Quantum Electrodynamics

Ling-Fong Li

National Center for Theoretical Science

Quantum Electrodynamics
Lagrangian density for QED ,

$$\mathcal{L} = \overline{\psi}(x)\gamma^{\mu}\left(i\partial_{\mu} - eA_{\mu}\right)\psi(x) - m\overline{\psi}(x)\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Equations of motion are

$$\begin{aligned}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) &= eA_{\mu}\gamma^{\mu}\psi \qquad \text{non-linear coupled equations}\\ \partial_{v}F^{\mu\nu} &= e\overline{\psi}\gamma^{\mu}\psi\end{aligned}$$

Quantization
Write $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

$$\begin{aligned}\mathcal{L}_0 &= \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\ \mathcal{L}_{int} &= -e\overline{\psi}\gamma^{\mu}\psi A_{\mu}\end{aligned}$$

where $\mathcal{L}_0$,free field Lagrangian, $\mathcal{L}_{int}$ is interaction part.
Conjugate momenta for fermion

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_0\psi_{\alpha}\right)} = i\psi_{\alpha}^{\dagger}(x)$$

For em fields choose the gauge

$$\overrightarrow{\nabla}\cdot\overrightarrow{A} = 0$$

Conjugate mometa

$$\pi^i = \frac{\partial \mathcal{L}}{\partial (\partial_0 A^i)} = -F^{0i} = E^i$$

From equation of motion

$$\partial_\nu F^{0\nu} = e\psi^\dagger \psi \qquad \Longrightarrow \qquad -\nabla^2 A^0 = e\psi^\dagger\psi$$

A^0 is not an independent field ,

$$A^0 = e\int d^3x' \, \frac{\psi^\dagger(x',t)\,\psi(x',t)}{4\pi|\overrightarrow{x'} - \overrightarrow{x}|} = e\int \frac{d^3x'\,\rho(x',t)}{|\overrightarrow{x} - \overrightarrow{x'}|}$$

Commutation relations

$$\begin{aligned} \left\{\psi_\alpha(\overrightarrow{x},t), \psi^\dagger_\beta\left(\overrightarrow{x'},t\right)\right\} &= \delta_{\alpha\beta}\delta^3\left(\overrightarrow{x} - \overrightarrow{x'}\right) \qquad \left\{\psi_\alpha(\overrightarrow{x},t), \psi_\beta\left(\overrightarrow{x'},t\right)\right\} = \ldots = 0 \\ \left[\dot{A}_i(\overrightarrow{x},t), A_j\left(\overrightarrow{x'},t\right)\right] &= i\delta^{tr}_{ij}\left(\overrightarrow{x} - \overrightarrow{x'}\right) \end{aligned}$$

where

$$\delta^{tr}_{ij}(\vec{x} - \vec{y}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}(\delta_{ij} - \frac{k_i k_j}{k^2})$$

Commutators involving A_0

$$\left[A_0(\overrightarrow{x},t), \psi_\alpha\left(\overrightarrow{x'},t\right)\right] = e\int \frac{d^3x''}{4\pi|\overrightarrow{x} - \overrightarrow{x''}|}\left[\psi^\dagger\left(\overrightarrow{x''},t\right)\psi\left(\overrightarrow{x''},t\right), \psi_\alpha\left(\overrightarrow{x'},t\right)\right] = -\frac{e}{4\pi}\frac{\psi_\alpha\left(\overrightarrow{x'},t\right)}{|\overrightarrow{x} - \overrightarrow{x'}|}$$

Hamiltonian density

$$\begin{aligned}\mathcal{H} &= \frac{\partial\mathcal{L}}{\partial\left(\partial_0\psi_\alpha\right)}\dot{\psi}_\alpha + \frac{\partial\mathcal{L}}{\partial\left(\partial_0 A^k\right)}\dot{A}_k - \mathcal{L} \\ &= \psi^\dagger\left(-i\overrightarrow{\alpha}\cdot\overrightarrow{\nabla}+\beta m\right)\psi + \frac{1}{2}\left(\overrightarrow{E}^2+\overrightarrow{B}^2\right) + \overrightarrow{E}\cdot\overrightarrow{\nabla}A_0 + e\overline{\psi}\gamma^\mu\psi A_\mu\end{aligned}$$

and

$$H = \int d^3x\mathcal{H} = \int d^3x\{\psi^\dagger\left[\overrightarrow{\alpha}\cdot(-i\overrightarrow{\nabla}-e\overrightarrow{A})+\beta m\right]\psi + \frac{1}{2}\left(\overrightarrow{E}^2+\overrightarrow{B}^2\right)\}$$

A_0 does not appear in the interaction,
But if we write

$$\overrightarrow{E} = \overrightarrow{E_l}+\overrightarrow{E_t} \qquad \text{where} \quad \overrightarrow{E_l} = -\overrightarrow{\nabla}A_0 \qquad ,\overrightarrow{E_t} = -\frac{\partial\overrightarrow{A}}{\partial t}$$

Then

$$\frac{1}{2}\int d^3x\left(\overrightarrow{E}^2+\overrightarrow{B}^2\right) = \frac{1}{2}\int d^3x\overrightarrow{E_l}^2 + \int d^3x\left(\overrightarrow{E_t}^2+\overrightarrow{B}^2\right)$$

longitudinal part is

$$\frac{1}{2}\int d^3x\overrightarrow{E_l}^2 = \frac{e}{4\pi}\int d^3x d^3y\frac{\rho\left(\overrightarrow{x},t\right)\rho\left(\overrightarrow{y},t\right)}{\left|\overrightarrow{x}-\overrightarrow{y}\right|} \qquad \text{Coulomb interaction}$$

Without classical solutions, can not do mode expansion to get creation and annihilation operators We can only do perturbation theory.

