Quantum Field Theory

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Quantization of Free Fields

Non-relativistic quantum mechanics

$$[q_i, p_j] = i\hbar\delta_{ij}$$

where p_j is defined by

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \qquad L : \text{Lagrangian}$$

Hamiltonian is

$$H = \sum_i p_i \dot{q}_i - L$$

In field theroy

$$q_i \longrightarrow \phi(x), \qquad L\left(q_i, \dot{q}_j\right) \longrightarrow \mathcal{L}\left(\phi, \partial_\mu \phi\right)$$

replace q_i by $\phi(x)$ and $L\left(q_i, \dot{q}_j\right)$ by $\mathcal{L}\left(\phi, \partial_\mu \phi\right)$.

Scalar field

Consider a scalar field ϕ satifies the Klein-Gordon equation

$$\left(\partial^\mu\partial_\mu + \mu^2\right)\phi = 0$$

Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu\phi\right)\left(\partial_\mu\phi\right) - \frac{\mu^2}{2}\phi^2$$

the Euler-Lagrange equation for this $\mathcal{L}$

$$\partial^\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\phi\right)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

gives the Klein-Gordon equation.

$$\partial^\mu\partial_\mu\phi + \mu^2\phi = 0$$

Canonical quantization

Conjugate momentum

$$\pi\left(\overrightarrow{x}, t\right) = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi\right)} = \left(\partial_0 \phi\right)$$

Impose commutation relations,

$$\begin{gathered}\left[\phi\left(\overrightarrow{x}, t\right), \pi\left(\overrightarrow{y}, t\right)\right] = i\delta^3\left(\overrightarrow{x} - \overrightarrow{y}\right), \qquad \left[\phi\left(\overrightarrow{x}, t\right), \phi\left(\overrightarrow{y}, t\right)\right] = 0, \\ \left[\pi\left(\overrightarrow{x}, t\right), \pi\left(\overrightarrow{y}, t\right)\right] = 0\end{gathered} \tag{1}$$

Hamiltonian density is

$$\mathcal{H} = \pi\partial_0\phi - \mathcal{L} = \frac{1}{2}\left[\left(\partial^0\phi\right)^2 + \left(\vec{\nabla}\phi\right)^2\right] + \frac{1}{2}\mu^2\phi^2$$

Mode expansion

To find physical consequence, expand in classical solutions,

$$\phi\left(\overrightarrow{x},t\right)=\int\frac{d^3k}{\sqrt{(2\pi)^3\,2w_k}}\left[a(\overrightarrow{k})e^{-ik\cdot x}+a^\dagger(\overrightarrow{k})e^{ik\cdot x}\right],\quad k_0=\sqrt{\overrightarrow{k}^2+\mu^2}$$

$a(k)$ and $a^\dagger(k)$ are operators
Solve $a(k)$ and $a^\dagger(k)$ in ϕ and $\partial_0\phi$,

$$a(k)=i\int\frac{d^3x\ e^{ik\cdot x}}{\sqrt{(2\pi)^3\,2w_k}}\overleftrightarrow{\partial_0}\,\phi\left(x\right)\qquad a^\dagger(k)=-i\int\frac{d^3x\ e^{-ik\cdot x}}{\sqrt{(2\pi)^3\,2w_k}}\overleftrightarrow{\partial_0}\,\phi\left(x\right)$$

where

$$f\overleftrightarrow{\partial_0}\,g\equiv f\partial_0 g-\left(\partial_0 f\right)g$$

Commutators

$$\left[a\left(\overrightarrow{k}\right), a^{\dagger}\left(\overrightarrow{k'}\right)\right] = \delta^{3}\left(\overrightarrow{k} - \overrightarrow{k'}\right)\ , \qquad \left[a\left(\overrightarrow{k}\right), a\left(\overrightarrow{k'}\right)\right] = 0$$

