1 Group Theory

The most useful tool for studying symmetry is the group theory. We will give a simple discussion of the parts of the group theory which are commonly used in high energy physics.

1.1 Elements of group theory

A group G is a collection of elements (a, b, c$\cdots$) with a multiplication laws having the following properties;

1. Closure. If $a, b \in G$, $c = ab \in G$
2. Associative $a(bc) = (ab)c$
3. Identity $\exists e \in G \ \ni \ a = ea = ae \quad \forall a \in G$
4. Inverse For every $a \in G$, $\exists a^{-1} \ \ni \ aa^{-1} = e = a^{-1}a$

Examples of groups frequently used in physics are :

1. **Abelian group** —— group multiplication commutes, i.e. $ab = ba \quad \forall a, b \in G$
   e.g. cyclic group of order n, Z_n, consists of $a, a^2, a^3, \cdots, a^n = E$
2. **Orthogonal group** —— $n \times n$ orthogonal matrices, $RR^T = R^T R = 1$, $R : n \times n$ matrix
   e. g. the matrices representing rotations in 2-dimesions,
   $$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
3. **Unitary group** ——— $n \times n$ unitary matrices,

We can built larger groups from smaller ones by direct product:
Direct product group —— Given any two groups , $G = \{g_1, g_2 \cdots\}$, $H = \{h_1, h_2 \cdots\}$ and if $g's$ commute with $h's$ we can define a direct product group by $G \times H = \{g_i h_j\}$ with multiplication law

$$(g_i h_j)(g_m h_n) = (g_i g_m)(h_j h_n)$$

1.2 Theory of Representation

Consider a group $G = \{g_1 \cdots g_n \cdots\}$. If for each group element g_i,there is an $n \times n$ matrix $D(g_i)$ such that it preserves the group multiplication, i.e.

$$D(g_1)D(g_2) = D(g_1 g_2) \qquad \forall \;\; g_1, g_2 \in G$$

then $D's$ forms a representation of the group G (n-dimensional representation). In other words, $g_i \longrightarrow D(g_i)$ is a homomorphism. If there exists a non-singular matric M such that all matrices in the representation can be transformed into block diagonal form,

$$MD(a)M^{-1} = \begin{pmatrix} D_1(a) & 0 & 0 \\ 0 & D_2(a) & 0 \\ 0 & 0 & \ddots \end{pmatrix} \qquad for\ all\ a \in G.$$

$D(a)$ is called reducible representation. If representation is not reducible, then it is irreducible representation (irrep)

Continuous group: groups parametrized by set of continuous parameters

Example: Rotations in 2-dimensions can be parametrized by $0 \le \theta < 2\pi$

1.3 SU(2) group

Set of 2×2 unitary matrices with determinant 1 is called $SU(2)$ group.

In general, $n \times n$ unitary matrix U can be written as

$$U = e^{iH} \qquad H : n \times n \;\; \text{hermitian matrix}$$

From

$$\det U = e^{iTrH}$$

we get

$$TrH = 0 \qquad if \qquad detU = 1$$

Thus $n \times n$ unitary matrices U can be written in terms of $n \times n$ traceless Hermitian matrices.

Note that Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;\; , \;\; \sigma_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \;\; , \;\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a complete set of 2×2 hermitian traceless matrices. We can use them to describe $SU(2)$ matrices.

Define $J_i = \frac{\sigma_i}{2}$. We can compute the commutators

$$[J_1, J_2] = iJ_3 \quad , \quad [J_2, J_3] = iJ_1 \quad , \quad [J_3, J_1] = iJ_2$$

This is the *Lie* algebra of $SU(2)$ symmetry. This is exactly the same as the commutation relation of angular momentum.