Recall that the free field part $\overrightarrow{A}_0$ satisfy massless Klein-Gordon equation

$$\Box \overrightarrow{A}^{(0)} = 0$$

The solution is

$$\overrightarrow{A}^{(0)}(\vec{x}, t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_\lambda \overrightarrow{\epsilon}(k, \lambda)[a(k, \lambda)e^{-ikx} + a^+(k, \lambda)e^{ikx}] \qquad w = k_0 = |\overrightarrow{k}|$$

$$\vec{\epsilon}(k, \lambda), \ \lambda = 1, 2 \quad \text{with} \qquad \vec{k} \cdot \vec{\epsilon}(k, \lambda) = 0$$

Standard choice

$$\vec{\epsilon}(k, \lambda) \cdot \vec{\epsilon}(k, \lambda') = \delta_{\lambda\lambda'}, \quad \vec{\epsilon}(-k, 1) = -\vec{\epsilon}(k, 1), \quad \vec{\epsilon}(-k, 2) = \vec{\epsilon}(-k, 2)$$

It is convienent to write the mode expansion as,

$$A_\mu(\vec{x}, t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_\lambda \epsilon_\mu(k, \lambda)[a(k, \lambda)e^{-ikx} + a^+(k, \lambda)e^{ikx}]$$

where

$$\epsilon_\mu(k, \lambda) = (0, \vec{\epsilon}(k, \lambda))$$

Photon Propagator

Feynman propagatpr for photon is

$$\begin{aligned} iD_{\mu\nu}(x,x') &= \left\langle 0 \left| T\left(A_\mu(x) A_\nu(x')\right) \right| 0 \right\rangle \\ &= \theta(t-t') \left\langle 0 \left| A_\mu(x) A_\nu(x') \right| 0 \right\rangle + \theta(t'-t) \left\langle 0 \left| A_\nu(x') A_\mu(x) \right| 0 \right\rangle \end{aligned}$$

Using mode expansion,

$$D_{\mu\nu}(x,x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x'-x)}}{k^2+i\varepsilon} \sum_{\lambda=1}^{2} \epsilon_\nu(k,\lambda)\epsilon_\mu(k,\lambda)$$

polarization vectors $\epsilon_\mu(k,\lambda)$, $\lambda = 1,2$ are perpendicular to each other. Add 2 more unit vectors to form a complete set

$$\eta^\mu = (1,0,0,0), \qquad \hat{k}^\mu = \frac{k^\mu - (k\cdot\eta)\eta^\mu}{\sqrt{(k\cdot\eta)^2 - k^2}}$$

completeness relation is then,

$$\begin{aligned} \sum_{\lambda=1}^{2} \epsilon_\nu(k,\lambda)\epsilon_\mu(k,\lambda) &= -g_{\mu\nu} - \eta_\mu\eta_\nu - \hat{k}_\mu\hat{k}_\nu \\ &= -g_{\mu\nu} - \frac{k_\mu k_\nu}{(k\cdot\eta)^2 - k^2} + \frac{(k\cdot\eta)\left(k_\mu\eta_\nu + \eta_\mu k_\nu\right)}{(k\cdot\eta)^2 - k^2} - \frac{k^2\eta_\mu\eta_\nu}{(k\cdot\eta)^2 - k^2} \end{aligned}$$

If we define propagator in momentum space as

$$D_{\mu\nu}(x,x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x'-x)} D_{\mu\nu}(k)$$

then

$$D_{\mu\nu}(k) = \frac{1}{k^2 + i\varepsilon}\left[-g_{\mu\nu} - \frac{k_\mu k_\nu}{(k\cdot\eta)^2 - k^2} + \frac{(k\cdot\eta)\left(k_\mu\eta_\nu + \eta_\mu k_\nu\right)}{(k\cdot\eta)^2 - k^2} - \frac{k^2\eta_\mu\eta_\nu}{(k\cdot\eta)^2 - k^2}\right]$$

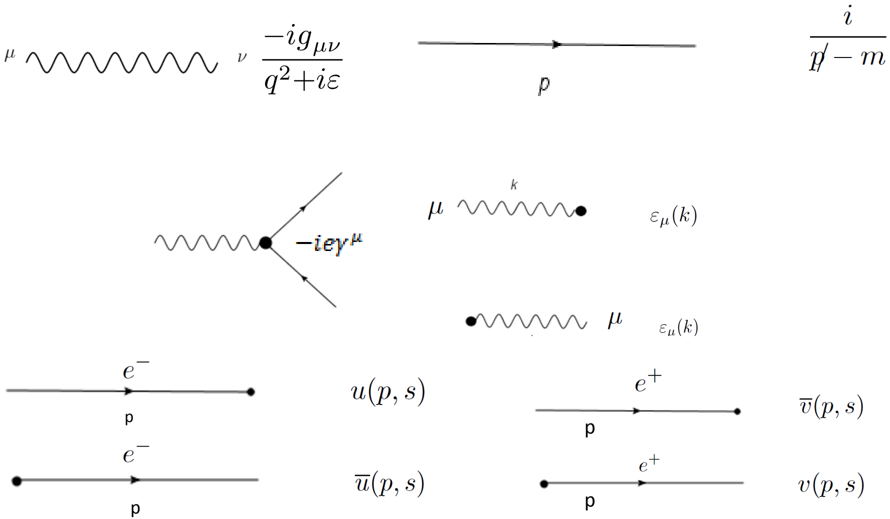
terms proportional to k_μ will not contribute to physical processes and the last term is of the form $\delta_{\mu 0}\ \delta_{\nu 0}$ will be cancelled by the Coulom interaction..

