^2\left(k\cdot k'\right)\right\}$$

Here it is more useful to use the laboratoy frame where

$$p_\mu = (M,0,0,0)\,, \qquad k_\mu = \left(E,\vec{k}\right), \qquad k'_\mu = \left(E',\vec{k'}\right)$$

Then

$$p\cdot k = ME, \qquad p.k' = ME', \qquad k\cdot k' = EE'\left(1-\cos\theta\right)$$

$$p'\cdot k' = \left(p+k-k'\right).k' = p.k' + k\cdot k', \qquad p'\cdot k = \left(p+k-k'\right).k = p.k - k\cdot k'$$

$$q^2 = \left(k-k'\right)^2 = -2k\cdot k' = -2EE'\left(1-\cos\theta\right)$$

$$d\sigma = \frac{1}{I}\frac{1}{2p_0}\frac{1}{2k_0}(2\pi)^4\delta^4(p+k-p'-k')\frac{1}{4}\sum_{spin'}|M|^2\frac{d^3p'}{(2\pi)^3 2p'_0}\frac{d^3k'}{(2\pi)^3 2k'_0}$$

The phase space is

$$\begin{aligned}\rho &= \int (2\pi)^4\delta^4(p+k-p'-k')\frac{d^3p'}{(2\pi)^3 2p'_0}\frac{d^3k'}{(2\pi)^3 2k'_0} \\ &= \frac{1}{4\pi^2}\int \delta\left(p_0+k_0-p'_0-k'_0\right)\frac{d^3k'}{2p'_0 2k'_0}\end{aligned} \qquad (1)$$

where

$$p'_0 = \sqrt{M^2 + \left(\vec{p}+\vec{k}-\vec{k'}\right)^2} = \sqrt{M^2 + \left(\vec{k}-\vec{k'}\right)^2}$$

Use the momenta configuration in the lab frame, we get

$$\begin{aligned}\rho &= \frac{1}{4\pi^2}\int \delta\left(M+E-p'_0-E'\right)\frac{k'^2dk'd\Omega}{2p'_0 2E'} \\ &= \frac{1}{4\pi^2}\int \delta\left(M+E-p'_0-E'\right)\frac{d\Omega E'dE'}{p'_0}\end{aligned}$$

Let

$$x = -E + p'_0 + E'$$

Then

$$dx = dE'(1+\frac{dp'_0}{dE'}) = dE'\left(\frac{p'_0+E'-E\cos\theta}{p'_0}\right)$$

and

$$\rho = \frac{1}{4\pi^2}\int \delta\left(x-M\right)\frac{d\Omega E'dx}{\left(p'_0+E'-E\cos\theta\right)} = \frac{1}{4\pi^2}\frac{d\Omega E'}{M+E\left(1-\cos\theta\right)}$$

From the argument of the $\delta-$function we get the relation,

$$M = x = -E + p'_0 + E'$$

We can use this to solve for E',

$$E' = \frac{ME}{E\left(1-\cos\theta\right)+M} = \frac{E}{1+\left(\frac{2E}{M}\right)\sin^2\frac{\theta}{2}}$$

The phase space is then

$$\rho = \frac{d\Omega}{4\pi^2} \frac{ME}{\left(M + E\left(1-\cos\theta\right)\right)^2} = \frac{d\Omega}{4\pi^2} \frac{E'^2}{ME}$$

and the flux factor is

$$I = \frac{1}{ME} p \cdot k = 1$$

The differential cross section is then

$$d\sigma = \frac{1}{I} \frac{1}{2p_0} \frac{1}{2k_0} (2\pi)^4 \delta^4(p + k - p' - k') \frac{1}{4} \sum_{spin'} |M|^2 \frac{d^3p'}{(2\pi)^3 2p_0'} \frac{d^3k'}{(2\pi)^3 2k_0'}$$

Or

$$\frac{d\sigma}{d\Omega} = \frac{1}{4ME} \frac{1}{4\pi^2} \frac{E'^2}{ME} \frac{1}{4} \sum_{spin'} |M|^2 = \left(\frac{E'}{E}\right)^2 \frac{1}{16\pi^2 M^2} \frac{e^4}{q^4} \left\{8\left[\left(p\cdot k\right)\left(p'\cdot k'\right) + \left(p'\cdot k\right)\left(p\cdot k'\right)\right] - 8M^2\left(k\cdot k'\right)\right\}$$