Same as harmonic oscillators.
The Hamiltonian is

$$H = \int d^{3}k\mathcal{H}_{k} = \frac{1}{2}\int d^{3}kw_{k}\left[a^{\dagger}\left(\overrightarrow{k}\right)a\left(\overrightarrow{k}\right) + a\left(\overrightarrow{k}\right)a^{\dagger}\left(\overrightarrow{k}\right)\right]$$

superposition of oscillators with frequency w_k.
momentum operator

$$\overrightarrow{p} = \frac{1}{2}\int d^{3}k\overrightarrow{k}\left[a^{\dagger}(k)\,a(k) + a(k)\,a^{\dagger}(k)\right] = \int d^{3}k\overrightarrow{p_{k}}$$

with

$$\overrightarrow{p_{k}} = \frac{\overrightarrow{k}}{2}\left[a^{\dagger}(k)\,a(k) + a(k)\,a^{\dagger}(k)\right]$$

Note

$$a(k)\, a^{\dagger}(k) = a^{\dagger}(k)\, a(k) + \delta^{3}(0)$$

Interpret $\delta^{3}(0)$ as

$$\delta^{3}\left(\overrightarrow{k}\right) = \int \frac{d^{3}x}{(2\pi)^{3}} e^{i\overrightarrow{k}\cdot\overrightarrow{x}}$$

as $\overrightarrow{k} \rightarrow 0$

$$\delta^{3}(0) = (2\pi)^{-3} \int d^{3}x = \frac{V}{(2\pi)^{3}}$$

V total volume of the system. Then

$$H = \int d^{3}k w_{k} \left[a^{\dagger}(k)\, a(k) + \frac{(2\pi)^{-3}}{2} V\right]$$

Last term will be dropped.
To achieve this more formally, use normal ordering.

Normal ordering

In normal ordering : $(\cdots)$: move all a†(k) to the left of a(k) .
For example,

.

$$: \quad a(k)a\dagger(k) := a\dagger(k)a(k)$$
$$: \quad a\dagger(k)a(k) := a\dagger(k)a(k)$$

Vaccum is defined by

$$a(k)|0\rangle = 0 \qquad \forall \overrightarrow{k} \qquad \Longrightarrow \langle 0|a\dagger(k) = 0$$

Then

$$\langle 0| : f\left(a, a^{\dagger}\right) : |0\rangle = 0$$

Define Hamiltonican by normaling ordering

$$H = \frac{1}{2}\int d^3 k w_k : [a\dagger(k)a(k) + a(k)a\dagger(k)] := \int d^3 k w_k a\dagger(k)a(k)$$

Similarly,

$$\overrightarrow{p} = \frac{1}{2}\int d^3 k \overrightarrow{p_k} : [a\dagger(k)a(k) + a(k)a\dagger(k)] := \int d^3 k \overrightarrow{p_k} a\dagger(k)a(k)$$

Then vacuum has zero energy and momentum.

Particle interpretation

State defined by

$$|\overrightarrow{k}\rangle = \sqrt{(2\pi)^3 2w_k} a^{\dagger}(k)|0\rangle$$

is eigenstate of H & $\overrightarrow{p}$,

$$H|\overrightarrow{k}\rangle = w_k|\overrightarrow{k}\rangle, \qquad \overrightarrow{p}|\overrightarrow{k}\rangle = \overrightarrow{k}|\overrightarrow{k}\rangle \qquad \text{where } w_k = \sqrt{\overrightarrow{k}^2 + \mu^2}$$

Interpret this as one-particle state because eigenvalues are related by

$$w_k^2 + \overrightarrow{k}^2 = \mu^2$$

Similarly,we can define 2 particle satate by

$$|\overrightarrow{k}_1, \overrightarrow{k}_2\rangle = \sqrt{(2\pi)^3\, 2w_{k_1}}\sqrt{(2\pi)^3\, 2w_{k_2}}\, a\dagger(\overrightarrow{k_1})a\dagger(\overrightarrow{k_2})|0\rangle$$