Feynman rule in QED

The interaction Hamiltonian is ,

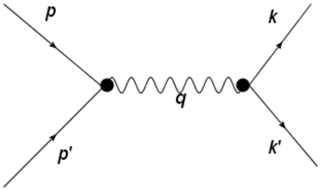
$$H_{int} = e \int d^3x \bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

The Feynman propagators, vertices and external wave functions are given below.



$e^+e^- \to \mu^+\mu^-$
Total Cross Section
momenta for this reaction

$$e^+(p') + e^-(p) \to \mu^+(k') + \mu^-(k)$$



Use Feynman rule to write the matrix element as

$$\begin{aligned} M(e^+e^- \quad &\to \quad \mu^+\mu^-) = \bar{v}(p',s')\left(-ie\gamma^\mu\right)u(p,s)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(k',r')\left(-ie\gamma^\nu\right)v(k,r) \\ &= \quad \frac{ie^2}{q^2}\bar{v}(p',s')\gamma^\mu u(p,s)\,\bar{u}(k',r')\gamma_\mu v(k,r) \end{aligned}$$

where $q = p + p'$. Notet that electron vertex have property,

$$q_\mu \bar{v}(p')\gamma^\mu u(p) = (p+p')_\mu \bar{v}(p')\gamma^\mu u(p) = \bar{v}(p')\left(\not{p} + \not{p}'\right)u(p) = 0$$

This shows the term proportional to photon momentum q^μ will not contribute in the physical processes. For cross section, we need M^* which contains factor $\left(\bar{v}\gamma^\mu u\right)^*$

$$\left(\bar{v}\gamma^\mu u\right)^* = u^\dagger \left(\gamma^\mu\right)^\dagger \left(\gamma_0\right)^\dagger v = u^\dagger \gamma_0 \gamma^\mu v = \bar{u}\gamma^\mu v$$

More generally,

$$\left(\bar{v}\Gamma u\right)^* = \bar{u}\bar{\Gamma} v, \qquad \text{with} \;\; \bar{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$$

It is easy to see

$$\bar{\gamma}_\mu = \gamma_\mu$$

$$\overline{\gamma_\mu \gamma_5} = -\gamma_\mu \gamma_5$$

$$\overline{\not{a}\not{b}\cdots\not{p}} = \not{p}\cdots\not{b}\not{a}$$

unpolarized cross section which requires the spin sum,

$$\sum_s u_\alpha\left(p,s\right)\bar{u}_\beta\left(p,s\right) = \left(\not{p}+m\right)_{\alpha\beta}$$

$$\sum_s v_\alpha\left(p,s\right)\bar{v}_\beta\left(p,s\right) = \left(\not{p}-m\right)_{\alpha\beta}$$

A typical calculation is,

$$\begin{aligned}
& \sum_{s,s'} \bar{v}_\alpha(p',s') \left(\gamma^\mu\right)_{\alpha\beta} u_\beta\left(p,s\right) \bar{u}_\rho(p,s) \left(\gamma^\nu\right)_{\rho\sigma} v_\sigma\left(p,s\right) \\
= \quad & \sum_{s'} \bar{v}_\alpha(p',s') \left(\gamma^\mu\right)_{\alpha\beta} \left(\not{p}+m\right)_{\beta\rho} \left(\gamma^\nu\right)_{\rho\sigma} v_\sigma\left(p,s\right) \\
= \quad & \left(\gamma^\mu\right)_{\alpha\beta} \left(\not{p}+m\right)_{\beta\rho} \left(\gamma^\nu\right)_{\rho\sigma} \left(\not{p}-m\right)_{\sigma\alpha} \\
= \quad & Tr\left[\gamma^\mu\left(\not{p}+m\right)\gamma^\nu\left(\not{p}-m\right)\right]
\end{aligned}$$

trace of product of γ matrices.

$$Tr\left(\gamma^\mu\right) = 0$$

$$Tr\left(\gamma^\mu\gamma^\nu\right) = 4g^{\mu\nu}$$

$$Tr\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = 4\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}\right)$$

$$\begin{aligned}
& Tr\left(\not{a}_1\not{a}_2\cdots\not{a}_n\right) \\
= \quad & \left(a_1\cdot a_2\right) Tr\left(\not{a}_3\cdots\not{a}_n\right) - \left(a_1\cdot a_3\right) Tr\left(\not{a}_2\cdots\not{a}_n\right) + \cdots + \left(a_1\cdot a_n\right) Tr\left(\not{a}_2\not{a}_3\cdots\not{a}_{n-1}\right), \qquad n \;\; \text{even} \\
= \quad & 0 \qquad n \quad \text{odd}
\end{aligned}$$