It is straightforward to compute the combination,

$$\begin{aligned}
& \left[\left(p\cdot k\right)\left(p'\cdot k'\right) + \left(p'\cdot k\right)\left(p\cdot k'\right)\right] - M^2\left(k\cdot k'\right) \\
= & \left[\left(p\cdot k\right)\left(p + k - k'\right)\cdot k' + \left(p\cdot k'\right)\left(p + k - k'\right)\cdot k - M^2\left(k\cdot k'\right)\right] \\
= & \; 2EE'M^2 + \left(k\cdot k'\right)\left(p\cdot q - M^2\right) \\
= & \; 2EE'M^2 + M^2EE'\left(1-\cos\theta\right)\left(-\frac{q^2}{2M^2} - 1\right) \\
= & \; 2EE'M^2\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]
\end{aligned}$$

Then the differential crosse section is,

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left(\frac{E'}{E}\right)^2 \frac{\alpha^2}{M^2} \frac{1}{\left(4EE'\sin^2\frac{\theta}{2}\right)^2} 2EE'M^2\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \\
&= \frac{\alpha^2}{4} \frac{E'}{E^3} \frac{1}{\sin^4\frac{\theta}{2}} \left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]
\end{aligned}$$

Or

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2} \frac{1}{\sin^4\frac{\theta}{2}} \frac{\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]}{\left[1 + \left(\frac{2E}{M}\right)\sin^2\frac{\theta}{2}\right]}$$

Here we treat the proton as elementary particle like muon to get the cross section. In reality the proton has strong interaction which is not easy to calculate. But we can make use of the fact that the γpp interaction is local to parametrize the γpp matrix element as

$$\left\langle p' \left| J_\mu \right| p \right\rangle = \bar{u}(p', s')\left[\gamma^\mu F_1\left(q^2\right) + \frac{i\sigma_{\mu\nu}q^\nu}{2M} F_2\left(q^2\right)\right] u\left(p, s\right) \qquad \text{with} \qquad q = p - p' \tag{2}$$

where we have used Lorentz covariance and current conservation in writing the matrix element in this form. Another useful relation is the Gordon decomposition

$$\bar{u}(p')\gamma_\mu u\left(p\right) = \bar{u}(p')\left[\frac{\left(p + p'\right)^\mu}{2m} + \frac{i\sigma^{\mu\nu}\left(p' - p\right)_\nu}{2m}\right] u\left(p\right)$$

Here $F_1(q^2)$, called charge form factor and $F_2(q^2)$, called the magnetic form factor contain all the strong interaction effect. Note that $F_1(q^2) = 1$ and $F_2(q^2) = 0$ correspond to the case where proton is treated as point particle. The charge form factor satifies the condition $F_1(0) = 1$. This is coming from the fact that the total charge of the proton and can be seen as follows. From

$$Q|p\rangle = |p\rangle$$

we get

$$\langle p'|Q|p\rangle = \langle p'|p\rangle = 2E(2\pi)^3\delta^3\left(\vec{p} - \vec{p}'\right)$$

On the other hand from Eq(2) we see that

$$\begin{aligned}\langle p'|Q|p\rangle &= \int d^3x \langle p'|J_0(x)|p\rangle = \int d^3x \langle p'|J_0(0)|p\rangle e^{i(p'-p)\cdot x} \\ &= (2\pi)^3\delta^3\left(\vec{p} - \vec{p}'\right)\bar{u}(p', s')\gamma_0 u(p, s) F_1(0) \\ &= 2E(2\pi)^3\delta^3\left(\vec{p} - \vec{p}'\right) F_1(0)\end{aligned}$$

This implies that $F_1(0) = 1$.To gain more insight on these form factors, write the charge operator Q in terms of charge density as

$$Q = \int d^3x\rho(x) = \int d^3x J_0(x)$$

we have

$$\langle p'|J_0(x)|p\rangle = e^{iq\cdot x}\langle p'|J_0(0)|p\rangle = e^{iq\cdot x}F_1(q^2)\bar{u}(p', s')\gamma_0 u(p, s)$$

This suggests that $F_1(q^2)$ can be interpreted as the Fourier transform of the charge density distribution i.e.

$$F_1(q^2) \sim \int d^3x\rho(x)e^{-i\vec{q}\cdot\vec{x}}$$

If we expand the form factor $F_1(q^2)$ in powers of q^2,

$$F_1(q^2) = F_1(0) + q^2F_1'(0) + \cdots$$

then we see that $F_1(0)$ is just the total charge and $F_1'(0)$ is related to the charge radius.

