Quantum Field Theory

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Schrodinger equation $\Rightarrow$ conservation of particle number.

$$H\psi = i\hbar\frac{\partial\psi}{\partial t} \qquad \Rightarrow \qquad \frac{d}{dt}\int d^3x\psi^\dagger\psi = 0 \rightarrow \int d^3x(\psi^\dagger\psi) \quad \text{indep of time}$$

if Hamiltonian is hermitian, $H = H^\dagger$. Then number of particles is conserved and no particle creation or annihilation.

Canonical commutation relation gives uncertainty relation,

$$[x, p] = -i\hbar, \qquad \Rightarrow \qquad \triangle x \triangle p \geqslant \hbar$$

From

$$p^2c^2 + m^2c^4 = E^2$$

to give

$$\triangle E = \frac{p\triangle p}{E}c^2 \geqslant \frac{p\hbar c^2}{E\triangle x} \quad or \quad \triangle x \geqslant \frac{pc}{E}(\frac{\hbar c}{\triangle E})$$

To avoid new particle creation we require $\triangle E \leqslant mc^2$. Then we get a lower bound on $\triangle x$

$$\triangle x \geqslant \frac{pc}{E}\frac{\hbar}{mc} = (\frac{v}{c})(\frac{\hbar}{mc})$$

For relativistic particle $\frac{v}{c} \approx 1$, then

$$\triangle x \geqslant (\frac{\hbar}{mc}) \qquad \text{Compton wavelength}$$

$\Rightarrow$ Particle can not be confined to a interval smaller than its Compton wavelength

Gauge Theory–Quantum Field Theory with Local Symmetry

Gauge principle

All fundamental Interactions are descibed in terms of gauge theories;

1. Strong Interaction-QCD;
   gauge theory based on SU(3) symmetry
2. Electromagnetic and Weak interaction-
   gauge theory based on $SU(2)\times\ U(1)$ symmetry
3. Gravitational interaction-
   Einstein's theory-gauge theory of local coordinate transformation.

Natural unit

$$\hbar = c = 1$$

In MKS units

$$\hbar = 1.055 \times 10^{-34} J\,\text{sec}, \quad c = 2.99 \times 10^{8} m/sec$$

In this unit, at the end of the calculation one puts back the factors of $\hbar$ and c depending on the physical quantities in the problem.
For example, the quantity m_e can have following different meanings depending on the contexts;

1. Reciprocal length

$$m_e = \frac{1}{\frac{\hbar}{m_e c}} = \frac{1}{3.86 \times 10^{-11} cm}$$

2. Reciprocal time

$$m_e = \frac{1}{\frac{\hbar}{m_e c^2}} = \frac{1}{1.29 \times 10^{-21} sec}$$

3. energy

$$m_e = m_e c^2 = 0.511 Mev$$

4. momentum

$$m_e = m_e c = 0.511 Mev/c$$

The following conversion relations

$$\hbar = 6.58 \times 10^{-22} Mev - sec \qquad \hbar c = 1.973 \times 10^{-11} Mev - cm$$

are quite useful in getting the physical quantities in the right units.
Example: Thomson cross section

$$\sigma = \frac{8\pi\alpha^2}{3m_e^2} = \frac{8\pi\alpha^2(\hbar c)^2}{3m_e^2 c^4} = (\frac{1}{137})^2 \times \frac{(1.973 \times 10^{-11} Mev - cm)^2}{(0.5 Mev)^2} \times (\frac{8\pi}{3}) \simeq$$

Useful convertion factor

$$1ev = 1.6 \times 10^{-19} J, \qquad 1 Gev = 1.6 \times 10^{-10} J \quad or \quad 1J = \frac{1}{1.6} \times 10^{10} Gev$$

Review of Special Relativity

Basic principles of special relativity :

1. The speed of light : same in all inertial frames.
2. Physical laws: same forms in all inertial frames.

   Lorentz transformation–relate coordinates in different inertial frame

$$x' = \frac{x - vt}{\sqrt{1 - v^2}} \quad y' = y, \qquad z' = z, \qquad t' = \frac{t - vx}{\sqrt{1 - v^2}}$$

$\Rightarrow$

$$t^2 - x^2 - y^2 - z^2 = t'^2 - x'^2 - y'^2 - z'^2$$

Proper time $\tau^2 = t^2 - \overrightarrow{r}^2$ invariant under Lorentz transfomation.
Particle moves from $\overrightarrow{r_1}(t_1)$ to $\overrightarrow{r_2}(t_2)$ .The speed is

$$|\overrightarrow{v}| = \frac{1}{|t_2 - t_1|}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

For $|\overrightarrow{v}| = 1$,

$$(t_1 - t_2)^2 = |\overrightarrow{r_1} - \overrightarrow{r_2}|^2$$

this is invariant under Lorentz transformation $\Rightarrow$ speed of light same in all inertial frames.