Recall that to construct the irrep of $SU(2)$ algebra, we define

$$J^2 = J_1^2 + J_2^2 + J_2^3 \quad , \quad \text{with property} \quad [J^2, J_i] = 0 \quad , \quad i = 1, 2, 3$$

Also define

$$J_\pm \equiv J_1 \pm iJ_2 \quad \text{then} \quad J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2 \quad \text{and} \quad [J_+, J_-] = 2J_3$$

For convenience, we choose simultaneous eigenstates of J^2, J_3,

$$J^2|\lambda, m\rangle = \lambda|\lambda, m\rangle \quad , \quad \lambda_3|\lambda, m\rangle = m|\lambda, m\rangle$$

From

$$[J_+, J_3] = -J_+$$

we get

$$(J_+J_3 - J_3J_+)|\lambda, m\rangle = -J_+|\lambda, m\rangle$$

Or

$$J_3(J_+|\lambda, m\rangle) = (m + 1)(J_+|\lambda, m\rangle)$$

Thus J_+ raises the eigenvalue from m to $m+1$ and is called *raising operator*. Similarly, J_- lowers m to $m - 1$,

$$J_3(J_-|\lambda, m\rangle) = (m - 1)(J_-|\lambda, m\rangle)$$

Since

$$J^2 \geq J_3^2 \quad , \quad \lambda - m^2 \geq 0$$

we see that m is bounded above and below. Let j be the largest value of m, then

$$J_+|\lambda, j\rangle = 0$$

Then

$$0 = J_-J_+|\lambda, j\rangle = (J_3^2 - J_3^2 - J_3)|\lambda, j\rangle = (\lambda - j^2 - j)|\lambda, j\rangle$$

and

$$\lambda = j(j + 1)$$

Similarly, let j' be the smallest value of m, then

$$J_-|\lambda, j'\rangle = 0 \qquad \lambda = j'(j'-1)$$

Combining these 2 relations, we get

$$j(j+1) = j'(j'-1) \quad \Rightarrow \quad j' = -j \;\; and \;\; j - j' = 2j = integer$$

We will use j, m to label the states. Assume that the states are normalized,

$$\langle jm|jm'\rangle = \delta_{mm'}$$

Write

$$J_\pm|jm\rangle = C_\pm(jm)|j, m \pm 1\rangle$$

then

$$\langle jm|J_-J_+|jm\rangle = |C_+(j,m)|^2$$

$$LHS = \langle j,m|(J^2 - J_3^2 - J_3)|jm\rangle = j(j+1) - m^2 - m$$

This gives

$$C_+(j,m) = \sqrt{(j-m)(j+m+1)}$$

Similarly

$$C_-(j,m) = \sqrt{(j+m)(j-m+1)}$$

Summary: eigenstates $|jm\rangle$ have the properties

$$J_3|j,m\rangle = m|j,m\rangle \quad J_\pm|j,m\rangle = \sqrt{(j \mp m)(j \pm m+1)}|jm{\pm}1\rangle \;\; , \;\; J^2|j,m\rangle = j(j{+}1)jm\rangle$$

$J|j,m\rangle$, $m = -j, -j+1, \cdots, j$ are the basis for irreducible representation of SU(2) group. From these relations we can construct the representation matrices. We will illustrate these by following examples.

Example: $j = \frac{1}{2}$, $m = \pm\frac{1}{2}$

$$J_3 = |\frac{1}{2}, \pm\frac{1}{2}\langle = \pm\frac{1}{2}|\frac{1}{2}, \pm\frac{1}{2}\rangle$$

$$J_+|\frac{1}{2},\frac{1}{2}\rangle = 0 \;\; , \;\; J_+|\frac{1}{2}, -\frac{1}{2} = |\frac{1}{2},\frac{1}{2}\rangle \;\; , \;\; J_-|\frac{1}{2},\frac{1}{2} = |\frac{1}{2},-\frac{1}{2}\rangle \;\; , \;\; J_-|\frac{1}{2},-\frac{1}{2}\rangle = 0$$

If we write

$$|\frac{1}{2},\frac{1}{2}\rangle = \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\frac{1}{2},-\frac{1}{2}\rangle = \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then we can represent $J's$ by matrices,

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J_1 = \frac{1}{2}(J_+ + J_-) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad J_2 = \frac{1}{2i}(J_+ - J_-) = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Within a factor of $\frac{1}{2}$, these are just Pauli matrices

Product representation

Let α be the spin-up and β the spin-down states. Then for 2 spin $\frac{1}{2}$ particles, the total wavefunction is product of wavefunctions of the form, $\alpha_1\alpha_2, \alpha_1\beta_2 \cdots$

Define $\vec{J}^{(1)}$ acts only on particle 1 and $\vec{J}^{(2)}$ acts only on particle 2.

