Renormalization

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Renormalization

Renomalization is a general physical phenomena Consider an electron moving inside a solid. Due to interaction of electorn with ions on the lattice, the effective mass of the elelctron $m^* \neq m$. Electron mass is changed (renormalized) from m to m^*. Clearly both m and m^* are finite and measurable.
In relativistic field theory the concept of renormalization is the same.
Two important distinctions.

1. Modification due to interaction is infinite.
2. Can't turn off interaction to measure bare mass

Technically, the theory of renormalization is quite complicated. We will explain the principal ideas

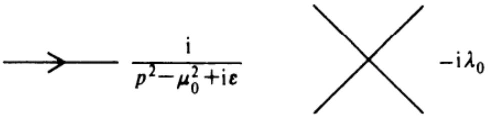
Renormalization in $\lambda\phi^4$ Theory

Consider $\lambda\phi^4$ theory

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

$$\mathcal{L}_0 = \frac{1}{2}[(\partial_\mu\phi_0)^2 - \mu_0^2\phi_0^2] \quad , \quad \mathcal{L}_I = -\frac{\lambda_0}{4!}\phi_0^4$$

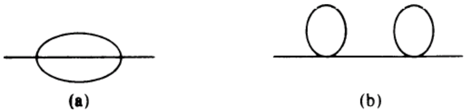
Feynman rule
vertex and propagator are



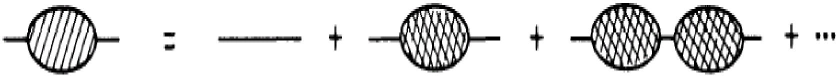
1. 4-momentum conservation at each vertex.
2. Integrate over internal momenta not fixed by momentum conservation
3. no propagator for external line

Simple example

2-point function has contribution from following graphs,



Define **1PI**: one-particle irreducible graphs— graphs which can not be disconnected by cutting any one line. Complete 2 point function in terms of 1PI graphs

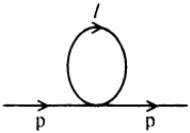


$$\frac{i}{(p^2-\mu_0^2+i\varepsilon)}+\frac{i}{(p^2-\mu_0^2+i\varepsilon)}(-i\Sigma(p^2))\frac{i}{(p^2-\mu_0^2+i\varepsilon)}+\ldots\ldots$$

$$=\frac{i}{(p^2-\mu_0^2+i\varepsilon)}[\frac{1}{1+i\Sigma(p^2)\frac{i}{p^2-\mu_0^2+i\varepsilon}}]=\frac{i}{p^2-\mu_0^2-\Sigma(p^2)+i\varepsilon}$$

1-loop diagrams

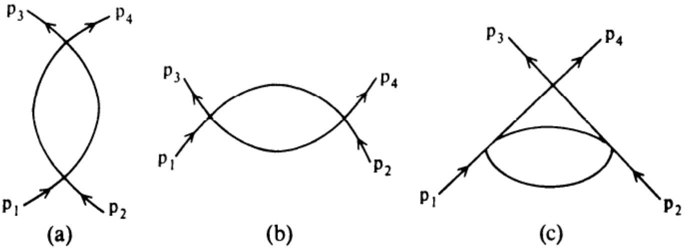
In one-loop we have



The self energy

$$-i\Sigma(p) = -\frac{i\lambda_0}{2}\int \frac{d^4 l}{(2\pi)^4}\,\frac{i}{l^2-\mu_0^2+i\varepsilon}$$

is quadratically divergent.
The 4-point function are



Graph (a) gives

$$\Gamma(p^2) = \frac{(i\lambda_0)^2}{2} \int \frac{d^4 l}{(2\pi)^4} \frac{i}{(l-p)^2 - \mu_0^2 + i\varepsilon} \frac{i}{l^2 - \mu_0^2 + i\varepsilon}$$