Bose statistics

Expand arbitrary state

$$|\Phi\rangle = \left[C_0 + \sum_{i=1}^{\infty}\int d^3k_1 ... d^3k_n\, C_n\left(k_1, k_2, ..., k_n\right) a^\dagger(\overrightarrow{k_1}) ... a^\dagger(\overrightarrow{k_n})|0\rangle\right]$$

$C_n\left(k_1, k_2, ..., k_n\right)$ the momentum space wavefunction.
Since

$$\left[a^{\dagger}(k_i), a^{\dagger}(k_j)\right] = 0$$

$$C_n(k_1, ..., k_i, ..., k_j ..., k_n) = C_n(k_1, ..., k_j, ..., k_i ..., k_n)$$

$C_n(k_1, k_2, ..., k_n)$ satisfies Bose statistics

Fermion fields

Dirac equation for free particles

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0 \qquad \text{or} \qquad \overline{\psi}\left(-i\gamma^\mu \overleftarrow{\partial_\mu} - m\right) = 0$$

Lagrangian density

$$\mathcal{L} = \overline{\psi_\alpha}\left(i\gamma^\mu \partial_\mu - m\right)_{\alpha\beta}\psi_\beta$$

Conjugate momentum

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \psi_\alpha\right)} = i\psi_\alpha^\dagger$$

If quantize with commutation relation will get Bose statistics
Impose anticommutation relations to get Fermi-Dirac statistics,

$$\left\{\pi_\alpha\left(\overrightarrow{x}, t\right), \psi_\beta\left(\overrightarrow{y}, t\right)\right\} = i\delta^3\left(\overrightarrow{x} - \overrightarrow{y}\right)\delta_{\alpha\beta} \ , \quad \left\{\psi_\alpha\left(\overrightarrow{x}, t\right), \psi_\beta\left(\overrightarrow{y}, t\right)\right\} = 0$$

Hamiltonian density

$$\mathcal{H} = \sum_{\alpha} \pi_\alpha \dot{\psi}_\alpha - \mathcal{L} = i\psi^\dagger \gamma_0 \gamma_0 \partial_0 \psi - \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi = \overline{\psi}\left(i\overrightarrow{\gamma} \cdot \overrightarrow{\nabla} + m\right)\psi$$

Mode expansion

Expansion in terms of classical solutions,

$$\psi(\overrightarrow{x},t) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left[b(p,s)\, u(p,s)\, e^{-ip\cdot x} + d^\dagger(p,s)\, \upsilon(p,s) \right.$$

$$\psi^\dagger(\overrightarrow{x},t) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left[b^\dagger(p,s)\, u^\dagger(p,s)\, e^{ip\cdot x} + d(p,s)\, \upsilon^\dagger(p,s) \right.$$

Invert these relations

$$b(p,s) = \int \frac{d^3x e^{ip\cdot x}}{(2\pi)^{\frac{3}{2}} \sqrt{2E_p}} u^\dagger(p,s)\, \psi(\overrightarrow{x},t)$$

$$d^\dagger(p,s) = \int \frac{d^3x e^{-ip\cdot x}}{(2\pi)^{\frac{3}{2}} \sqrt{2E_p}} \upsilon^\dagger(p,s)\, \psi(\overrightarrow{x},t)$$

Compute the anti-commutation relations,

$$\left\{b\left(p,s\right), b^{\dagger}\left(p',s'\right)\right\} = \delta_{ss'}\delta^{3}\left(\overrightarrow{p} - \overrightarrow{p}'\right), \qquad \left\{d\left(p,s\right), d^{\dagger}\left(p',s'\right)\right\} = \delta_{ss'}\delta$$

Hamiltonian

$$\begin{aligned} H &= \sum_{s} \int d^{3}p \mathcal{H}_{ps} \\ \text{where} \quad \mathcal{H}_{ps} &= E_{p}\left[b^{\dagger}\left(p,s\right) b\left(p,s\right) - d\left(p,s\right) d^{\dagger}\left(p,s\right)\right] \end{aligned}$$