$$\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$$

Use

$$J_3 = J_3^{(1)} + J_3^{(2)} \quad , \quad J_3(\alpha_1\alpha_2) = (J_3^{(1)} + J_3^{(2)})(\alpha_1\alpha_2) = (\alpha_1\alpha_2)$$

From

$$\vec{J}^2 = (\vec{J}^{(1)} + \vec{J}^{(2)})^2 = (\vec{J}^{(1)})^2 + (\vec{J}^{(2)})^2 + 2[\frac{1}{2}(J_+^{(1)}J_-^{(2)} + J_-^{(1)}J_+^{(2)} + J_3^{(1)}J_3^{(2)}]$$

$$\vec{J}^2(\alpha_1\alpha_2) = (\frac{3}{4} + \frac{3}{4} + \frac{2}{4})|\alpha_1\alpha_2\rangle = 2|\alpha_1\alpha_2\rangle$$

This means $|1,1\rangle = \alpha_1\alpha_2$ is a $j = 1$ state. To get other $j = 1$ states, we can use the lowering operator

$$J_-(\alpha_1\alpha_2) = (J_-^{(1)} + J_-^{(2)})(\alpha_1\alpha_2) = (\beta_1\alpha_2 + \alpha_1\beta_2)$$

On the other hand

$$J_-(\alpha_1\alpha_2) = J_-|11\rangle = \sqrt{(1+1)(1-1+1)}|1,0\rangle = \sqrt{2}|1,0\rangle$$

$$\Rightarrow \quad |1,0\rangle = \frac{1}{\sqrt{2}}(\beta_1\alpha_2 + \alpha_1\beta_2)$$

The only state left-over is

$$\frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2)$$

This is a $|0,0\rangle$ state.

Summary:

1. Among the generator only J_3 is diagonal, — SU(2) is a rank-1 group
2. Irreducible representation is labeled by j and the dimension is $2j + 1$
3. Basis states $|j,m\rangle \quad m = j, j-1 \cdots (-j)$ representation matrices can be obtained from

$$J_3|j,m\rangle = m|j,m\rangle \qquad J_\pm|j,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

1.4 SU(2) and rotation group

The generators of $SU(2)$ group are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let $\vec{r} = (x, y, z)$ be arbitrary vector in R_3 (3 dimensional coordinate space). Define a 2×2 matrix h by

$$h = \vec{\sigma} \cdot \vec{r} = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

h has the following properties

1. $h^{+} = h$
2. $Trh = 0$
3. $det\ h = -(x^2 + y^2 + z^2)$

Let U be a 2×2 unitary matrix with $detU = 1$. Consider the transformation

$$h \rightarrow h' = UhU^{\dagger}$$

Then we have

1. $h'^{+} = h'$
2. $Trh' = 0$
3. $\det h' = \det h$

$$, \quad (3)$$

Properties (1)&(2) imply that h' can also be expanded in terms of Pauli matrices

$$h' = \vec{r}' \cdot \vec{\sigma} \ \vec{r} = (x', y', z')$$

$$\det h' = \det h \quad \Rightarrow \quad x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$$

Thus relation between $\vec{r}$ and $\vec{r}'$ is a rotation. This means that an arbitrary 2×2 unitary matrix U induces a rotation in R_3. This provides a connection between $SU(2)$ and $O(3)$ groups.

1.5 Rotation group & QM

Rotation in R_3 can be represented as linear transformations on

$$\vec{r}(x,y,z) = (r_1, r_2, r_3) \quad , \quad r_i \to r'_i = R_{ij} X_j \qquad RR^T = 1 = R^T R$$

Consider an arbitary function of coordinates, $f(\vec{r}) = f(x,y,z)$. Under the rotation, the change in f

$$f(r_i) \to \; f(R_{ij} r_j) = f'(r_i)$$

If $f = f'$ we say f is invariant under rotation, eg $f(r_i) = f(r), r = \sqrt{x^2+y^2+z^2}$

