Quantum Field Theory

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Klein Gordon Equation

Classically,

$$E = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r})$$

Quantization : $E \to i\frac{\partial}{\partial t}$, $\vec{p} \to -i\vec{\nabla}$ and act on ψ

$$i\frac{\partial \psi}{\partial t} = [-\frac{1}{2m}\nabla^2 + V(\vec{r})]\psi \quad \text{Schrodinger equation}$$

Not good for relativistic system because x and time t are not on equal footing. For relativistic case, use

$$E^2 = \vec{p}^2 + m^2$$

$\Longrightarrow$

$$(\nabla^2 + m^2)\psi = -\partial_0^2\psi \tag{1}$$

Or

$$(\Box + m^2)\psi = 0, \text{ where } \quad \Box = \partial_0^2 - \nabla^2 = \partial^\mu\partial_\mu = \partial^2$$

This is known as Klein-Gordon equation.

Probablity interpretation

Klein-Gordon equation

$$(\partial_0^2 - \nabla^2 + m^2)\psi = 0$$

complex conjugate,

$$(\partial_0^2 - \nabla^2 + m^2)\psi^* = 0$$

gives the continuity equation,

$$\frac{\partial\rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

where

$$\rho = i(\psi\partial_0\psi^* - \psi\partial_0\psi^*), \qquad \vec{j} = (\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi)$$

Then

$$\frac{dP}{dt} = \int_V \frac{\partial\rho}{\partial t} d^3x = -\int_V \vec{\nabla} \cdot \vec{j}\, d^3x = -\oint_S \vec{j} \cdot \vec{ds} = 0 \qquad \text{if} \quad \vec{j} = 0, \quad \text{on} \quad S$$

P is conserved, probability ? But P is not positive For example,

$$\text{if} \quad \psi \tilde{} e^{iEt} \phi(x), \qquad \text{then} \qquad \rho = -2E \left| \phi(x) \right|^2 \leq 0$$

if we take the probabilty density to be $\rho = \psi\psi^*$ then it is not conserved,

$$\frac{d}{dt} \int \psi\psi^* d^3x \neq 0$$

Solutions to Klein-Gordon Equation

$$(\Box + m^2)\psi(x) = (-\nabla^2 + \partial_0^2 + m^2)\psi(x) = 0$$

plain wave solution,

$$\phi(x) = e^{-ipx} \quad if \qquad p_0^2 - P^2 - m^2 = 0 \quad \text{or} \quad p_0 = \pm\sqrt{\vec{p}^2 + m^2}$$

1. Positive energy solution: $P_0 = \omega_p = \sqrt{\vec{p}^2 + m^2}$, $\vec{p}$ *arbitrary*

$$\phi^{(+)}(x) = \exp\left(-i\omega_p t + i\overrightarrow{p}\cdot\overrightarrow{x}\right)$$

2. Negative energy solution: $P_0 = -\omega_p = -\sqrt{\vec{p}^2 + m^2}$

$$\phi^{(-)}(x) = \exp\left(i\omega_p t - i\overrightarrow{p}\cdot\overrightarrow{x}\right)$$

general solution is ,

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\left[a(k)e^{-ikx} + a(k)^+ e^{ikx}\right] \quad , \qquad kx = \omega_k t - \vec{k}\cdot\vec{x}$$

Dirac Equation

Dirac(1928) want first order in time derivative and first order in spatial coordinates. He assume an Ansatz

$$E = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta m = \vec{\alpha} \cdot \vec{p} + \beta m \tag{2}$$

where α_i, β are assumed to be matrices. Then

$$E^2 = \frac{1}{2}(\alpha_i \alpha_j + \alpha_j \alpha_i) p_i p_j + \beta^2 p^2 + (\alpha_i \beta + \beta \alpha_i) m$$

To get energy momentum relation, we require

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \tag{3}$$

$$\alpha_i \beta + \beta \alpha_i = 0 \tag{4}$$

$$\beta^2 = 1 \tag{5}$$

From Eq(3) we get

$$\alpha_i^2 = 1 \tag{6}$$

Togather with Eq(5) α_i, β all have eigenvalues ± 1. s

$$\alpha_1 \alpha_2 = -\alpha_2 \alpha_1 \implies \alpha_2 = -\alpha_1 \alpha_2 \alpha_1$$

Taking the trace

$$Tr \alpha_2 = -Tr\left(\alpha_1 \alpha_2 \alpha_1\right) = -Tr\left(\alpha_2 \alpha_1^2\right) = -Tr\left(\alpha_2\right)$$

Thus

$$Tr\left(\alpha_i\right) = 0 \tag{7}$$

Similarly,

$$Tr\left(\beta\right) = 0$$

α_i, β even dimension. Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are all traceless and anti-commuting. But we need 4 such matrices.