With these tools

$$\frac{1}{4}\sum_{spin}\left|M(e^+e^-\to\mu^+\mu^-)\right|^2 = \frac{e^4}{q^4} Tr\left[\left(\not{p}'-m_e\right)\gamma^\mu\left(\not{p}+m_e\right)\gamma^\nu\right] Tr\left[\left(\not{k}'+m_\mu\right)\gamma_\mu\left(\not{k}+m_\mu\right)\gamma^\nu\right]$$

$$Tr\left[\left(\not{p}'-m_e\right)\gamma^\mu\left(\not{p}+m_e\right)\gamma^\nu\right] = Tr\left[\not{p}'\gamma^\mu\not{p}\gamma^\nu\right]-m^2\,Tr\left[\gamma^\mu\gamma^\nu\right]$$
$$= 4\left[p'^\mu p^\nu - g^{\mu\nu}\left(p\cdot p'\right)+p^\mu p'^\nu\right]-4m_e^2 g^{\mu\nu}$$

$$Tr\left[\left(\not{k}'+m_\mu\right)\gamma_\mu\left(\not{k}-m_\mu\right)\gamma^\nu\right] = Tr\left[\not{k}'\gamma_\mu\not{k}\gamma^\nu\right]-m_\mu^2\,Tr\left[\gamma_\mu\gamma^\nu\right]$$
$$= 4\left[k'^\mu k^\nu - g^{\mu\nu}\left(k\cdot k'\right)+k^\mu k'^\nu\right]-4m_\mu^2 g^{\mu\nu}$$

for energies $\gg m_\mu$.

$$\frac{1}{4}\sum_{spin'}\left|M(e^+e^-\to\mu^+\mu^-)\right|^2 = 8\frac{e^4}{q^4}\left[(p\cdot k)\left(p'\cdot k'\right)+(p'\cdot k)\left(p\cdot k'\right)\right]$$

In center of mass,

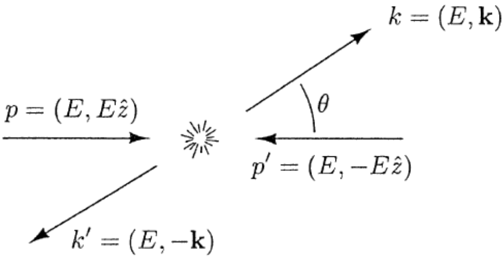
$$p_\mu = (E,0,0,E)\,,\qquad p'_\mu = (E,0,0,-E)$$

$$k_\mu = \left(E,\vec{k}\right),\qquad k'_\mu = \left(E,-\vec{k}\right),\qquad \text{with } \vec{k}\cdot\hat{z} = E\cos\theta$$

If we set $m_\mu = 0$, $E = \left|\vec{k}\right|$ and

$$q^2 = (p+p')^2 = 4E^2,\qquad p\cdot k = p'\cdot k' = E^2\left(1-\cos\theta\right),$$

$$p'\cdot k = p\cdot k' = E^2\left(1+\cos\theta\right)$$



Then

$$\frac{1}{4}\sum_{spin'} |M|^2 = \frac{8e^4}{16E^4}\left[E^4(1-\cos\theta)^2 + E^4(1+\cos\theta)^2\right]$$
$$= e^4\left(1+\cos^2\theta\right)$$

Note that under the parity $\theta \to \pi - \theta$. this matrix element conserves the parity
The cross section is

$$d\sigma = \frac{1}{I}\frac{1}{2E}\frac{1}{2E}(2\pi)^4\delta^4(p+p'-k-k')\frac{1}{4}\sum_{spin'}|M|^2\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}$$

use the $\delta-$function to carry out integrations . introduce the quantity ρ, called the **phase space**, given by

$$\rho = \int (2\pi)^4 \delta^4 (p + p' - k - k') \frac{d^3 k}{(2\pi)^3 2\omega} \frac{d^3 k'}{(2\pi)^3 2\omega'}$$
$$= \frac{1}{4\pi^2} \int \delta (2E - \omega - \omega') \frac{d^3 k}{4\omega\omega'} = \frac{1}{32\pi^2} \int \delta (E - \omega) \frac{k^2 dk d\Omega}{\omega^2} = \frac{d\Omega}{32\pi^2}$$

The flux factor is

$$I = \frac{1}{E_1 E_2} \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = \frac{1}{E^2} 2E^2 = 2$$

The differential crossection is then

$$d\sigma = \frac{1}{2} \frac{1}{4E^2} \left(\frac{1}{4} \sum_{spin'} |M|^2 \right) \frac{d\Omega}{32\pi^2}$$

Or

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2} \left(1 + \cos^2 \theta\right)$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant. The total cross section is

$$\sigma \left(e^+ e^- \to \mu^+ \mu^-\right) = \frac{\alpha^2 \pi}{3E^2}$$

Or

$$\sigma \left(e^+ e^- \to \mu^+ \mu^-\right) = \frac{4\alpha^2 \pi}{3s} \qquad \text{with} \qquad s = (p_1 + p_2)^2 = 4E^2$$

$e^+e^- \longrightarrow hadrons$

One of the interesting processess in e^+e^- collider is the reaction

$$e^+e^- \longrightarrow hadrons$$

According to QCD, theory of strong interaciton, this processes will go through

$$e^+e^- \longrightarrow q\bar{q}$$

and then $q\bar{q}$ trun into hadrons. Since coupling of γ to $q\bar{q}$ differs from the coupling to $\mu^+\mu^-$ only in their charges cross section for $q\bar{q}$ as

$$\sigma\left(e^+e^- \longrightarrow q\bar{q}\right) = 3\left(Q_q^2\right)\frac{4\alpha^2\pi}{3s} = 3\left(Q_q^2\right)\sigma\left(e^+e^- \longrightarrow \mu^+\mu^-\right)$$