In QM, we implement the rotation by

$$|\psi\rangle \to |\psi'\rangle = U|\psi\rangle, \quad O \to O' = UOU^\dagger$$

so that

$$\Rightarrow \langle\psi'|O'|\psi'\rangle = \langle\psi|O|\psi\rangle$$

If $O' = O$, we say the operator O is invariant under rotation

$$\to \quad UO = OU \quad [O, U] = 0$$

In terms of infinitesimal generators, we have

$$U = e^{-i\theta\vec{n}\cdot\vec{J}/\hbar}$$

This implies

$$[J_i, O] = 0, \;\; i = 1, 2, 3$$

For the case where O is the Hamiltonian H, this gives $[J_i, H] = 0$. Let $|\psi\rangle$ be an eigenstate of H with eigenvaule E,

$$H|\psi\rangle = E|\psi\rangle$$

then

$$(J_i H - H J_i)|\psi\rangle = 0 \quad \Rightarrow \quad H(J_i|\psi\rangle) = E(J_i|\psi\rangle)$$

$i.e$ $\quad |\psi\rangle$ & $J_i|\psi\rangle$ are degenerate. For example, let $|\psi\rangle = |j, m\rangle$ the eigenstates of angular momentum, then $J_\pm|j.m\rangle$ are also eigenstates if $|\psi\rangle$ is eigenstate of H. This means for a given j , the degeneracy is $(2j+1)$.

1.6 Gauge Theory

1.6.1 Abelian gauge theory(QED)

Maxwell Equation

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad , \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad , \quad \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J}$$

Source free equations can be solved by introducing $\vec{A}$ and ϕ

$$\vec{B} = \nabla \times \vec{A} \quad , \quad \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

$$\partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu} \; , \qquad \text{with} \quad F^{ij} \sim \epsilon^{ijk} B_h \quad F^{0i} \sim E^i$$

$\overrightarrow{E}$ and $\overrightarrow{B}$ are unchanged under the transformation

$$\phi \to \phi - \frac{\partial \alpha}{\partial t} \; , \qquad \vec{A} \to \vec{A} + \vec{\nabla} \alpha$$

Or

$$A^\mu \to A^\mu + \partial^\mu \alpha, \qquad \text{with} \quad A^\mu = (\frac{\phi}{c}, \vec{A})$$

This is called the **gauge invariance** of eletrodynamics. Classically, it not clear as to the physical significance of this invariance. It taks quantum theory to realize its meaning.

Schrodinger Equation for a charged particle is of the form,

$$[\frac{1}{2m}(\frac{\hbar}{i}\vec{\nabla} - e\vec{A})^2 - e\phi]\psi = i\hbar \frac{\partial \psi}{\partial t}$$

To get the same physics, we need to transform ψ by

$$\psi \to e^{ie\alpha/\hbar}\psi, \qquad U(1) \qquad \text{phase transformation}$$

This provides a **connection between gauge transformation with symmetry transformation**.

Consider the Lagrangian for a free electron field $\psi(x)$

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$$

This has global U(1) symmetry,

$$\psi(x) \to \psi'^{-i\alpha}\psi(x) \qquad \alpha : \text{constant}$$

$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)e^{i\alpha}$$

Suppose now the phase is space-time dependent, $\alpha = \alpha(x)$

$$\psi = e^{-i\alpha(x)}\psi(x) \quad , \quad \bar{\psi}'(x) = \bar{\psi}(x)e^{i\alpha(x)}$$

The transformation of derivative is

$$\bar{\psi}(x)\partial_\mu\psi(x) \;\; \to \;\; \bar{\psi}'(x)\partial_\mu\psi'(x) = \bar{\psi}(x)\partial_\mu\psi(x) - i(\partial_\mu\alpha)(\bar{\psi}\psi)$$

which is not invariant. Introduce gauge field $A_\mu(x)$ to form **covariant derivative**

$$D_\mu\psi \equiv (\partial_\mu + igA_\mu)\psi(x)$$

So that $D_\mu\psi$ transforms the same way as ψ,

$$(D_\mu\psi)' = e^{-i\alpha(x)}(D_\mu\psi)$$

This requires that

$$(\partial_\mu + igA'_\mu)\psi' = e^{-i\alpha}(\partial_\mu + igA_\mu)\psi$$

which implies

$$A'_\mu = A_\mu - \frac{1}{g}\partial_\mu\alpha$$

Then

$$\mathcal{L} = \;\bar{\psi}i\gamma^\mu(\partial_\mu + igA_\mu)\psi - m\bar{\psi}\psi$$

is invariant under local symmetry transformation (**local symmetry**)