Similarly,

$$\begin{aligned} \overrightarrow{p} &= \sum_{s} d^{3}p \, \overrightarrow{p}_{p} \\ \overrightarrow{p}_{p} &= \overrightarrow{p}\left[b^{\dagger}\left(p,s\right) b\left(p,s\right) - d\left(p,s\right) d^{\dagger}\left(p,s\right)\right] \end{aligned}$$

Commutators of H with $b^{\dagger}(p,s)$

$$\left[H, b^{\dagger}(p,s)\right] = \sum_{s'} d^3p' \left[b^{\dagger}(p',s')\, b(p',s'),\, b^{\dagger}(p,s)\right] E_p = b^{\dagger}(p,s)\, E_p$$

$$\left[\overrightarrow{p}, b^{\dagger}(p,s)\right] = \overrightarrow{p}\, b^{\dagger}(p,s)$$

$b^{\dagger}(p,s)$ creats a particle with E_p and $\overrightarrow{p}$ with relation $E_p = \sqrt{\overrightarrow{p}^2 + m^2}$.
$d^{\dagger}(p,s)$ creates a particle with same mass opposite charge as $b^{\dagger}(p,s)$.

Symmetry

The Lagrangian

$$\mathcal{L} = \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi$$

is invariant under,

$$\psi(x) \to e^{i\alpha}\psi(x) \implies \psi^{\dagger}(x) \to \psi^{\dagger}(x)\, e^{-i\alpha} \qquad \alpha : \text{some real constant}$$

Noether's theorem,$\implies$ conserved current

$$j_{\mu} = \overline{\psi}\gamma_{\mu}\psi$$

Conserved charge

$$Q = \int j_0(x)\, d^3x = \sum_s \int d^3p \left[N^{+}(p,s) - N^{-}(p,s)\right]$$

where

$$N^{+}_{ps} = b^{\dagger}(p,s)\, b(p,s) \qquad N^{-}_{ps} = d^{\dagger}(p,s)\, d(p,s)$$

are the number operators $\Longrightarrow$ particle and anti-particle have opposite "charge".

Electromagnetic fields

Start with free Maxwell's equations,

$$\nabla \cdot \overrightarrow{B} = 0, \quad \nabla \times \overrightarrow{E} + \frac{\partial \overrightarrow{B}}{\partial t} = 0, \tag{2}$$

$$\nabla \cdot \vec{E} = 0, \qquad \frac{1}{\mu_0} \nabla \times \overrightarrow{B} - \epsilon_0 \frac{\partial \overrightarrow{E}}{\partial t} = 0 \tag{3}$$

Introduce vector and scalar potentials $\overrightarrow{A}, \phi$ by

$$\overrightarrow{B} = \nabla \times \overrightarrow{A}, \qquad \overrightarrow{E} = -\nabla\phi - \frac{\partial \overrightarrow{A}}{\partial t} \tag{4}$$

These solve equations given in Eq(2). Write the relations in Eq(4) as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \text{with} \quad F^{0i} = \partial^0 A^i - \partial^i A^0 = -E^i, \quad F^{ij} = \partial^i A^j - \partial^j A^i =$$

Other two equations in Eq(3) are

$$\partial_\nu F^{\mu\nu} = 0, \quad \mu = 0, 1, 2, 3$$

For example

$$\mu = 0, \quad \partial_i F^{0i} = 0 \quad \Rightarrow \quad \nabla \cdot \overrightarrow{E} = 0$$

$$\mu = i, \quad \partial_\nu F^{i\nu} = 0 \quad \Rightarrow \quad \nabla \times \overrightarrow{B} - \frac{\partial \overrightarrow{E}}{\partial t} = 0$$

Note $c^2 = \dfrac{1}{\mu_0 \epsilon_0} = 1$. $F^{\mu\nu}$ is invariant under the transformation,

$$A^\mu \longrightarrow A^\mu + \partial^\mu \alpha \qquad \alpha = \alpha(x)$$

$\alpha(x)$ is arbitrary function.This is called gauge transformation. Given a set of $\overrightarrow{B}$ and $\overrightarrow{E}$ fields, $\overrightarrow{A}$,and ϕ are not unique. Different $\alpha(x)$ gives same $\overrightarrow{B}$ and $\overrightarrow{E}$ fields This property is usually called the **gauge invariance**.