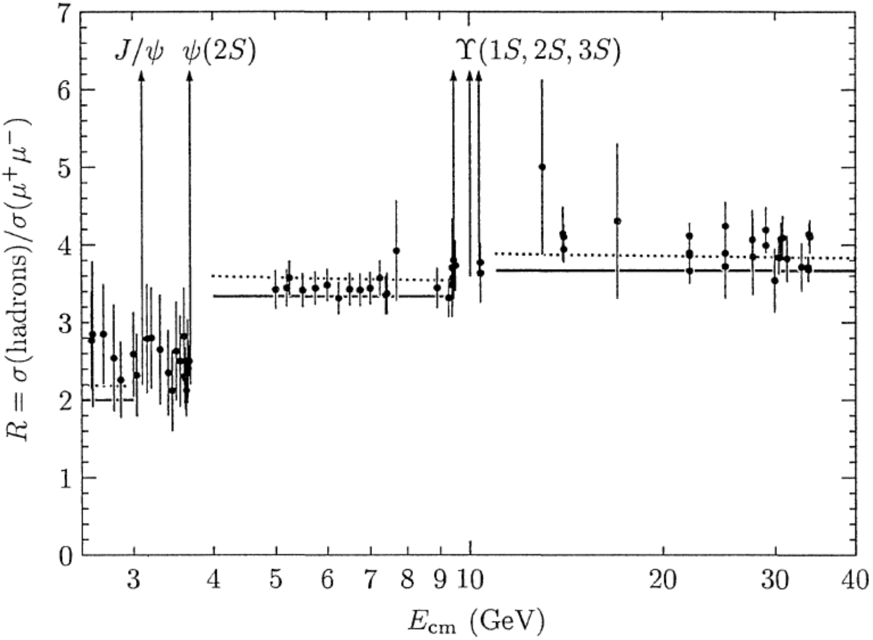
Q_q is electric charge of quark q. The factor of 3 because each quark has 3 colors. Then

$$\frac{\sigma\left(e^+e^- \longrightarrow hadrons\right)}{\sigma\left(e^+e^- \longrightarrow \mu^+\mu^-\right)} = 3\left(\sum_i Q_i^2\right)$$

Summation is over quarks which are allowed by the avaliable energies. e. g., for energy below the the charm quark only u, d, and s quarks should be included,

$$\frac{\sigma\left(e^+e^- \longrightarrow hadrons\right)}{\sigma\left(e^+e^- \longrightarrow \mu^+\mu^-\right)} = 3\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] = 2$$

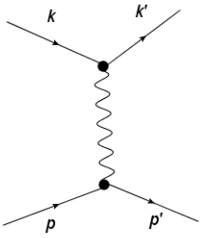
which is not far from the reality.



$R = \sigma(\text{hadrons})/\sigma(\mu^+\mu^-)$
J/ψ
$\psi(2S)$
$\Upsilon(1S, 2S, 3S)$
E_{cm} (GeV)
0
1
2
3
4
5
6
7
3
4
5
6
7
8
9
10
20
30
40

$\underline{ep \to ep}$,

$$e\,(k) + p\,(p) \longrightarrow e\,(k') + p\,(p')$$



Proton has strong interaction. First consider proton has no strong interaction and include strong interaction later. The lowest order contribution is ,

$$\begin{aligned} M(e+p \quad &\rightarrow \quad e+p) = \bar{u}(p',s')\,(-ie\gamma^{\mu})\,u\,(p,s)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(k',r')\,(-ie\gamma^{\nu})\,u\,(k,r) \\ &= \quad \frac{ie^2}{q^2}\bar{u}(p',s')\gamma^{\mu}u\,(p,s)\,\bar{u}(k',r')\gamma_{\mu}u\,(k,r) \end{aligned}$$

where $q = k - k'$. For unploarized cross section, sum over the spins ,

$$\frac{1}{4}\sum_{spin}|M(e+p\rightarrow e+p)|^2 = \frac{e^4}{q^4}Tr\left[\left(\not{p}'+M\right)\gamma^{\mu}\left(\not{p}+M\right)\gamma^{\nu}\right]Tr\left[\left(\not{k}'+m_e\right)\gamma_{\mu}\left(\not{k}+m_e\right)\gamma^{\nu}\right]$$

Again neglect m_e. Compute the traces

$$Tr\left[\not{k}'\gamma_\mu\not{k}\gamma^\nu\right] = 4\left[k'^\mu k^\nu - g^{\mu\nu}\left(k\cdot k'\right) + k^\mu k'^\nu\right]$$

$$Tr\left[\left(\not{p}' + M\right)\gamma^\mu\left(\not{p} + M\right)\gamma^\nu\right] = 4\left[p'^\mu p^\nu - g^{\mu\nu}\left(p\cdot p'\right) + p^\mu p'^\nu\right] + 4M^2 g^{\mu\nu}$$

Then

$$\frac{1}{4}\sum_{spin}\left|M(e+p\to e+p)\right|^2 = \frac{e^4}{q^4}\left\{8\left[\left(p\cdot k\right)\left(p'\cdot k'\right) + \left(p'\cdot k\right)\left(p\cdot k'\right)\right] - 8M^2\left(k\cdot k'\right)\right\}$$

More useful to use the laboratoy frame

$$p_\mu = (M, 0, 0, 0)\,, \qquad k_\mu = \left(E, \vec{k}\right), \qquad k'_\mu = \left(E', \vec{k'}\right)$$