The Lagrangian for gauge field is of the form,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

One useful relation is to write $F_{\mu\nu}$ in terms of covariant derivative,

$$\begin{aligned} D_\mu D_\nu\psi &= (\partial_\mu + igA_\mu)(\partial_\nu + igA_\nu)\psi \\ &= \partial_\mu\partial_\nu\psi - g^2A_\mu A_\nu\psi + ig(A_\mu\partial_\nu + A_\nu\partial_\mu)\psi + ig(\partial_\mu A_\nu)\psi \end{aligned}$$

Antisymmetrization gives

$$(D_\mu D_\nu - D_\nu D_\mu)\psi = ig(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi = ig(F_{\mu\nu})\psi$$

From

$$[(D_\mu D_\nu - D_\nu D_\mu)\psi]' = e^{-i\alpha}(D_\mu D_\nu - D_\nu D_\mu)\psi$$

we get

$$F'_{\mu\nu} = F_{\mu\nu}$$

The advantage of this relation is that the gauge transformation of $F_{\mu\nu}$ is automatically determined by the covariant derivative. Thus the Lagrangian of the form

$$\mathcal{L} = \bar{\psi} i\gamma^\mu(\partial_\mu + igA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

is invariant under gauge transformation

$$\psi(x) \to \psi' = e^{-i\alpha(x)}\psi(x)$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x)$$

Remarks:

1. $A_\mu A^\mu$ term is not gauge invariant $\Rightarrow$ gauge field massless.

2. $D_\mu\psi = (\partial_\mu + igA_\mu)\psi \Rightarrow$ minimal coupling determined by U(1) transformation is universal

3. no gauge self coupling because A_μ does not carry U(1) charge.

1.6.2 Non-Abelian symmetry-Yang Mills fields

1954: Yang-Mills generalized U(1) local symmetry to SU(2) local symmetry.

Consider an isospin doublet $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

Under SU(2) transformation

$$\psi(x) \to \psi'(x) = exp\{-\frac{i\vec{\tau}\cdot\vec{\theta}}{2}\}\psi(x)$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are Pauli matrices, with

$$[\frac{\tau_i}{2}, \frac{\tau_j}{2}] = i\epsilon_{ijk}(\frac{\tau_k}{2})$$

Start with free Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi$$

which is invariant under global $SU(2)$ transformation.

Under local symmetry transformation, we have

$$\psi(x) \to \psi'(x) = U(\theta)\psi(x) \qquad U(\theta) = exp\{-\frac{i\vec{\tau}\cdot\vec{\theta(x)}}{2}\}$$

Derivative term

$$\partial_\mu\psi(x) \to \partial_\mu\psi'(x) = U\partial_\mu\psi + (\partial_\mu U)\psi$$

is not invariant. Introduce gauge fields $\vec{A_\mu}$ to form the covariant derivative,

$$D_\mu\psi(x) \equiv (\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A_\mu}}{2})\psi$$

Require that

$$[D_\mu\psi]' = U[D_\mu\psi]$$

Or

$$(\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A_\mu}'}{2})(U\psi) = U(\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A_\mu}}{2})\psi$$

This gives the transformation of gauge field,

$$\boxed{\frac{\vec{\tau}\cdot\vec{A_\mu}'}{2} = U(\frac{\vec{\tau}\cdot\vec{A_\mu}}{2})U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}}$$

We can use covariant derivatives to construct field tensor

$$D_\mu D_\nu\psi = (\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A_\mu}}{2})(\partial_\nu - ig\frac{\vec{\tau}\cdot\vec{A_\nu}}{2})\psi = \partial_\mu\partial_\nu\psi - ig(\frac{\vec{\tau}\cdot\vec{A_\mu}}{2}\partial_\nu\psi + \frac{\vec{\tau}\cdot\vec{A_\nu}}{2}\partial_\mu\psi)$$

$$-ig\partial_\mu(\frac{\vec{\tau}\cdot\vec{A_\nu}}{2})\psi + (-ig)^2(\frac{\vec{\tau}\cdot\vec{A_\mu}}{2})(\frac{\vec{\tau}\cdot\vec{A_\nu}}{2})\psi$$