Lagrangian density given by,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\overrightarrow{E}^2 - \overrightarrow{B}^2)$$

will give Maxwell equations as consequence of Euler-Lagrange equations.
Conjugate momenta

$$\pi_0 = \frac{\partial L}{\partial(\partial_0 A_0)} = 0, \qquad \pi^i(x) = \frac{\partial L}{\partial(\partial_0 A_i)} = -F^{0i} = E^i$$

A_0 does not have conjugate momenta and is not a dynamical degree of freedom.
Hamiltanian density is of the form,

$$\mathcal{H} = \pi^k \dot{A}_k - \mathcal{L} = \frac{1}{2}(\overrightarrow{E}^2 + \overrightarrow{B}^2) + (\overrightarrow{E} \cdot \nabla)A_0$$

Using $\overrightarrow{\nabla} \cdot \overrightarrow{E} = 0$, can write Hamiltonian as,

$$H = \int d^3x \mathcal{H} = \frac{1}{2}\int d^3x(\overrightarrow{E}^2 + \overrightarrow{B}^2)$$

Impose the commutation relation,

$$[\pi^i(\overrightarrow{x},t),\ A^j(\overrightarrow{y},t)] = -i\delta_{ij}\delta^3(\overrightarrow{x}-\overrightarrow{y}), \quad ...$$

But this is not consistent with $\overrightarrow{\nabla}\cdot\vec{E}=0$ because

$$[\nabla\cdot E(x,t),\ A_j(x,t)] = -i\partial_j\delta^3(x-y) \neq 0$$

$\delta-$function in momentum space

$$\partial_j\delta^3(\overrightarrow{x}-\overrightarrow{y}) = i\int\frac{d^3k}{(2\pi)^3}e^{i\overrightarrow{k}\cdot(\overrightarrow{x}-\overrightarrow{y})}k_j$$

To get zero for the commutator of $\nabla\cdot E$, replace,

$$\delta_{ij}\delta^3(\vec{x}-\vec{y}) \rightarrow \delta^{tr}_{ij}(\vec{x}-\vec{y}) = \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}(\delta_{ij}-\frac{k_ik_j}{k^2})$$

then

$$\partial_i\delta^{tr}_{ij}\delta^3(\vec{x}-\vec{y}) = i\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}(\cdot\vec{x}-\vec{y})}k_i(\delta_{ij}-\frac{k_ik_j}{k^2}) = 0$$

So commutator is modified to,

$$[E^i(x,t),\ A_j(y,t)] = -i\delta^{tr}_{ij}(\vec{x}-\vec{y})$$

which implies

$$[E^i(x,t),\ \overrightarrow{\nabla}\cdot\vec{A}(y,t)] = 0$$

Now that A_0 and $\overrightarrow{\nabla}\cdot\vec{A}$ commute with all operators, they must be C-number.Choose a gauge such that

$$A_0 = 0 \; and \; \nabla\cdot\vec{A} = 0 \qquad \text{radiation gauge}$$

In this gauge

$$\pi^i = \partial^i A^0 - \partial^0 A^i = -\partial^0 A^i$$

$$[\partial_0 A^i(\vec{x},t),\ A^j(\vec{y},t)] = i\delta^{tr}_{ij}(\vec{x}-\vec{y})$$