Then

$$p \cdot k = ME, \qquad p.k' = ME', \qquad k \cdot k' = EE'\,(1 - \cos\theta)$$

$$p' \cdot k' = (p + k - k')\,.k' = p.k' + k \cdot k', \qquad p' \cdot k = (p + k - k')\,.k = p.k - k \cdot k'$$

$$q^2 = (k - k')^2 = -2k \cdot k' = -2EE'\,(1 - \cos\theta)$$

Differential cross section is

$$d\sigma = \frac{1}{I}\frac{1}{2p_0}\frac{1}{2k_0}(2\pi)^4\delta^4(p + k - p' - k')\frac{1}{4}\sum_{spin'}|M|^2\frac{d^3p'}{(2\pi)^3 2p'_0}\frac{d^3k'}{(2\pi)^3 2k'_0}$$

The phase space is

$$\begin{aligned} \rho &= \int (2\pi)^4\delta^4(p + k - p' - k')\frac{d^3p'}{(2\pi)^3 2p'_0}\frac{d^3k'}{(2\pi)^3 2k'_0} \qquad (1) \\ &= \frac{1}{4\pi^2}\int \delta\left(p_0 + k_0 - p'_0 - k'_0\right)\frac{d^3k'}{2p'_0 2k'_0} \end{aligned}$$

where

$$p'_0 = \sqrt{M^2 + \left(\vec{p} + \vec{k} - \vec{k'}\right)^2} = \sqrt{M^2 + \left(\vec{k} - \vec{k'}\right)^2}$$

Use the momenta in lab frame,

$$\begin{aligned}\rho &= \frac{1}{4\pi^2}\int \delta\left(M+E-p_0'-E'\right)\frac{k'^2 dk' d\Omega}{2p_0' 2E'} \\ &= \frac{1}{4\pi^2}\int \delta\left(M+E-p_0'-E'\right)\frac{d\Omega E' dE'}{p_0'}\end{aligned}$$

Let

$$x = -E + p_0' + E'$$

Then

$$dx = dE'(1+\frac{dp_0'}{dE'}) = dE'\left(\frac{p_0'+E'-E\cos\theta}{p_0'}\right)$$

and

$$\rho = \frac{1}{4\pi^2}\int \delta\left(x-M\right)\frac{d\Omega E' dx}{\left(p_0'+E'-E\cos\theta\right)} = \frac{1}{4\pi^2}\frac{d\Omega E'}{M+E\left(1-\cos\theta\right)}$$

From the argument of the $\delta-$function we get the relation, $M = x = -E + p_0' + E'$
Solve for E',

$$E' = \frac{ME}{E\left(1-\cos\theta\right)+M} = \frac{E}{1+\left(\frac{2E}{M}\right)\sin^2\frac{\theta}{2}}$$

The phase space is then

$$\rho = \frac{d\Omega}{4\pi^2}\frac{ME}{\left(M+E\left(1-\cos\theta\right)\right)^2} = \frac{d\Omega}{4\pi^2}\frac{E'^2}{ME}$$

The flux factor is

$$I = \frac{1}{ME}p\cdot k = 1$$

The differential cross section is then

$$d\sigma = \frac{1}{I}\frac{1}{2p_0}\frac{1}{2k_0}(2\pi)^4\delta^4(p+k-p'-k')\frac{1}{4}\sum_{spin'}|M|^2\frac{d^3p'}{(2\pi)^3 2p_0'}\frac{d^3k'}{(2\pi)^3 2k_0'}$$

Or

$$\frac{d\sigma}{d\Omega} = \frac{1}{4ME}\frac{1}{4\pi^2}\frac{E'^2}{ME}\frac{1}{4}\sum_{spin'}|M|^2 = \left(\frac{E'}{E}\right)^2\frac{1}{16\pi^2 M^2}\frac{e^4}{q^4}\left\{8\left[(p\cdot k)\left(p'\cdot k'\right)+(p'\cdot k)\left(p\cdot k'\right)\right]-8M^2(k\cdot k')\right\}$$

It is staightforward to get

$$
\begin{aligned}
& \left[(p\cdot k)\left(p'\cdot k'\right)+(p'\cdot k)\left(p\cdot k'\right)\right]-M^2\left(k\cdot k'\right) \\
= \quad & \left[(p\cdot k)\left(p+k-k'\right)\cdot k'+(p\cdot k')\left(p+k-k'\right)\cdot k-M^2\left(k\cdot k'\right)\right] \\
= \quad & 2EE'M^2+(k\cdot k')\left(p\cdot q-M^2\right) \\
= \quad & 2EE'M^2+M^2EE'\left(1-\cos\theta\right)\left(-\frac{q^2}{2M^2}-1\right) \\
= \quad & 2EE'M^2\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left(\frac{E'}{E}\right)^2\frac{\alpha^2}{M^2}\frac{1}{\left(4EE'\sin^2\frac{\theta}{2}\right)^2}2EE'M^2\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \\
&= \frac{\alpha^2}{4}\frac{E'}{E^3}\frac{1}{\sin^4\frac{\theta}{2}}\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]
\end{aligned}
$$

Or

$$
\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{4E^2}\frac{1}{\sin^4\frac{\theta}{2}}\frac{\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]}{\left[1+\left(\frac{2E}{M}\right)\sin^2\frac{\theta}{2}\right]}
$$