and is logarithmically divergent. If we differentiate $\Gamma\left(p^2\right)$ with respect to p, power of l will increase in denominator and make the integral more convergent,

$$\frac{\partial}{\partial p^2}\Gamma(p^2) = \frac{1}{2p^2} p_\mu \frac{\partial}{\partial p_\mu}\Gamma(p^2) = \frac{\lambda_0^2}{p^2} \int \frac{d^4 l}{(2\pi)^4} \frac{(l-p)\cdot p}{[(l-p)^2 - \mu_0^2 + i\varepsilon]^2} \frac{1}{l^2 - \mu_0^2 + i\varepsilon} \rightarrow \quad \textit{convergent}$$

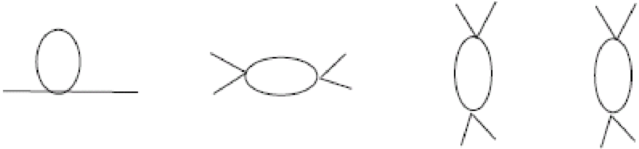
If expand $\Gamma(p^2)$ in Taylor series,

$$\Gamma(p^2) = a_0 + a_1 p^2 + \ldots$$

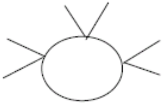
divergences are contained in first few terms. In our simple case,

$$\Gamma(p^2) = \Gamma(0) + \tilde{\Gamma}(p^2)$$

$\tilde{\Gamma}(p^2)$ is finite.
In 1-loop, the divergent graphs are (1PI)

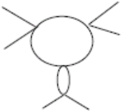


Other 1-loop graphs are either finite or contain the above graphs as subgraphs



Finite

Self energy



Vertex correction

Mass and wavefunction renormalization

Taylor expansionof 1PI self energy, has 2 divergent terms,

$$\Sigma(p^2) = \Sigma(\mu^2) + (p^2 - \mu^2)\Sigma'^2(p^2) + \tilde{\Sigma}(p^2) \qquad \mu^2 : \text{arbitrary}$$

$\Sigma(\mu^2)$ is quadratically, $\Sigma'^2(\mu^2)$ logarithmically divergnet, 3rd term $\tilde{\Sigma}(p^2)$ is finite and ,

$$\tilde{\Sigma}(\mu^2) = 0, \qquad \tilde{\Sigma}'^2(p^2) = 0$$

Complete propagator is

$$i\Delta(p^2) = \frac{i}{p^2 - \mu_0^2 - \Sigma(\mu^2) - (p^2 - \mu^2)\Sigma'^2) - \tilde{\Sigma}(p^2)}$$

Choose μ^2 such that

$$\mu_0^2 - \Sigma(\mu^2) = \mu^2 \qquad \text{mass renormalization}$$

then $\Delta(p^2)$ has a pole at $p^2 = \mu^2$. $\Longrightarrow \mu^2$ physical mass and μ_0^2 bare mass.
Full propagator is

$$i\Delta(p^2) = \frac{i}{(p^2 - \mu^2)[1 - \Sigma'^2\,(\mu^2))] - \tilde{\Sigma}(p^2)}$$

$\Sigma'(\mu^2)$ and $\tilde{\Sigma}(p^2)$ are both of order λ_0 or higher, we can approximate

$$\tilde{\Sigma}(p^2) \to (1 - \Sigma'^2\,(\mu^2))\tilde{\Sigma}(p^2)$$

Then

$$i\Delta(p^2) = \frac{iZ_\phi}{p^2 - \mu^2 - \tilde{\Sigma}(p^2) + i\varepsilon} \quad \text{with} \qquad Z_\phi = \frac{1}{1 - \Sigma'^2(p^2)} \approx 1 + \Sigma'^2(p^2)$$

Get rid of Z_ϕ by defining renormalized field ϕ by

$$\phi = \frac{1}{\sqrt{Z_\phi}} \phi_0$$

propagator for ϕ is

$$i\Delta_R(p) = \int d^4 x e^{-px} \; \langle 0| T(\phi(x)\phi(0))|0\rangle = \frac{i}{P^2 - \mu^2 - \tilde{\Sigma}(p^2) + i\varepsilon}$$