Antisymmetrize this to get the field tensor,

$$(D_\mu D_\nu - D_\nu D_\mu)\psi \equiv ig(\frac{\vec{\tau}\cdot\vec{F_{\mu\nu}}}{2})\psi$$

then

$$\frac{\vec{\tau}\cdot\vec{F_{\mu\nu}}}{2} = \frac{\vec{\tau}}{2}\cdot(\partial_\mu\vec{A_\nu} - \partial_\nu\vec{A_\mu}) - ig[\frac{\vec{\tau}\cdot\vec{A_\mu}}{2}, \frac{\vec{\tau}\cdot\vec{A_\nu}}{2}]$$

Or in terms of components,

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk}A^i_\mu A^k_\nu$$

The the term quadratic in A is new in Non-Abelian symmetry. Under the gauge transformation we have

$$\vec{\tau} \cdot \vec{F_\mu \nu}' = U(\vec{\tau} \cdot \vec{F_\mu \nu})U^{-1}$$

Infinitesmal transformation $\theta(x) \ll 1$

$$A^{i/\mu} = A^\mu + \epsilon^{ijk}\theta^j A_\mu^k - \frac{1}{g}\partial_\mu \theta^i$$

$$F_{\mu\nu}^{/i} = F_{\mu\nu}^i + \epsilon^{ijk}\theta^j F_{\mu\nu}^k$$

Remarks

1. Again $A_\mu^a A^{a\mu}$ is not gauge invariant⇒gauge boson massless⇒long range force
2. A_μ^a carries the symmetry charge (e.g. color —)
3. The quadratic term in $F^{a\mu\nu} \sim \partial A - \partial A + gAA$ is for asymptotic freedom.

2 Spontaneous symmetry breaking

Spontaneous symmetry breaking—-ground state does not have the symmetry of the Hamiltonian

⇒If the symmetry is continuous one, there will be massless scalar fields–Goldstone boson

Example:ferromagnetism

$T > T_c$(Curie temp) all dipoles are randomly oriented–rotational invariant

$T < T_c$ all dipoles are oriented in some direction

Ginzburgh-Landau theory

Free energy as function of magnetization $\vec{m}$ (averaged)

$$\mu(\vec{M}) = (\partial_t \vec{M})^2 + \alpha_1(T)\vec{M} \cdot \vec{M} + \alpha_2(\vec{M} \cdot \vec{M})^2$$

We take $\alpha_2 > 0$ so that the free energy is positive for large M and $\alpha_1(T) = \alpha(T - T_c) \quad \alpha > 0$ so that there is a transition going through Curie temperature T_c. It is easy to see that the ground state is governed by

$$\vec{M}(\alpha_1 + 2\alpha_2 \vec{M} \cdot \vec{M}) = 0$$

For $T > T_c$ only solution is $\vec{M} = 0$ and $T < T_c$ non-trivial sol $|\vec{M}| = +\sqrt{\frac{\alpha_1}{2\alpha_2}} \neq 0$

⇒ ground state with $\vec{M}$ in some direction is no longer rotational invariant.

2.1 Nambu-Goldstone theorem

Recall that Noether's theorem says that a continuous symmetry will give conserved charge Q. Suppose there are 2 local operators A, B with property

$$[Q,B]=A \qquad Q=\int d^3\times j_0(x) \quad \textit{indep of time}$$

Suppose $\langle 0|A|0\rangle = V \neq 0$ (symmetry breaking condition)

$$\Rightarrow 0\neq \langle 0|[Q,B]|\rangle = \int d^3\times \langle O|[j_0(x),BJ]|0\rangle$$

$$=\sum_n (2\pi)^3\delta^3(\vec{P}_n)\{\langle 0|j_0(0)|n\rangle\langle n|B|0\rangle e^{-iE_nt}-\langle n|B|0\rangle\langle 0|j_0(0)|n\rangle e^{-iE_nt}\}=U$$

Since $U\neg 0$ and time-independent, we need to a state such that

$$E_n\to 0 \;\; for \;\; \vec{P}_n=0$$

massless excitation. For the case of relativistic particle with energy momentum rotation $E=\sqrt{\vec{P}^2+m^2}$ this implies massless particle- Goldstone boson.