Now include the strong interaction. Use the fact that the γpp interaction is local to parametrize the γpp matrix element as

$$\left\langle p' \left| J_\mu \right| p \right\rangle = \bar{u}(p', s') \left[\gamma^\mu F_1\left(q^2\right) + \frac{i\sigma_{\mu\nu} q^\nu}{2M} F_2\left(q^2\right) \right] u\left(p, s\right) \qquad \text{with} \qquad q = p - p' \tag{2}$$

Lorentz covariance and current conservation have been used. Another useful relation is the Gordon decomposition

$$\bar{u}(p')\gamma_\mu u\left(p\right) = \bar{u}(p') \left[\frac{\left(p + p'\right)^\mu}{2m} + \frac{i\sigma^{\mu\nu}\left(p' - p\right)_\nu}{2m} \right] u\left(p\right)$$

$F_1\left(q^2\right)$, charge form factor
$F_2\left(q^2\right)$, magnetic form factor .
Note that $F_1\left(q^2\right) = 1$ and $F_2\left(q^2\right) = 0$ correspond to point particle.
The charge form factor satifies the condition $F_1\left(0\right) = 1$. From

$$Q\left|p\right\rangle = \left|p\right\rangle$$

we get

$$\left\langle p' \left| Q \right| p \right\rangle = \left\langle p' | p \right\rangle = 2E\left(2\pi\right)^3 \delta^3\left(\vec{p} - \vec{p}'\right)$$

On the other hand from Eq(2) we see that

$$\begin{aligned}\langle p' |Q| p\rangle &= \int d^3x \langle p' |J_0(x)| p\rangle = \int d^3x \langle p' |J_0(0)| p\rangle e^{i(p'-p)\cdot x} \\ &= (2\pi)^3 \delta^3\left(\vec{p}-\vec{p}'\right) \bar{u}(p',s')\gamma_0 u(p,s) F_1(0) \\ &= 2E(2\pi)^3 \delta^3\left(\vec{p}-\vec{p}'\right) F_1(0)\end{aligned}$$

compare two equations $\Longrightarrow F_1(0) = 1$. To gain more insight, write Q in terms of charge density

$$Q = \int d^3x \rho(x) = \int d^3x J_0(x)$$

Then

$$\langle p' |J_0(x)| p\rangle = e^{iq\cdot x}\langle p' |J_0(0)| p\rangle = e^{iq\cdot x} F_1\left(q^2\right) \bar{u}(p',s')\gamma_0 u(p,s)$$

$F_1\left(q^2\right)$ is the Fourier transform of charge density distribution i.e.

$$F_1\left(q^2\right) \sim \int d^3x \rho(x) e^{-i\vec{q}\cdot\vec{x}}$$

Expand $F_1\left(q^2\right)$ in powers of q^2,

$$F_1\left(q^2\right) = F_1(0) + q^2 F_1'(0) + \cdots$$

$F_1(0)$ is total charge and $F_1'(0)$ is related to the charge radius.
Calulate cross section as before,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2}\frac{\left[\cos^2\frac{\theta}{2}\left(\frac{1}{1-q^2/4M^2}\right)\left[G_E^2 - \left(q^2/4M^2\right)G_M^2\right] - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}G_M^2\right]}{\sin^4\frac{\theta}{2}\left[1+\left(\frac{2E}{M}\right)\sin^2\frac{\theta}{2}\right]}$$

where

$$G_E = F_1 + \frac{q^2}{4M^2} F_2$$

$$G_M = F_1 + F_2$$

Experimentally, G_E and G_M have the form,

$$G_E\left(q^2\right) \approx \frac{G_M\left(q^2\right)}{\varkappa_p} \approx \frac{1}{\left(1 - q^2/0.7\,Gev^2\right)^2} \tag{3}$$

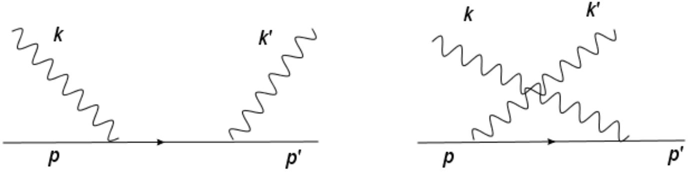
where $\varkappa_p = 2.79$ magnetic moment of the proton. If proton were point like, we would have $G_E\left(q^2\right)$ $= G_M\left(q^2\right) = 1$

Dependence of q^2 in Eq(3) $\Longrightarrow$ proton has a structure. For large q^2 the elastic cross section falls off rapidly as $G_E \approx G_M \sim q^{-4}$.

Compton Scattering

$$\gamma(k) + e(p) \longrightarrow \gamma(k') + e(p')$$

Two diagrams contribute,



The amplitude is given by

$$\begin{aligned} M(\gamma e \longrightarrow \gamma e) &= \bar{u}(p')(-ie\gamma^\mu)\varepsilon'_\mu(k') \frac{i}{\not{p} + \not{k} - m} (-ie\gamma^\nu)\varepsilon_\nu(k)\, u(p) \\ &+ \bar{u}(p')(-ie\gamma^\mu)\varepsilon_\mu(k) \frac{i}{\not{p} - \not{k}' - m} (-ie\gamma^\nu)\varepsilon'_\nu(k')\, u(p) \end{aligned}$$

Put the γ- matrices in the numerator,

$$M = -ie^2 \varepsilon'_\mu \varepsilon_\nu \left[\bar{u}(p')\gamma^\mu \frac{\not{p} + \not{k} + m}{2p \cdot k} \gamma^\nu u(p) + \bar{u}(p')\gamma^\nu \frac{\not{p} - \not{k}' + m}{-2p \cdot k'} \gamma^\mu u(p) \right]$$