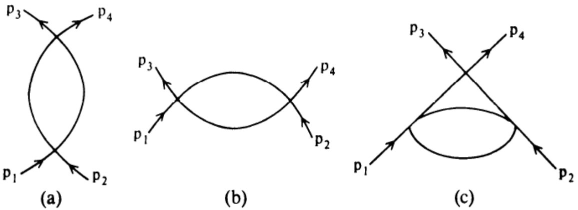
which is completely finite. Z_ϕ is called the *wave function renormalization constant*.
For general Green's functions of renomalized fields,

$$G_R^{(n)}(x_1 \ldots x_n) = \langle 0| T(\phi(x_1) \cdots \phi(x_n))|0\rangle$$

$$= Z_\phi^{-n/2} \langle 0| T(\phi_0(x_1) \cdots \phi_0(x_n))|0\rangle = Z_\phi^{-n/2} G_0^{(n)}(x_1 \ldots x_n)$$

Coupling constant renormalization

1PI 4-point functions $\Gamma^{(4)}(p_1 \cdots p_4)$, there are ,



Include tree diagram,

$$\Gamma_0^{(4)}(s,t,u) = -i\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u)$$

$$s = (p_1+p_2)^2, \qquad t = (p_1-p_3)^2, \qquad u = (p_1-p_4)^2, \qquad s+t+u = 4\mu^2$$

These are logarithmically divergent, one substraction to make this finite.

Choose $s_0 = t_0 = u_0 = \dfrac{4\mu^2}{3}$,

$$\Gamma_0^{(4)}(s,t,u) = -i\lambda_0 + 3\Gamma(s_0) + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)$$

where

$$\tilde{\Gamma}(s) = \Gamma(s) - \Gamma(s_0),$$

is finite. Define Z_λ by

$$-i\lambda_0 + 3\Gamma(s_0) = -iZ_\lambda^{-1}\lambda_0$$

Thus

$$\Gamma_0^{(4)}(s,t,u) = -iZ_\lambda^{-1}\lambda_0 + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)$$

At the symmetric point

$$\Gamma_0^{(4)}(s_0,t_0,u_0) = -iZ_\lambda^{-1}\lambda_0$$

with $\tilde{\Gamma}(s_0) = \tilde{\Gamma}(t_0) = \tilde{\Gamma}(u_0) = 0$. Renormalized 1PI 4 point function $\Gamma^{(4)}$ is related to Green's function by

$$\Gamma_R^{(4)} = \prod_{j=1}^{4}[i\Delta_R(p_j)]^{-1} G_R^{(4)}$$

which implies

$$\Gamma_R^{(4)}(s,t,u) = -Z_\phi^2\Gamma_0^{(4)}(s,t,u)$$

Define renormalized coupling constant λ by

$$\lambda = Z_\phi^2 Z_\lambda^{-1}\lambda_0$$

then

$$\Gamma_R^{(4)}(p_1,\cdots,p_4\ ) = Z_\phi^2\Gamma_0^{(4)} = -iZ_\lambda^{-1}Z_\varphi^2\lambda_0 + Z_\varphi^2[\tilde{\Gamma}(s)+\tilde{\Gamma}(t)+\tilde{\Gamma}(u)] = -i\lambda + Z_\varphi^2[\tilde{\Gamma}(s)+\tilde{\Gamma}(t)+\tilde{\Gamma}(u)]$$