Mode expansion

Equation of motion $\partial_\nu F^{\mu\nu} = 0$ can be written as

$$\partial_\nu \left(\partial^\nu A^\mu - \partial^\mu A^\nu\right) = \Box A^\mu - \partial^\mu \left(\partial_\upsilon A^\upsilon\right) = 0$$

In radiaiton gauge ,

$$A_0 = 0, \qquad \overrightarrow{\nabla} \cdot \overrightarrow{A} = 0$$

the wave equation become

$$\Box\ \overrightarrow{A} = 0 \qquad \text{massless Klein-Gordon equation}$$

The general solution is

$$A(\vec{x}, t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_\lambda \epsilon(\vec{k}, \lambda)\left[a(k, \lambda)e^{-ikx} + a^+(k, \lambda)e^{ikx}\right] \qquad w = k_0$$

There are only two degree of freedom

$$\vec{\epsilon}(k, \lambda), \lambda = 1, 2 \quad \text{with } \vec{k} \cdot \vec{\epsilon}(k, \lambda) = 0$$

Standard choice

$$\vec{\epsilon}(k,\lambda)\cdot\vec{\epsilon}(k,\lambda') = \delta_{\lambda\lambda'}, \quad \vec{\epsilon}(-k,1) = -\vec{\epsilon}(k,1), \quad \vec{\epsilon}(-k,2) = \vec{\epsilon}(-k,2)$$

Solve for $a(k,\lambda)$ and $a^{+}(k,\lambda)$

$$a(k,\lambda) = i\int \frac{d^3x}{\sqrt{(2\pi)^3 2\omega}}[e^{ik\cdot x}\overleftrightarrow{\partial_0}\vec{\epsilon}(k,\lambda)\cdot\vec{A}(x)] \tag{5}$$

$$a^{+}(k,\lambda) = -i\int \frac{d^3x}{\sqrt{(2\pi)^3 2\omega}}[e^{-ik\cdot x}\overleftrightarrow{\partial_0}\vec{\epsilon}(k,\lambda)\cdot\vec{A}(x)] \tag{6}$$

Commutation relations are ,

$$[a(k,\lambda),\ a^{+}(k',\lambda')] = \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}'), \quad [a(k,\lambda),\ a(k',\lambda')] = 0,$$

The normal ordered form for the Hamiltonian and momentum operators are

$$H = \frac{1}{2}\int d^3x : (E^2 + B^2) := \int d^3k\omega \sum_{\lambda} a^+(k,\lambda)a(k,\lambda) \tag{7}$$

$$\vec{P} = \int d^3x : E \times B := \int d^3k\vec{k} \sum_{\lambda} a^+(k,\lambda)a(k,\lambda) \tag{8}$$

The vaccum is defined by

$$a(\vec{k},\lambda)|0>= 0 \quad \forall\vec{k},\lambda$$

Lorentz group

Dirac γ matrices are related to representations of Lorentz group.
Lorentz group : linear transformations

$$x^{\mu} \rightarrow x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$

leaves invariant the proper time

$$\tau^2 = (x^o)^2 - (\overrightarrow{x})^2 = x^{\mu} x^{\nu} g_{\mu\nu} = x^2$$

. This requires pseudo-orthogonality relation

$$\Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} g_{\mu\nu} = g_{\alpha\beta}$$

Generators

For infinitesmal transformation

$$\Lambda^{\mu}_{\alpha} = g^{\mu}_{\alpha} + \epsilon^{\mu}_{\alpha} \qquad \text{with} \quad |\varepsilon^{\mu}_{\alpha}| \ll 1$$

pseudo-orthogonality, implies, $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$.