Using the relations,

$$\left(\not{p}+m\right)\gamma^{\nu}u\left(p\right)=2p^{\nu}u\left(p\right),$$

we get

$$M=-ie^{2}\bar{u}(p')\left[\frac{\not{\varepsilon}'\not{k}\not{\varepsilon}+2\left(p\cdot\varepsilon\right)\not{\varepsilon}'}{2p\cdot k}+\frac{-\not{\varepsilon}\not{k}'\not{\varepsilon}'+2\left(p\cdot\varepsilon\right)\not{\varepsilon}'}{-2p\cdot k'}\right]u\left(p\right)$$

The photon polarizations are,

$$\varepsilon_\mu = \left(0, \vec{\varepsilon}\right), \qquad \text{with} \quad \vec{\varepsilon}\cdot\vec{k} = 0, \qquad \varepsilon'_\mu = \left(0, \vec{\varepsilon}'\right), \qquad \text{with} \quad \vec{\varepsilon}'\cdot\vec{k}' = 0,$$

Lab frame , $p_\mu = (m, 0, 0, 0), \implies (p\cdot\varepsilon) = (p\cdot\varepsilon') = 0$ and

$$M = -ie^2\bar{u}(p')\left[\frac{\not{\varepsilon}'\not{k}\not{\varepsilon}}{2p\cdot k} + \frac{\not{\varepsilon}\not{k}'\not{\varepsilon}'}{2p\cdot k'}\right]u(p)$$

Summing over spin of the electron

$$\frac{1}{2}\sum_{spin}|M|^2 = e^4\,Tr\left\{(\not{p}' + m)\left[\frac{\not{\varepsilon}'\not{k}\not{\varepsilon}}{2p\cdot k} + \frac{\not{\varepsilon}\not{k}'\not{\varepsilon}'}{2p\cdot k'}\right](\not{p} + m)\left[\frac{\not{\varepsilon}'\not{k}\not{\varepsilon}}{2p\cdot k} + \frac{\not{\varepsilon}\not{k}'\not{\varepsilon}'}{2p\cdot k'}\right]\right\}$$

The cross section is given by

$$d\sigma = \frac{1}{I}\frac{1}{2p_0}\frac{1}{2k_0}(2\pi)^4\delta^4(p + k - p' - k')\frac{1}{4}\sum_{spin'}|M|^2\frac{d^3p'}{(2\pi)^3 2p'_0}\frac{d^3k'}{(2\pi)^3 2k'_0}$$

phase space

$$\rho = \int (2\pi)^4\delta^4(p + k - p' - k')\frac{d^3p'}{(2\pi)^3 2p'_0}\frac{d^3k'}{(2\pi)^3 2k'_0}$$

is exactly the same as the case for *ep* scattering and the result is

$$\rho = \frac{d\Omega}{4\pi^2}\frac{\omega'^2}{m\omega}$$

It is straightforward to compute the trace with result,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4\left(\varepsilon\cdot\varepsilon'\right)^2 - 2\right]$$

This is **Klein-Nishima** relation. In the limit $\omega \to 0$,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{m^2}\left(\varepsilon\cdot\varepsilon'\right)^2$$

here $\frac{\alpha}{m}$ is classical electron radius.

For unpolarized cross section, sum over polarization of photon,

$$\sum_{\lambda\lambda'}\left[\varepsilon\left(k,\lambda\right)\cdot\varepsilon'\left(k',\lambda'\right)\right]^2=\sum_{\lambda\lambda'}\left[\vec{\varepsilon}\left(k,\lambda\right)\cdot\vec{\varepsilon}'\left(k',\lambda'\right)\right]^2$$

Since $\vec{\varepsilon}\left(k,1\right)$, $\vec{\varepsilon}\left(k,2\right)$ and $\vec{k}$ form basis in 3-dimension, completeness relation is

$$\sum_{\lambda}\varepsilon_i\left(k,\lambda\right)\varepsilon_j\left(k,\lambda\right)=\delta_{ij}-\hat{k}_i\hat{k}_j$$

Then

$$\sum_{\lambda\lambda'}\left[\vec{\varepsilon}\left(k,\lambda\right)\cdot\vec{\varepsilon}'\left(k',\lambda'\right)\right]^2=\left(\delta_{ij}-\hat{k}_i\hat{k}_j\right)\left(\delta_{ij}-\hat{k}'_i\hat{k}'_j\right)=1+\cos^2\theta$$

where $\hat{k}\cdot\hat{k}'=\cos\theta$. The cross section is

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right]$$

The total cross section,

$$\sigma=\frac{\pi\alpha^2}{m^2}\int_{-1}^{1}dz\{\frac{1}{\left[1+\frac{\omega}{m}\left(1-z\right)\right]^3}+\frac{1}{\left[1+\frac{\omega}{m}\left(1-z\right)\right]}-\frac{1-z^2}{\left[1+\frac{\omega}{m}\left(1-z\right)\right]^2}\}$$

At low energies, $\omega\rightarrow 0$, we

$$\sigma=\frac{8\pi\alpha^2}{3m^2}$$

and at high energies

$$\sigma=\frac{\pi\alpha^2}{\omega m}\left[\ln\frac{2\omega}{m}+\frac{1}{2}+O\left(\frac{m}{\omega}\ln\frac{m}{\omega}\right)\right]$$