Another form of the Lorentz transformation

$$x' = \cosh\omega\ x - \sinh\omega\ t, \quad y' = y, \quad z' = z, \quad t' = \sinh\omega\, x - \cosh\omega\, t$$

where

$$\tanh\omega = v$$

For infinitesmal interval (dt, dx, dy, dz), proper time is

$$(d\tau)^2 = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

Minkowski space,

$$x^\mu = (t, x, y, z) = (x^0, x^1, x^2, x^3), \qquad 4 - vector$$

Lorentz invariant product can be written as

$$x^2 = (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2 = x_\mu x_\nu g_{\mu\nu}$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Define another 4-vector

$$x_\mu = g_{\mu\nu} x^\nu = (t, -x^1, -x^2, -x^3) = (t, -\overrightarrow{r})$$

so that

$$x^2 = x^\mu x_\mu$$

For infinitesmal coordinates

$$(dx)^2 = (dx^\mu)(dx_\mu) = dx^\mu dx^\nu g_{\mu\nu} = (dx^0)^2 - (d\overrightarrow{x})^2$$

Write the Lorentz transformation as

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$$

For example for Lorentz transformation in the $x-$direction, we have

$$\Lambda^\mu_\nu = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & \frac{-\beta}{\sqrt{1-\beta^2}} & 0 & 0 \\ \frac{-\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Write

$$x'^2 = x'^\mu x'^\nu g_{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \, g_{\mu\nu} x^\alpha x^\beta$$

then $x^2 = x'^2$ implies

$$\Lambda^\mu_\alpha \Lambda^\nu_\beta \, g_{\mu\nu} = g_{\alpha\beta}$$

and is called **pseudo-orthogonality** relation.

Energy and Momentum

Start from

$$dx^{\mu} = (dx^0, dx^1, dx^2, dx^3)$$

Proper time is Lorentz invariant and has the form,

$$(d\tau)^2 = dx^{\mu} dx_{\mu} = (dt)^2 - (\frac{d\overrightarrow{x}}{dt})^2 (dt)^2 = (1 - \overrightarrow{v}^2)(dt)^2$$

$4 - velocity$,

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} = (\frac{dx^0}{d\tau}, \frac{d\overrightarrow{x}}{d\tau})$$

there is a constraint

$$u^{\mu} u_{\mu} = \frac{dx^{\mu}}{d\tau} \frac{dx_{\mu}}{d\tau} = 1$$

Note that

$$\overrightarrow{u} = \frac{d\overrightarrow{x}}{d\tau} = \frac{d\overrightarrow{x}}{dt} (\frac{dt}{d\tau}) = \frac{1}{\sqrt{1 - v^2}} \overrightarrow{v} \approx \overrightarrow{v}, \qquad \text{for } v \ll 1$$

$4-velocity \qquad \Longrightarrow \qquad 4-momentum$

$$p^\mu = mu^\mu = (\frac{m}{\sqrt{1-v^2}}, \frac{m\overrightarrow{v}}{\sqrt{1-v^2}})$$

For $\quad v \ll 1,$

$$p^0 = \frac{m}{\sqrt{1-v^2}} = m(1+\frac{1}{2}v^2+...) = m+\frac{m}{2}v^2+..., \qquad \text{energy}$$

$$\overrightarrow{p} = m\overrightarrow{v}\frac{1}{\sqrt{1-v^2}} = m\overrightarrow{v}+... \qquad \text{momentum}$$

$$p^\mu = (E, \overrightarrow{p})$$

Note that

$$p^2 = E^2 - \overrightarrow{p}^2 = \frac{m^2}{1-v^2}[1-v^2] = m^2$$

Tensor analysis

Physical laws take the same forms in all inertial frames, if we write them in terms of tensors in Minkowski space.
Basically, tensors are