Consider arbitrary function $f(x^\mu)$. Under infinitesimal Lorentz transformation,

$$\begin{aligned} f(x^\mu) \quad &\rightarrow \quad f(x'^\mu) = f(x^\mu + \varepsilon^\mu_\alpha x^\alpha) \approx f(x^\mu) + \varepsilon_{\alpha\beta} x^\beta \partial_\alpha f + \cdots \\ &= \quad f(x^\mu) + \frac{1}{2}\varepsilon_{\alpha\beta}[x^\beta \partial^\alpha - x^\alpha \partial^\beta] f(x) + \cdots \end{aligned}$$

Introduce operator $M_{\mu\nu}$,

$$f(x') = f(x) - \frac{i}{2}\varepsilon_{\alpha\beta} M^{\alpha\beta} f(x) + \cdots$$

then

$$M^{\alpha\beta} = -i(x^\alpha \partial^\beta - x^\beta \partial^\alpha), \qquad \text{generators of Lorentz group} \tag{9}$$

generators for $\alpha, \beta = 1, 2, 3$ these are angular momentum operator.
Commutators of generators,

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i\{g_{\beta\gamma} M_{\alpha\delta} - g_{\alpha\gamma} M_{\beta\delta} - g_{\beta\delta} M_{\alpha\gamma} + g_{\alpha\delta} M_{\beta\gamma}\}$$

Define

$$M_{ij} = \epsilon_{ijk} J_k, \qquad M_{oi} = K_i$$

Solve for J_i

$$J_i = \frac{1}{2}\epsilon_{ijk} M_{jk}$$

Commutator of $J_i's$,

$$[J_i, J_j] = \left(\frac{1}{2}\right)^2 \varepsilon_{ikl}\varepsilon_{jmn}[M_{kl}, M_{mn}] = i\epsilon_{ijk} J_k$$

Thus we can identify J_i as the angular momentum operator.
Similarly,

$$[K_i, K_j] = -i\epsilon_{ijk} J_k, \qquad [J_i, K_j] = i\epsilon_{ijk} K_k \tag{10}$$

Eqs(**??**,10) are called the Lorentz algebra.
Define the combinations

$$A_i = \frac{1}{2}(J_i + iK_i) \quad , B_i = \frac{1}{2}(J_i - iK_i)$$

commutation relations,

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \qquad [B_i, B_j] = i\epsilon_{ijk}B_k, \qquad [A_i, B_j] = 0$$

Lorentz algebra factorizes into 2 SU(2) algebras.
Representations~tensor products of SU(2) representation. (j_1, j_2).

Simple representations

1. $(\frac{1}{2}, 0)$ representation χ_a
   This 2-component object transform

$$A_i \chi_a = (\frac{\sigma_i}{2})_{ab} \chi_b \qquad \Longrightarrow \qquad \frac{1}{2}(J_i + iK_i)\chi_a = (\frac{\sigma_i}{2})_{ab}\chi_b$$

$$B_i \chi_a = 0 \qquad \Longrightarrow \qquad \frac{1}{2}(J_i - iK_i)\chi_a = 0$$

   Combining these realtions

$$\vec{J}\chi = (\frac{\vec{\sigma}}{2})\chi, \qquad \vec{K}\chi = -i(\frac{\vec{\sigma}}{2})\chi$$

2. $(0, \frac{1}{2})$ representation η_a

Similarly,

$$A_i \eta_a = 0 \qquad \Rightarrow \qquad \frac{1}{2}(J_i + iK_i)\eta_a = 0$$

$$B_i \eta_a = (\frac{\sigma_i}{2})_{ab} \qquad \Longrightarrow \quad \frac{1}{2}(J_i - iK_i)\eta_a = (\frac{\sigma_i}{2})_{ab}\eta_b$$

$$\vec{J}\eta = (\frac{\vec{\sigma}}{2})\eta, \qquad \vec{K}\eta = i(\frac{\vec{\sigma}}{2})\eta$$

Define a 4-component ψ as,

$$\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$$

Action of the Lorentz generators

$$\vec{J}\psi = \begin{pmatrix} \frac{\vec{\sigma}}{2} & 0 \\ 0 & \frac{\vec{\sigma}}{2} \end{pmatrix} \psi, \qquad \vec{K}\psi = \begin{pmatrix} -i\frac{\vec{\sigma}}{2} & 0 \\ 0 & i\frac{\vec{\sigma}}{2} \end{pmatrix} \psi \qquad (11)$$