Discrete symmetry case

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2-\frac{\mu^2}{2}\phi^2-\frac{\lambda}{4}\phi^4 \qquad \phi\to-\phi \;\; symmetry$$

The Hamiltonian density

$$H=\frac{1}{2}(\partial_0\phi)^2+\frac{1}{2}(\vec{\nabla}\phi)^2+\frac{\mu^2}{2}\phi^2+\frac{\lambda}{4}\phi^4$$

Effective energy

$$\mu(\phi)=\frac{1}{2}(\vec{\nabla}\phi)^2+V(\phi) \quad , \quad V(\phi)=\frac{\mu^2}{2}\phi^2+\frac{\lambda}{4}\phi^4$$

For $\mu^2<0$ the ground state has $\phi=\pm\sqrt{\frac{-\mu^2}{\lambda}}$ classically.

This means the quantum ground state $|0\rangle$ will have the property

$$\langle 0|\phi|0\rangle=\nu\neq 0 \;\; \textit{symmetry breaking condition}$$

Define quantum field ϕ' by $\phi'=\phi-\nu$

$$then \quad \pounds = \frac{1}{2}(\partial_\mu \phi'^2 - (-\mu^2)\phi'^2 - \lambda\nu\phi'^3 - \frac{\lambda}{4}\phi'^4$$

No Goldstone boson—-discrete symmetry

Abelian symmetry case

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2] - V(\sigma^2 + \pi^2)$$

$$with \;\; V(\sigma^2 + \pi^2) = -\frac{\mu^2}{2}(\sigma^2 + \pi^2) + \frac{\lambda}{4}(\sigma^2 + \pi^2)^2$$

$$O(2) \; symmetry \quad \begin{pmatrix} \sigma \\ \pi \end{pmatrix} \rightarrow \begin{pmatrix} \sigma' \\ \pi' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix}$$

$$minimum \quad \sigma^2 + \pi^2 = \frac{\mu^2}{\lambda} = \nu^2 \quad circle \; in \; \sigma - \pi \; plane$$

$$For \; convenience \; choose \quad \langle 0|\sigma|0\rangle = \nu \quad \langle 0|\pi|0\rangle = 0$$

$$New \; quantum \; field \quad \sigma' = \sigma - \nu \;\; , \;\; \pi' = \pi$$

$$New \; Lagrangian \;\; \mathcal{L} = \frac{1}{2}[(\partial_\mu \sigma'^2 + (\partial_\mu \pi)^2] - \mu^2\sigma'^2 - \lambda\nu\sigma'(\sigma'^2 + \pi'^2) - \frac{\lambda}{4}(\sigma'^2 + \pi'^2)^2 \quad O(2)$$

no π'^2 term, $\Rightarrow \pi'$ massless Goldstone boson

Non-Abelian case-σmodel

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \sigma'^2 + (\partial_\mu \vec{\pi})^2] + \bar{N} i\gamma^\mu \partial_\mu N + g\bar{N}(\sigma + it\vec{au}\cdot\vec{\pi}\gamma_5)N - V(\sigma^2 + \vec{\pi}^2) + (f_\pi m_\pi^2 \sigma)$$

$$V(\sigma^2 + \vec{\pi}^2) = -\frac{\mu^2}{2}(\sigma^2 + \vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2$$

$$minimum \quad \sigma^2 + \vec{\pi}^2 = \nu^2 = \frac{\mu^2}{\lambda}$$

$$choose \quad \langle\sigma\rangle = \nu \;\; , \;\; \langle\vec{\pi}\rangle = 0$$

Then $\vec{\pi}$ are Goldstone bosons.