Since $Z_\varphi = 1 + O(\lambda_0)$, $\tilde{\Gamma} = O(\lambda_0^2)$ $\lambda = \lambda_0 + O(\lambda_0^2)$, we can approximate

$$\Gamma_R^{(4)}(\ p_1,\cdots,p_4) = -i\lambda + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) + 0(\lambda^3)$$

which is completely finite. From the original Lagrangian (unrenormalized Lagrangian)

$$\mathcal{L}_0 = \frac{1}{2}[(\partial_\mu \phi_0)^2 - \mu_0^2 \phi_0^2] - \frac{\lambda_0}{4!}\phi^4$$

we can write

$$\mathcal{L}_0 = \mathcal{L} + \Delta\mathcal{L}$$

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi)^2 - \mu^2 \phi^2] - \frac{\lambda}{4!}\phi^4$$

$$\Delta\mathcal{L} = \mathcal{L}_0 - \mathcal{L} = \frac{1}{2}(Z_\phi - 1)[(\partial_\mu \phi)^2 - \mu^2\phi^2] + \frac{\delta\mu^2}{2}\phi^2 - \frac{-\lambda(Z_\lambda - 1)}{4!}\phi^4$$

where

$$\mu^2 = \delta\mu^2 + \mu_0^2 \quad , \quad \phi = Z_\phi^{-\frac{1}{2}}\phi_0 \quad , \quad \lambda = Z_\lambda^{-1} Z_\phi^2 \lambda_0$$

Here $\mathcal{L}$ is usually called renormalized Lagrangian and $\Delta\mathcal{L}$ the counterterms.

BPH renormalization

An equivalent scheme is BPH (Bogoliubov, Parasiuk and Hepp) renormalization scheme.
Essential idea: use counter terms Lagrangian $\Delta\mathcal{L}$ as a device to cancel the divergences.

1. Starts with renormalized Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

   Generate free propagator and vertices from this Lagrangian.

2. The divergent parts of one-loop 1PI diagrams are isolated by Taylor expansion. Construct a set of counter terms $\Delta\mathcal{L}^{(1)}$ to cancel these divergences.
3. A new Lagrangian $\mathcal{L}^{(1)} = \mathcal{L} + \Delta\mathcal{L}^{(1)}$ is used to generate 2-loop diagrams and to counter terms $\Delta\mathcal{L}^{(2)}$ to cancel 2-loops divergences. This sequence of operation is iteratively applied.

To illustrate the usefulness of BPH scheme, we need to make use of the power counting method.

Power counting

Superficial degree of divergence D is defined as

$$D = (\sharp \;\; of \;\; loop \;\; momenta \;\; in \;\; numerator) - (\sharp \;\; of \;\; loop \;\; momenta \;\; indenominator)$$

We define the following quantities,
B= number of external lines
IB= number of internal lines
n= number of vertices
Counting the lines in the graph, we get

$$4n = 2(IB) + B$$

4-momentum conseravation at each vertex and overall 4-momentum conseravation which do not depend on the internal momentum.
number of loops L is

$$L = IB - n + 1$$

Eliminating n, L and (IB),

$$D = 4 - B$$

Thus $D < 0$ for $B > 4$. The $\lambda\phi^4$ theory has the symmetry $\phi \rightarrow -\phi$. which implies that $B = even$ and only $B = 2, 4$ are superficially divergent.

Comments on subgraph divergences

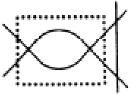
Convergence of Feynman integrals (Weinberg's Theorem): Feynman integral converges if the superficial degree of divergence of of **all** subgraphs are negative.
More explicitly, consider a Feynman graph with n external lines and l loops,

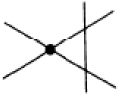
$$\Gamma^{(n)}(p_1, p_2, \cdots, p_n) = \int^{\Lambda} d^4q_1 \cdots d^4q_l I(p_1, p_2, \cdots, p_n; q_1, \cdots q_l)$$

where we have used a cutoff Λ to make estimate of the divergence. The integrand I is the product of vertices and propagators. Take a subset $S = \{q_1', \cdots q_m'\}$ of the loop momenta $\{q_1, q_2 \cdots q_l\}$ and scale them to infinity with all other momenta fixed. Let $D(S)$ be the superficial degree of divergence for integration over this set, namely

$$\left| \int^{\Lambda} d^4 q_1' \cdots d^4 q_m' I \right| \leq \Lambda^{D(S)} \left| \ln \Lambda \right|$$