$$\text{tensors} \sim \text{product of vectors}$$

2 different types of vectors,

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}, \qquad x'_{\mu} = \Lambda_{\mu}^{\ \nu} x_{\nu}$$

multiply these vectors to get 2*nd* rank tensors,

$$T'^{\mu\nu} = \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta} T^{\alpha\beta}, \qquad T'_{\mu\nu} = \Lambda_{\mu}^{\ \alpha}\Lambda_{\nu}^{\ \beta} T_{\alpha\beta}, \qquad T'^{\mu}_{\nu} = \Lambda^{\mu}_{\ \alpha}\Lambda_{\nu}^{\ \beta} T^{\alpha}_{\beta}$$

In general,

$$T'^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m} = \Lambda^{\mu_1}_{\ \alpha_1}\cdots\Lambda^{\mu_n}_{\ \alpha_n}\Lambda_{\nu_1}^{\ \beta_1}\cdots\Lambda_{\nu_m}^{\ \beta_m} T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}$$

transformation of tensor components is **linear and homogeneous**.

Tensor operations; operation which preserves the tensor property

1. Multiplication by a constant, (cT) has the same tensor properties as T
2. Addition of tensor of same rank
3. Multiplication of two tensors
4. Contraction of tensor indices. For example, $T_{\mu}^{\mu\alpha\beta\gamma}$ is a tensor of rank 3 while $T_{\nu}^{\mu\alpha\beta\gamma}$ is a tensor or rank 5. This follows from the psudo-orthogonality relation
5. Symmetrization or anti-symmetrization of indices. This can be seen as follows. Suppose $T^{\mu\nu}$ is a second rank tensor,

$$T'^{\mu\nu} = \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}T^{\alpha\beta}$$

interchanging the indices

$$T'^{\nu\mu} = \Lambda^{\nu}_{\ \alpha}\Lambda^{\mu}_{\ \beta}T^{\alpha\beta} = \Lambda^{\nu}_{\ \beta}\Lambda^{\mu}_{\ \alpha}T^{\beta\alpha}$$

Then

$$T'^{\mu\nu} + T'^{\nu\mu} = \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\left(T^{\alpha\beta} + T^{\beta\alpha}\right)$$

symmetric tensor transforms into symmetric tensor. Similarly, the anti-symmetric tensor transforms into antisymmetic one.

6. $g_{\mu\nu}$, and $\varepsilon^{\alpha\beta\gamma\delta}$ have the property

$$\Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\, g_{\mu\nu} = g_{\alpha\beta}, \qquad \varepsilon^{\alpha\beta\gamma\delta}\det(\Lambda) = \Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu}\Lambda^{\gamma}_{\rho}\Lambda^{\delta}_{\sigma}\varepsilon^{\mu\nu\rho\sigma}$$

$g_{\mu\nu}$, and $\varepsilon^{\alpha\beta\gamma\delta}$ transform in the same way as tensors if $\det(\Lambda) = 1$.

Example: $M^{\mu\nu} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu}$, $\qquad F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ $\qquad$ second rank antisymmetric tensor.

Note that if all components of a tensor vanish in one inertial frame they vanish in all inertial frame. Suppose

$$f^{\mu} = ma^{\mu}$$

Define

$$t^{\mu} = f^{\mu} - ma^{\mu}$$

then t^{μ} vanish in this inertial frame. From

$$t'^{\mu} = f^{\mu'} - ma'^{\mu} = 0$$

we get

$$f^{\mu'} = ma'^{\mu}$$

Thus physical laws in tensor form are same in all inertial frames .

Action principle: actual trajectory of a partilce minimizes the action

Particle mechanics

A particle moves from x_1 at t_1 to x_2 at t_2. Write the action as

$$S = \int_{t_1}^{t_2} L(x, \dot{x})\, dt \qquad L : \text{Lagrangian}$$

$$\delta S = 0$$

For the least action, make a small change $x(t)$,

$$x(t) \longrightarrow x'(t) = x(t) + \delta x(t)$$

with end points fixed

$$i.e. \quad \delta x(t_1) = \delta x(t_2) = 0 \qquad \text{initial conditions}$$

Then

$$\delta S = \int_{t_1}^{t_2} [\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta(\dot{x})]\, dt$$