ψ related to 4-component Dirac field , but with different representation for the γ matrices. .
Consider Dirac matrices in the form

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix} \quad \text{where} \quad \sigma^{\mu} = (1, \vec{\sigma}) \quad , \bar{\sigma}^{\mu} = (1, -\vec{\sigma})$$

More explicitly,

$$\gamma^{o} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

We can check that .

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In $\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$, χ is right-handed and η is left-handed. We can check that

$$\sigma_{0i} = i\gamma_0\gamma_1 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix}$$

$$\sigma_{ij} = i\gamma_i\gamma_j = i \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

In the Lorentz transformation of Dirac field,

$$\psi'(x') = S\psi = exp\{-\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}\} = exp\{-\frac{i}{4}(2\sigma_{0i}\varepsilon^{0i} + \sigma_{ij}\varepsilon^{ij})\}$$

Write $\varepsilon^{0i} = \beta^i, \quad \varepsilon^{ij} = \varepsilon^{ijk}\theta^k$

$$\sigma_{ij}\epsilon^{ij} = \varepsilon^{ijk}\theta^k\epsilon_{ijl} \begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix} = 2 \begin{pmatrix} \vec{\sigma}\cdot\vec{\theta} & 0 \\ 0 & \vec{\sigma}\cdot\vec{\theta} \end{pmatrix}$$

$$\sigma_0 i \varepsilon^0 i = \begin{pmatrix} -i\vec{\sigma}\cdot\vec{\beta} & 0 \\ 0 & i\vec{\sigma}\cdot\vec{\beta} \end{pmatrix}$$

$\Rightarrow$

$$-\frac{i}{4}(2\sigma_{0i}\varepsilon^{0i} + \sigma_{ij}\varepsilon^{ij}) = \frac{-i}{2}\begin{pmatrix} \vec{\sigma}\cdot\vec{\theta} - i\vec{\sigma}\cdot\vec{\beta} & 0 \\ 0 & \vec{\sigma}\cdot\vec{\theta} + i\vec{\sigma}\cdot\vec{\beta} \end{pmatrix}$$

More precisely,

$$\psi'(x') = S\psi = exp\{-\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}\}\psi \quad (12)$$

$$= \exp\left[\frac{-i}{2}\begin{pmatrix} \vec{\sigma}\cdot\vec{\theta} - i\vec{\sigma}\cdot\vec{\beta} & 0 \\ 0 & \vec{\sigma}\cdot\vec{\theta} + i\vec{\sigma}\cdot\vec{\beta} \end{pmatrix}\right]\psi \quad (13)$$

Write the Lorentz transformations in terms of generators,

$$L = \exp(-iM_{\mu\nu}\varepsilon^{\mu\nu})$$

then in terms of generators $\overrightarrow{J}$, $\overrightarrow{K}$

$$L = \exp\left[(-i)\left(\overrightarrow{J}\cdot\overrightarrow{\theta} + \overrightarrow{K}\cdot\overrightarrow{\beta}\right)\right]$$

From Eq(12) that for this ψ, $\overrightarrow{J}$, $\overrightarrow{K}$ are of the form,

$$\overrightarrow{J} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \qquad \overrightarrow{K} = \frac{1}{2}\begin{pmatrix} -i\vec{\sigma} & 0 \\ 0 & i\vec{\sigma} \end{pmatrix}$$

These are the same as those in Eq(11). $\Longrightarrow$ Diracwavefunction is just the representation $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ under the Lorentz group.

Futhermore, the right-handed components ~ $\left(\frac{1}{2}, 0\right)$ represenation, while left-handed components~ $\left(0, \frac{1}{2}\right)$ representation.