α_i, β all have to be 4×4 matrices. Bjoken and Drell choice

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Dirac equation ; $E \to i\frac{\partial}{\partial t}$, $\vec{p} \to -i\vec{\nabla}$

$$(-i\vec{\alpha} \cdot \nabla + \beta m)\psi = i\frac{\partial \psi}{\partial t}$$

For conveient, define a new set of matrices

$$\gamma^0 = \beta, \qquad \gamma^i = \beta\alpha_i$$

and in Bjorken and Drell notation,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{8}$$

Dirac equation

$$(-i\gamma^i\partial_i - i\gamma^0\partial_0 + m)\psi = 0, \qquad \text{or} \qquad (-i\gamma^\mu\partial_\mu + m)\psi = 0$$

Dirac equation in covariant form. Note that the anti-commutations are

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

Probability interpretation

From Dirac equation

$$-i\frac{\partial\psi^\dagger}{\partial t} = (\{-i\vec{\alpha}\cdot\vec{\nabla}+\beta m)\psi\})^\dagger$$

and

$$i(\frac{\partial\psi^\dagger}{\partial t}\psi+\psi^\dagger\frac{\partial\psi}{\partial t}) = \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla}+\beta m)\psi-\{(-i\vec{\alpha}\cdot\vec{\nabla}+\beta m)\psi\}^\dagger\psi$$

Integrate over space, we get

$$\begin{aligned} i\frac{d}{dt}\int d^3x(\psi^\dagger\psi) &= \int\{-i\psi^\dagger(\vec{\alpha}\cdot\vec{\nabla})\psi-i\{(\vec{\alpha}\cdot\vec{\nabla})\psi\}^\dagger\psi\}d^3x \\ &= -i\int\vec{\nabla}\psi^\dagger(\vec{\alpha}\cdot\vec{\nabla})\psi d^3x=0 \end{aligned}$$

The probability $\int d^3x(\psi^\dagger\psi)$ is conserved and positive.

Solution to Dirac equation

plane wave solution

$$\psi(x) = e^{-ipx} \begin{pmatrix} u \\ l \end{pmatrix}$$

u and l are 2 components column vector. Then

$$(\not{p} - m) \begin{pmatrix} u \\ l \end{pmatrix} = 0 \qquad \text{where} \qquad \not{p} = \gamma^\mu p_\mu$$

In Bjorken-Drell representation,

$$\begin{pmatrix} m & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m \end{pmatrix} \begin{pmatrix} u \\ l \end{pmatrix} = p_0 \begin{pmatrix} u \\ l \end{pmatrix}$$

Or

$$\begin{cases} (p_0 - m)u - (\vec{\sigma} \cdot \vec{p})l = 0 \\ -(\vec{\sigma} \cdot \vec{p})u + (p_0 + m)l = 0 \end{cases} \tag{9}$$

Homogeneous linear equations, non-trivial solution exists if

$$\begin{vmatrix} p_0 - m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & p_0 + m \end{vmatrix} = 0$$

$\Longrightarrow$

$$p_0^2 = \vec{p}^2 + m^2 \qquad \text{or} \qquad p_0 = \pm\sqrt{\vec{p}^2 + m^2}$$

1. $p_0 = E = \sqrt{\vec{p}^2 + m^2}$,

$$l = \frac{\vec{\sigma}\cdot\vec{p}}{E+m} u$$

solution,

$$\psi = e^{-ipx} \begin{pmatrix} u \\ l \end{pmatrix} = e^{-ipx} N \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \chi$$

χ arbitrary 2 components vector, and N is normalization constant .

2. Negative energy solution $p_0 = -E = -\sqrt{\vec{p}^2 + m^2}$,
solution,

$$\psi = e^{-ipx} N \begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix} \chi$$

The standard notation for these 4-component column vector, *spinors* are,

$$u(p.s) = N \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \chi_s \quad v(p,s) = e^{-ipx} N \begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix} \chi_s \quad N = \sqrt{E +$$

Dirac conjugate

Dirac equation in momentum space

$$(\not{p} - m)\psi(p) = 0$$

is not hermitian. In the Hermitian conjugate

$$\psi^\dagger(p)(\not{p}^\dagger - m) = 0$$

$\gamma_\mu' s$ are not hermitian,

$$\gamma_0^\dagger = \gamma_0 \quad \gamma_i^\dagger = -\gamma_i$$

But we can write

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0$$

Then

$$\psi^\dagger(p)(\gamma_0\gamma_\mu\gamma_0 p^\mu - m) = 0 \qquad \text{or} \qquad \psi^\dagger(p)\gamma_0(\gamma_\mu p^\mu - m) = 0$$

Or

$$\bar{\psi}(\not{p} - m) = 0 \quad \text{where} \quad \bar{\psi} = \psi^\dagger \gamma_0 \quad \text{Dirac conjugate}$$

Dirac equation under Lorentz transformation

Dirac equation is not invariant under Lorentz transformation. How Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

behaves under Lorentz transformation

$$x^\mu \rightarrow x^{'\mu} = \Lambda^\mu_\nu x^\nu$$

In the new coordinate system, the Dirac equation is of the form

$$(i\gamma^\mu \partial^{'}_\mu - m)\psi^{'}(x^{'}) = 0 \quad (10)$$

Assume

$$\psi^{'}(x^{'}) = S\psi(x)$$

Invert the Lorentz transformation

$$x^\gamma = \Lambda^\gamma_\mu x^{'\mu} \quad \Longrightarrow \quad \frac{\partial}{\partial x^{'\mu}} = \frac{\partial}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^{'\mu}} = \Lambda^\gamma_\mu \frac{\partial}{\partial x^\gamma}$$