Then the convergence theorem says that the integral onver $\{q_1, q_2 \cdots q_l\}$ converges if the $D(S)$ for all possible choices of S are negative. For example, in the graph on the left below, we have $D = -2$. But the integration inside the box having $D = 0$ is logarithmically divergent. In the BPH procedure these subdiagram divergences are in fact renormalized by low-order counterterm. For example, the graph on the right below with its counter term vertex will cancel the subgraph divergence of the graph on the left.





Here we see that even though some graphs are not convergent according to Weinberg's theorem, in BPH scheme the divergences associated with some subgraphs are systematically canceled by lower order counter terms.

Regularization
Need first to make divergent integral finite before we can do any manipulation.
2 different schemes: Pauli-Villars regularization and dimensional regularization.

Pauli-Villars Regularization
Repalce the propagator by

$$\frac{1}{k^2-\mu_0^2} \rightarrow \left(\frac{1}{k^2-\mu_0^2}-\frac{1}{k^2-\Lambda^2}\right)=\frac{(\mu_0^2-\Lambda^2)}{(k^2-\mu_0^2)(k^2-\Lambda^2)} \rightarrow \frac{1}{k^4} \quad \text{for large } k$$

will make the integral more convergent.
. 4-point function from the following graph,

$$\Gamma\left(p^2\right) = \Gamma\left(s\right) = \frac{\left(-i\lambda\right)^2}{2}\int \frac{d^4 l}{\left(2\pi\right)^4}\frac{i}{\left(l-p\right)^2-\mu^2}\frac{i}{l^2-\mu^2}$$

With Pauli-Villars regularization this becomes,

$$\Gamma\left(p^2\right) = \frac{-\lambda^2\Lambda^2}{2}\int \frac{d^4 l}{\left(2\pi\right)^4}\frac{1}{\left[\left(l-p\right)^2-\mu^2\right]\left(l^2-\mu^2\right)\left(l^2-\Lambda^2\right)}$$

Taylor expansion around $p^2 = 0$,

$$\Gamma\left(p^2\right) = \Gamma\left(0\right) + \widetilde{\Gamma}\left(p^2\right)$$

with

$$\Gamma\left(0\right) = \frac{-\lambda^2\Lambda^2}{2}\int \frac{d^4 l}{\left(2\pi\right)^4}\frac{1}{\left(l^2-\mu^2\right)^2\left(l^2-\Lambda^2\right)}$$

$$\widetilde{\Gamma}\left(p^2\right) = \frac{\lambda^2}{2}\int \frac{d^4 l}{\left(2\pi\right)^4}\frac{2l\cdot p-p^2}{\left[\left(l-p\right)^2-\mu^2\right]\left(l^2-\mu^2\right)^2}$$

take the limit $\Lambda^2 \to \infty$ inside in $\widetilde{\Gamma}\left(p^2\right)$.
Combine the denominators by using the identities,

$$\frac{1}{a_1 a_2\cdots a_n} = \left(n-1\right)!\int_0^1 \frac{dz_1 dz_2\cdots dz_n}{\left(a_1 z_1+\cdots+a_n z_n\right)^n}\delta\left(1-\sum_{i=1}^n z_i\right)$$

$$\frac{1}{a_1^2 a_2\cdots a_n} = n!\int_0^1 \frac{z_1 dz_1 dz_2\cdots dz_n}{\left(a_1 z_1+\cdots+a_n z_n\right)^{n+1}}\delta\left(1-\sum_{i=1}^n z_i\right)$$