2.2 Higgs Phenomena

When we combine spontaneous symmetry breaking with local symmetry, a very interesting phenomena occurs. This was discovered in the $60's$ by Higgs, Englert & Brout, Guralnik, Hagen & Kibble independently

Abelian case

Consider the Lagrangian given by

$$\mathcal{L} = (D_\mu \phi)^+ (D^\mu \phi) + \mu^2 \phi^\phi - \lambda(\phi^+\phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$where \quad D^\mu \phi = (\partial^\mu - igA^\mu)\phi \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The Lagrangian is invariant under the local gauge transformation

$$\phi(x) \to \phi'^{-i\alpha(x)} \phi(x)$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{g} \partial_\mu \alpha(x)$$

The spontaneous symm. breaking is generated by the potential

$$V(\phi) = -\mu^2 \phi^+ \phi + \lambda(\phi^+\phi)^2$$

which has a minimum at

$$\phi^+\phi = \frac{\nu^2}{2} = \frac{1}{2}(\frac{\mu^2}{\lambda})$$

For the quantum theory, we can choose

$$|\langle 0|\phi|0\rangle| = \frac{\nu}{\sqrt{2}}$$

Or if we write

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

this corresponds to

$$\langle \phi_1 \rangle = \nu \quad , \quad \langle \phi_2 \rangle = 0 \qquad \phi_2 : Goldstone\ boson$$

Define the quantum fields by

$$\phi'_1 = \phi_1 - \nu \quad , \quad \phi'_2 = \phi_2$$

Covariant derivative terms gives

$$(D_\mu \phi)^+ (D^\mu \phi) = [(\partial_\mu + igA_\mu)\phi^+][(\partial^\mu - igA^\mu)\phi]$$

$$\frac{-1}{2}(\partial_\mu\phi_1'+gA_\mu\phi_2')^2+\frac{1}{2}(\partial_\mu\phi_2'-gA_\mu\phi_1')^2+\underline{\frac{g^2\nu^2}{2}A^\mu A_\mu}+\cdots\ mass\,terms\ for A^\mu$$

Write the scalar field as

$$\phi(x)=\frac{1}{\sqrt{2}}(\nu+\eta(x))e^{i\xi(x)/\nu}$$

"Gauge" transformation:

$$\phi'^{-i\xi(x)/\nu}\phi(x)\quad,\quad B_\mu=A_\mu(x)-\frac{1}{g\nu}\partial_\mu\xi$$

$\xi(x)$disappears from the Lagrangian

Roughly speaking, massless gauge field A_μ combine with Goldstone boson $\xi(x)$ to become massive gauge boson. As a consequence, two long range forces (from Goldstone boson $\xi(x)$ and $A_\mu(x)$) disappear.

Non-Abelian case

SU(2) group: $\phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ doublet

$$\mathcal{L}=(D_\mu\phi)^+(D^\mu\phi)-V(\phi)-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\quad,\quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

$$V(\phi)=-\mu^2(\phi^+\phi)+\lambda(\phi^+\phi)^2$$

Spontaneous symmetry breaking:

$$\langle\phi\rangle_0=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ \nu\end{pmatrix}\qquad \nu=\sqrt{\frac{\mu^2}{\lambda}}$$

Define $\phi'=\phi-\langle\phi\rangle_0$

From covariant derivative

$$(D_\mu\phi)^+(D^\mu\phi)=[\partial_\mu-ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}(\phi'+\langle\phi\rangle_0)]^+[\partial^\mu-ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}(\phi'+\langle\phi\rangle_0)]$$

$$\rightarrow\frac{1}{4}g^2\langle\phi\rangle_0(\vec{\tau}\cdot\vec{A}_\mu)(\vec{\tau}\cdot\vec{A}^\mu)\langle\phi\rangle_0=\frac{1}{2}(fracg\nu 2)^2\vec{A}_\mu\cdot\vec{A}^\mu$$

All gauge bosons get masses

$$M_A=\frac{1}{2}g\nu$$

The symmetry is completely broken.

$$Write\quad \phi(x)=exp\{\frac{i\vec{\tau}\cdot\vec{\xi}(x)}{\nu}\}\begin{pmatrix}0\\ \frac{\nu+\eta(x)}{\sqrt{2}}\end{pmatrix}$$

"gauge" transformation

$$\phi'(x) = U(x)\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \nu + \eta(x) \end{pmatrix}$$

$$\frac{\vec{\tau}\cdot\vec{B}_\mu}{2} = U(x)\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}U^{-1} - \frac{i}{g}[\partial_\mu U]U^{-1}(x)$$

$$where \quad U(x) = exp\{\frac{\vec{\tau}\cdot\vec{\xi}}{\nu}\}$$