Then Eq(10) becomes

$$(i\gamma^\mu \Lambda_\mu^\alpha \partial_\alpha - m) S\psi(x) = 0 \qquad \text{or} \qquad (i(S^{-1}\gamma^\mu S)\Lambda_\mu^\alpha \partial_\alpha - m)\psi(x) = 0$$

equivalent to the original Dirac equation, if

$$(S^{-1}\gamma^\mu S)\Lambda_\mu^\alpha = \gamma^\alpha \quad \text{or} \quad (S^{-1}\gamma^\mu S) = \Lambda_\alpha^\mu \gamma^\alpha \tag{11}$$

infinitesimal transformation

$$\Lambda_\nu^\mu = g_\nu^\mu + \epsilon_\nu^\mu + O(\epsilon^2) \qquad \text{with} \quad \left|\epsilon_\nu^\mu\right| << 1$$

Pseudo-othogonality implies

$$g_{\mu\nu}(g_\alpha^\mu + \epsilon_\alpha^\mu)(g_\beta^\nu + \epsilon_\beta^\nu) = g_{\alpha\beta}$$

Or

$$\epsilon_{\alpha\beta} + \epsilon_{\beta\alpha} = 0, \qquad \Longrightarrow \qquad \epsilon_{\alpha\beta} \text{ antisymmetric}$$

Write $S = 1 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu} + O(\epsilon^2)$ then $S^{-1} = 1 + \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}$ $\quad \sigma_{\mu\nu}$: 4×4 matrices. Then Eq(11) yields,

$$(1 + \frac{i}{4}\sigma_{\alpha\beta}\epsilon^{\alpha\beta})\gamma^{\mu}(1 - \frac{i}{4}\sigma_{\alpha\beta}\epsilon^{\alpha\beta}) = (g^{\mu}_{\alpha} + \epsilon^{\mu}_{\alpha})\gamma^{\alpha}$$

Or

$$\epsilon^{\alpha\beta}\frac{i}{4}[\sigma_{\alpha\beta}, \gamma^{\mu}] = \epsilon^{\mu}_{\alpha}\gamma^{\alpha} = \frac{1}{2}\epsilon^{\alpha\beta}(g^{\mu}_{\alpha}\gamma_{\beta} - g^{\mu}_{\beta}\gamma_{\alpha})$$

coefficient of $\varepsilon^{\alpha\beta}$

$$[\sigma_{\alpha\beta}, \gamma_{\mu}] = 2i(g_{\beta\mu}\gamma_{\alpha} - g_{\alpha\mu}\gamma_{\beta}) \tag{12}$$

Solution

$$\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_{\alpha}, \gamma_{\beta}]$$

satisfy Eq(12). Finite Lorentz transformation,

$$\psi^{'}(x^{'}) = S\psi(x), \qquad \text{with} \quad S = exp[-\frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}] \tag{13}$$

$$\sigma^{\dagger}_{\mu\nu} = \gamma_0\sigma_{\mu\nu}\gamma_0 \qquad \text{and} \qquad S^{\dagger} = \gamma^0 S^{-1}\gamma^0$$

S is not unitary. From $\psi^{'}(x^{'}) = S\psi$ we get

$$\psi^{\dagger'}(x^{'}) = \psi^{\dagger}S^{\dagger} = \psi^{\dagger}\gamma^0 S^{-1}\gamma^0, \qquad \text{or} \qquad \bar{\psi}'(x^{'}) = \bar{\psi}(x)S^{-1}$$

$\bar{\psi}$ Dirac conjugate

Fermion bilinears

The fermion bi-linears $\bar{\psi}_{\alpha}(x)\psi_{\beta}(x)$ has simple transformation. For example,

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)S^{-1}S\psi(x) = \bar{\psi}(x)\psi(x)$$

$\bar{\psi}(x)\psi(x)$ is Lorentz invariant. Similarly, .

$$
\begin{array}{ll}
\bar{\psi}\gamma_\mu\psi & \text{4-vector} \\
\bar{\psi}\gamma_\mu\gamma_5\psi & \text{axial vector} \\
\bar{\psi}\sigma_{\mu\nu}\psi & \text{2nd rank antisymmetric ensor} \\
\bar{\psi}\gamma_5\psi & \text{pseudo scalar}
\end{array}
$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

Hole Theory (Dirac 19)

To solve the problem with negative energy states, Dirac proposed that the vaccum is the one in which $E < 0$ states are all filled and $E > 0$ states are empty. Then Pauli exclusion principle will prevent an electron from moving into E<0 states. In this picture hole in the negative sea, i.e. absence of an electron with charge $-|e|$ with negative energy $-|E|$ is equivalent to a presence of a particle with energy $|E|$ and charge $+|e|$. This new particle is called "positron" and sometime also called $anti-particle$. This correspondence of particle and anti-particle is called *charge conjugation*.