Calulate the cross section as before, we get the cross section with form factors,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2}\frac{\left[\cos^2\frac{\theta}{2}\left(\frac{1}{1 - q^2/4M^2}\right)\left[G_E^2 - \left(q^2/4M^2\right)G_M^2\right] - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}G_M^2\right]}{\sin^4\frac{\theta}{2}\left[1 + \left(\frac{2E}{M}\right)\sin^2\frac{\theta}{2}\right]}$$

where

$$G_E = F_1 + \frac{q^2}{4M^2}F_2$$
$$G_M = F_1 + F_2$$

Experimentally, G_E and G_M for the proton are given by

$$G_E(q^2) \approx \frac{G_M(q^2)}{\varkappa_p} \approx \frac{1}{(1 - q^2/0.7Gev^2)} \tag{3}$$

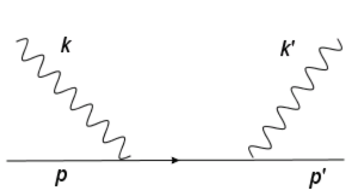
where $\varkappa_p = 2.79$ is the magnetic moment of the proton. Note that if the proton were point like, we would have $G_E(q^2) = G_M(q^2) = 1$. Thus the non-trivial dependence of q^2 in Eq(3) indicates that proton has a structure. Also for large q^2 the elastic cross section falls off rapidly as $G_E \approx G_M \sim q^{-4}$.

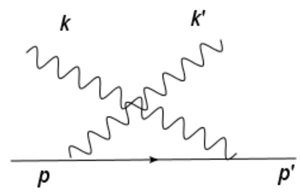
4 Compton Scattering

The momenta for γ and e are,

$$\gamma\left(k\right)+e\left(p\right)\longrightarrow\gamma\left(k'\right)+e\left(p'\right)$$

To lowest order in e, there are two Feynman diagrams contributing to this reaction,





The amplitude is given by

$$\begin{aligned} M\left(\gamma e\longrightarrow\gamma e\right) &= \bar{u}(p')(-ie\gamma^{\mu})\varepsilon'_{\mu}\left(k'\right)\frac{i}{\not{p}+\not{k}-m}\left(-ie\gamma^{\nu}\right)\varepsilon_{\nu}\left(k\right)u\left(p\right) \\ &\quad +\bar{u}(p')(-ie\gamma^{\mu})\varepsilon_{\mu}\left(k\right)\frac{i}{\not{p}-\not{k}'-m}\left(-ie\gamma^{\nu}\right)\varepsilon'_{\nu}\left(k'\right)u\left(p\right) \end{aligned}$$

Put the γ- matrices in the numerator,

$$M=-ie^{2}\varepsilon'_{\mu}\varepsilon_{\nu}\left[\bar{u}(p')\gamma^{\mu}\frac{\not{p}+\not{k}+m}{2p\cdot k}\gamma^{\nu}u\left(p\right)+\bar{u}(p')\gamma^{\nu}\frac{\not{p}-\not{k}'+m}{-2p\cdot k'}\gamma^{\mu}u\left(p\right)\right]$$

Using the relations,

$$\left(\not{p}+m\right)\gamma^{\nu}u\left(p\right)=2p^{\nu}u\left(p\right),$$

we get

$$M=-ie^{2}\bar{u}(p')\left[\frac{\not{\varepsilon}'\not{k}\not{\varepsilon}+2\left(p\cdot\varepsilon\right)\not{\varepsilon}'}{2p\cdot k}+\frac{-\not{\varepsilon}\not{k}'\not{\varepsilon}'+2\left(p\cdot\varepsilon\right)\not{\varepsilon}'}{-2p\cdot k'}\right]u\left(p\right)$$

The photon polarizations are of the form

$$\varepsilon_{\mu}=\left(0,\vec{\varepsilon}\right),\qquad\text{with}\quad\vec{\varepsilon}\cdot\vec{k}=0,\qquad\varepsilon'_{\mu}=\left(0,\vec{\varepsilon}'\right),\qquad\text{with}\quad\vec{\varepsilon}'\cdot\vec{k}'=0,$$

So in the lab frame where $p_{\mu}=(m,0,0,0)$, we have $(p\cdot\varepsilon)=(p\cdot\varepsilon')=0$ and

$$M=-ie^{2}\bar{u}(p')\left[\frac{\not{\varepsilon}'\not{k}\not{\varepsilon}}{2p\cdot k}+\frac{\not{\varepsilon}\not{k}'\not{\varepsilon}'}{2p\cdot k'}\right]u\left(p\right)$$