Note that

$$\delta\dot{x} = \dot{x}'(t) - \dot{x}(t) = \frac{d}{dt}[\delta(x)]$$

Integrate by parts and used the initial conditions

$$\delta S = \int_{t_1}^{t_2} [\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\frac{d}{dt}(\delta x)]\, dt = \int_{t_1}^{t_2} [\frac{\partial L}{\partial x} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}})]\delta x\, dt$$

For S to be minimum, we require

$$\frac{\delta S}{\delta x} = 0,$$

$$i.e. \quad \frac{\partial L}{\partial x} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = 0 \qquad \text{Euler-Lagrange} \quad \text{equation}$$

Conjugate momentum is

$$p \equiv \frac{\partial L}{\partial \dot{x}}$$

Hamiltonian is ,

$$H(p,q) = p\dot{x} - L(x,\dot{x})$$

Consider the simple case

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x}$$

Suppose

$$L = \frac{m}{2}(\frac{dx}{dt})^2 - V(x)$$

then

$$\frac{\partial L}{\partial x} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}), \qquad \Rightarrow -\frac{\partial V}{\partial x} = m\frac{d^2x}{dt^2}$$

Hamiltonian

$$H = p\dot{x} - L = \frac{m}{2}(\dot{x})^2 + V(x) \quad \text{where} \quad p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

is just the total energy.
Generalization

$$x(t) \rightarrow q_i(t), \quad i = 1, 2, ..., n$$

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt$$

Euler-Lagrange equations

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0 \quad i = 1, 2, ..., n$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, H = \Sigma p_i \dot{q}_i - L$$

Example: harmonic oscillator in 3-dimensions
Lagrangian

$$L = T - V = \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2 + \dot{x_3}^2) - \frac{mw^2}{2}(x_1^2 + x_2^2 + x_3^2)$$

and

$$\frac{\partial L}{\partial x_i} = -mw^2 x_i \quad \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$$

Euler-Langarange equation

$$m\ddot{x}_i = -mw^2 x_i$$

same as Newton's second law.

Field Theory

Field theory $\sim$ limiting case where number of degrees of freedom is infinite. $q_i(t) \to \phi(\overrightarrow{x}, t)$.

Action

$$S = \int L(\phi, \partial_\mu \phi)\, d^3x dt \qquad L : \text{Lagrangian} \quad \text{density}$$

Variation of action

$$\delta S = \int [\frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)]\, dx^4 = \int [\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)}]\delta\phi\, dx^4$$

Use $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$ and do the integration by part. then $\delta S = 0$ implies

$$\Longrightarrow \frac{\partial L}{\partial \phi} = \partial_\mu(\frac{\partial L}{\partial(\partial_\mu \phi)}) \qquad \text{Euler-Lagrange} \quad \text{equation}$$

Conjugate momentum density

$$\pi(\overrightarrow{x}, t) = \frac{\partial L}{\partial(\partial_0 \phi)}$$

and Hamiltonian density

$$H = \pi\dot{\phi} - L$$

Generalization to more than one field

$$\phi(\overrightarrow{x}, t) \longrightarrow \phi_i(\overrightarrow{x}, t), \qquad i = 1, 2, ..., n$$

Equations of motion are

$$\frac{\partial L}{\partial \phi_i} = \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \phi_i)}\right) \quad i = 1, 2, ..., n$$

and conjugate momentum

$$\pi_i(\overrightarrow{x}, t) = \frac{\partial L}{\partial(\partial_0 \phi_i)}$$

Hamiltonian density is

$$H = \sum_i \pi_i \dot{\phi}_i - L$$

Symmetry and Noether's Theorem

Continuous symmetry $\Longrightarrow$ conservation law, e.g. invariance under time translation

$$t \longrightarrow t + a, \quad \text{a} \quad \text{is arbitrary} \quad \text{constant}$$

gives energy conservation. Newton's equation for a force derived from a potential $V(\overrightarrow{x}, t)$ is,

$$m\frac{d^2\overrightarrow{x}}{dt^2} = -\overrightarrow{\nabla} V(\overrightarrow{x}, t)$$

Suppose V($\overrightarrow{x}$,t)=V($\overrightarrow{x}$) invariant under time translation. Then

$$m\frac{d\overrightarrow{x}}{dt} \cdot \left(\frac{d^2\overrightarrow{x}}{dt^2}\right) = -\left(\frac{d\overrightarrow{x}}{dt}\right) \cdot \overrightarrow{\nabla} V = -\frac{d}{dt}[V(\overrightarrow{x})]$$

Or

$$\frac{d}{dt}[\frac{1}{2}m(\frac{d\overrightarrow{x}}{dt})^2 + V(\overrightarrow{x})] = 0, \qquad \text{energy conservation}$$

Similarity, invariance under spatial translation

$$\overrightarrow{x} \to \overrightarrow{x} + \overrightarrow{a}$$

gives momentum conservation and invariance under rotations gives angular momentum conservation. Noether's theorem : unified treatment of symmetries in the Lagrangian formalism.

Particle mechanics

Action in classical mech

$$S = \int L(q_i, \dot{q}_i)\, dt$$

Suppose S is invariant under a continuous symmetry transformation,

$$q_i \to q_i' = f_{ij}(\alpha) q_j, \qquad \text{with} \quad f_{ij}(0) = \delta_{ij}$$

For $\alpha \ll 1$ then

$$q_i \to q_i' \simeq q_i + \alpha f_{ij}'(0) q_j = q_i + \delta q_i \qquad \text{with} \quad \delta q_i = \alpha f_{ij}'(0) q_j$$

The change of S

$$\delta S = \int [\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i]\, dt \quad \text{where} \quad \delta \dot{q}_i \to \frac{d}{dt}(\delta q_i)$$

Using the equation of motion,

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})$$

we can write δS as

$$\delta S = \int [\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i)]\, dt = \int [\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}\delta q_i)]\, dt$$

Thus $\delta S = 0 \Rightarrow$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}\delta q_i) = 0 \quad or \quad \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}\alpha f'_{ij}(0) q_j) = 0$$

This can be written as

$$or \quad \frac{dA}{dt} = 0, \quad A = \frac{\partial L}{\partial \dot{q}_i} \alpha f'_{ij}(0) q_j$$

A is the conserved charge.

Example: rotational symmetry in 3-dimension
action

$$S = \int L(x_i, \dot{x}_i)\, dt$$

Suppose S is invariant under rotation,

$$x_i \to x_i' = R_{ij} x_j, \qquad RR^T = R^T R = 1 \quad or \quad R_{ij} R_{ik} = \delta_{jk}$$

For infinitesmal rotations

$$R_{ij} = \delta_{ij} + \varepsilon_{ij} \qquad\qquad |\varepsilon_{ij}| \ll 1$$

Orthogonality requires,

$$(\delta_{ij} + \varepsilon_{ij})(\delta_{ik} + \varepsilon_{ik}) = \delta_{jk} \Longrightarrow \varepsilon_{jk} + \varepsilon_{kj} = 0 \quad i, e, \quad \varepsilon_{jk} \quad \text{is} \quad \text{antisymmetric}$$

We can compute the conserved charges as

$$J = \frac{\partial L}{\partial \dot{x}} \varepsilon_{ij} x_j = \varepsilon_{ij} p_i x_j$$

If we write $\varepsilon_{ij} = -\varepsilon_{ijk}\theta_k$

$$J = -\theta_k \varepsilon_{ijk} p_i x_j = -\theta_k J_k \qquad J_k = \varepsilon_{ijk} x_i p_j$$

J_k k-th component of angular momentum.

Field Theory

Start from the action

$$S = \int L(\phi, \partial_\mu \phi)\, d^4x$$

Symmetry transformation,

$$\phi(x) \rightarrow \phi'(x'),$$

which includes the change of coordinates,

$$x^\mu \rightarrow x'^\mu \neq x^\mu$$

Infinitesmal transformation

$$\delta\phi = \phi'\left(x'\right) - \phi\left(x\right), \qquad \delta x'^\mu = x'^\mu - x^\mu$$

need to include the change in the volume element

$$d^4x' = Jd^4x \qquad \text{where} \quad J = \left|\frac{\partial(x'_0, x'_1, x'_2, x'_3)}{\partial(x_0, x_1, x_2, x_3)}\right|$$

J :Jacobian for the coordinate transformation. For infinitesmal transformation,

$$J = |\frac{\partial x'^{\mu}}{\partial x^{\nu}}| \approx |g^{\mu}_{\nu} + \frac{\partial(\delta x^{\mu})}{\partial x^{\nu}}| \approx 1 + \partial_{\mu}(\delta x^{\mu})$$

we have used the relation

$$det(1+\varepsilon) \approx 1 + Tr(\varepsilon) \qquad for \quad |\varepsilon| \ll 1$$