Summing over spin of the electron

$$\frac{1}{2}\sum_{spin}|M|^{2}=e^{4}Tr\left\{\left(\not{p}'+m\right)\left[\frac{\not{\varepsilon}'\not{k}\not{\varepsilon}}{2p\cdot k}+\frac{\not{\varepsilon}\not{k}'\not{\varepsilon}'}{2p\cdot k'}\right]\left(\not{p}+m\right)\left[\frac{\not{\varepsilon}'\not{k}\not{\varepsilon}}{2p\cdot k}+\frac{\not{\varepsilon}\not{k}'\not{\varepsilon}'}{2p\cdot k'}\right]\right\}$$

The cross section is given by

$$d\sigma=\frac{1}{I}\frac{1}{2p_{0}}\frac{1}{2k_{0}}(2\pi)^{4}\delta^{4}(p+k-p'-k')\frac{1}{4}\sum_{spin'}|M|^{2}\frac{d^{3}p'}{(2\pi)^{3}2p'_{0}}\frac{d^{3}k'}{(2\pi)^{3}2k'_{0}}$$

We compute the phase space first

$$\rho=\int(2\pi)^{4}\delta^{4}(p+k-p'-k')\frac{d^{3}p'}{(2\pi)^{3}2p'_{0}}\frac{d^{3}k'}{(2\pi)^{3}2k'_{0}}$$

Clearly this is exactly the same as the case for *ep* scattering and the result is

$$\rho = \frac{d\Omega}{4\pi^2}\frac{\omega'^2}{m\omega}$$

It is straightforward, but tedious to compute the trace, and the result is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4\left(\varepsilon\cdot\varepsilon'\right)^2 - 2\right]$$

This is the Klein-Nishima relation. In the low energy limit of $\omega \to 0$, this reduces to the classical Thomson scattering,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{m^2}\left(\varepsilon\cdot\varepsilon'\right)^2$$

Note that the combination $\frac{\alpha}{m}$ is just the classical electron radius.

For the case where photon is not polarized we need to sum over polarization of final state photon and average over polarization of phton in the initial state,

$$\sum_{\lambda\lambda'}\left[\varepsilon\left(k,\lambda\right)\cdot\varepsilon'\left(k',\lambda'\right)\right]^2 = \sum_{\lambda\lambda'}\left[\vec{\varepsilon}\left(k,\lambda\right)\cdot\vec{\varepsilon}'\left(k',\lambda'\right)\right]^2$$

Since $\vec{\varepsilon}\left(k,1\right), \vec{\varepsilon}\left(k,2\right)$ and $\vec{k}$ form an orthogonal basis in 3-dimension, we have the completeness relation,

$$\sum_{\lambda}\varepsilon_i\left(k,\lambda\right)\varepsilon_j\left(k,\lambda\right) = \delta_{ij} - \hat{k}_i\hat{k}_j$$

Then

$$\sum_{\lambda\lambda'}\left[\vec{\varepsilon}\left(k,\lambda\right)\cdot\vec{\varepsilon}'\left(k',\lambda'\right)\right]^2 = \left(\delta_{ij} - \hat{k}_i\hat{k}_j\right)\left(\delta_{ij} - \hat{k}'_i\hat{k}'_j\right) = 1 + \cos^2\theta$$

where $\hat{k}\cdot\hat{k}' = \cos\theta$. The cross section is then

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]$$

Integrating over the solid angle, we get the total cross section,

$$\sigma = \frac{\pi\alpha^2}{m^2}\int_{-1}^{1} dz\{\frac{1}{\left[1+\frac{\omega}{m}\left(1-z\right)\right]^3} + \frac{1}{\left[1+\frac{\omega}{m}\left(1-z\right)\right]} - \frac{1-z^2}{\left[1+\frac{\omega}{m}\left(1-z\right)\right]^2}\}$$

At low energies, $\omega \to 0$, we

$$\sigma = \frac{8\pi\alpha^2}{3m^2}$$

and at high energies

$$\sigma = \frac{\pi\alpha^2}{\omega m}\left[\ln\frac{2\omega}{m} + \frac{1}{2} + O\left(\frac{m}{\omega}\ln\frac{m}{\omega}\right)\right]$$