Then

$$d^4x' = d^4x(1 + \partial_{\mu}(\delta x^{\mu}))$$

change in the action is

$$\delta S = \int [\frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial(\partial_{\mu}\phi)}\delta(\partial_{\mu}\phi) + L\partial_{\mu}(x^{\mu})]\, dx^4$$

Define the change of ϕ for fixed x^μ,

$$\bar{\delta}\phi(x) = \phi'(x) - \phi(x) = \phi'(x) - \phi'(x') + \phi'(x') - \phi(x) = -\partial^\mu \phi' \delta x_\mu + \delta\phi$$

$$or \quad \delta\phi = \bar{\delta}\phi + (\partial_\mu \phi)\delta x^\mu$$

Similarly,

$$\delta(\partial_\mu \phi) = \bar{\delta}(\partial_\mu \phi) + \partial_\nu(\partial_\mu \phi)\delta x^\nu$$

Operator $\bar{\delta}$ commutes with the derivative operator ∂_μ,

$$\bar{\delta}(\partial_\mu \phi) = \partial_\mu(\bar{\delta}\phi)$$

Then

$$\delta S = \int [\frac{\partial L}{\partial \phi}(\bar{\delta}\phi + (\partial_\mu \phi)\delta x^\mu) + \frac{\partial L}{\partial(\partial_\mu \phi)}(\bar{\delta}(\partial_\mu \phi) + \partial_\nu(\partial_\mu \phi)\delta x^\nu) + L\partial_\mu(\delta x^\mu)]\, d$$

Use equation of motion

$$\frac{\partial L}{\partial \phi} = \partial^\mu(\frac{\partial L}{\partial(\partial_\mu \phi)})$$

we get

$$\frac{\partial L}{\partial \phi}\bar{\delta}\phi + \frac{\partial L}{\partial(\partial_\mu \phi)})\bar{\delta}(\partial_\mu \phi) = \partial^\mu (\frac{\partial L}{\partial(\partial_\mu \phi)}\bar{\delta}\phi + \frac{\partial L}{\partial(\partial_\mu \phi)}\partial_\mu(\bar{\delta}\phi) = \partial^\mu [\frac{\partial L}{\partial(\partial_\mu \phi)}\bar{\delta}\phi]$$

Combine other terms as

$$\begin{aligned} [\frac{\partial L}{\partial \phi}(\partial_\nu \phi) + \frac{\partial L}{\partial(\partial_\mu \phi)}\partial_\nu(\partial_\mu \phi)]\delta x^\nu + L\partial_\nu(\delta x^\nu) &= (\partial_\nu L)\delta x^\nu + L\partial_\nu(\delta x^\nu) \\ &= \partial_\nu (L\delta x^\nu) \end{aligned}$$

Then

$$\delta S = \int dx^4 \partial_\mu [\frac{\partial L}{\partial(\partial_\mu \phi)}\bar{\delta}\phi + L\delta x^\mu]$$

and if δS=0 under the symmetry ransformation, then

$$\partial^\mu J_\mu = \partial^\mu [\frac{\partial L}{\partial(\partial_\mu \phi)}\bar{\delta}\phi + L\delta x^\mu] = 0 \qquad \text{current conservation}$$

Simple case: space-time translation

Here the coordinate transformation is,

$$x^\mu \to x'^\mu = x^\mu + a^\mu \Longrightarrow \phi'(x+a) = \phi(x)$$

then

$$\bar{\delta}\phi = -a^\mu \partial_\mu \phi$$

and the conservation laws take the form

$$\partial^\mu [\frac{\partial L}{\partial(\partial_\mu \phi)}(-a^\nu \partial_\nu \phi) + La^\mu] = -\partial^\mu (T_{\mu\nu} a^\nu) = 0$$

where

$$T_{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu \phi)}\partial_\nu \phi - g_{\mu\nu} L \quad \text{energy momentum tensor}$$

In particular,

$$T_{0i} = \frac{\partial L}{\partial(\partial_0 \phi)}\partial_i \phi$$

and

$$P_i = \int dx^3 \, T_{0i} \quad \text{momentum of the fields}$$