Here $\alpha_1, \cdots \alpha_n$ are called the Feynman parameters. Then

$$\frac{1}{\left[(l-p)^2-\mu^2\right](l^2-\mu^2)^2} = 2\int \frac{(1-\alpha)\,d\alpha}{A^3}$$

where

$$A = (1-\alpha)\left(l^2-\mu^2\right) + \alpha\left[(l-p)^2-\mu^2\right] = (1-\alpha p)^2 - a^2$$

with

$$a^2 = \mu^2 - \alpha(1-\alpha)\,p^2$$

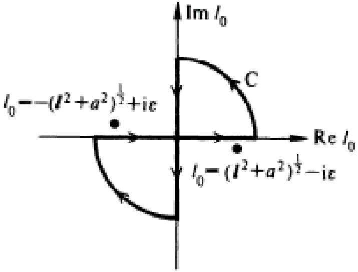
Thus

$$\begin{aligned}
\widetilde{\Gamma}\left(p^2\right) &= \lambda^2 \int_0^1 (1-\alpha)\,d\alpha \int \frac{d^4 l}{(2\pi)^4} \frac{2l\cdot p - p^2}{\left[(l-\alpha p)^2 - a^2\right]^3} \\
&= \lambda^2 \int_0^1 (1-\alpha)\,d\alpha \int \frac{d^4 l}{(2\pi)^4} \frac{(2\alpha-1)\,p^2}{(l^2-a^2+i\varepsilon)^3}
\end{aligned}$$

we have changed the variable $l \to l + \alpha p$ and drop terms linear in l. In the complex l_0 plane, poles at

$$l_0 = \pm\left[\sqrt{l^2+a^2} - i\varepsilon\right]$$

Do the integration by *Wick* rotation,



From Cauchy's theorem we have

$$\oint_C dl_0 f\left(l_0\right) = 0$$

where

$$f\left(l_0\right) = \frac{1}{\left[l_0^2 - \left(\sqrt{\vec{l}^{\,2} + a^2} - i\varepsilon\right)^2\right]^3}$$

Since $f\left(l_0\right) \to l_0^{-6}$ as $l_0 \to \infty$, circular part of contour C with very large radius vanishes and

$$\int_{-\infty}^{\infty} dl_0 f\left(l_0\right) = \int_{-i\infty}^{i\infty} dl_0 f\left(l_0\right)$$

Integration path has been rotated from along real axis to imaginary axis (Wick rotation). Changing the variable $l_0 = il_4$,

$$\int_{-i\infty}^{i\infty} dl_0 f(l_0) = i \int_{-\infty}^{\infty} dl_4 f(l_4) = -i \int_{-\infty}^{\infty} \frac{dl_4}{\left(l_1^2 + l_2^2 + l_3^2 + l_4^2 + a^2 - i\varepsilon\right)^3}$$

Define Euclidean momentum $k_i = (l_1, l_2, l_3, l_4)$ with $k^2 = l_1^2 + l_2^2 + l_3^2 + l_4^2$. The integral is then

$$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{\left(l^2 - a^2 + i\varepsilon\right)^3} = -i \int \frac{d^4 k}{(2\pi)^4} \frac{1}{\left(k^2 + a^2 - i\varepsilon\right)^3}$$

Using polar coordinates in 4-dim

$$\int d^4 k = \int_0^{\infty} k^3 dk \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\pi} \sin^2\chi d\chi$$

and integrating over angles

$$\begin{aligned} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{\left(k^2 + a^2 - i\varepsilon\right)^3} &= 2\pi^2 \int_0^{\infty} \frac{k^3 dk}{(2\pi)^4} \frac{1}{\left(k^2 + a^2 - i\varepsilon\right)^3} \\ &= \frac{1}{16\pi^2} \int_0^{\infty} \frac{k^2 dk^2}{\left(k^2 + a^2 - i\varepsilon\right)^3} \end{aligned}$$

Using the formula

$$\int \frac{t^{m-1} dt}{\left(t + a^2\right)^n} = \frac{1}{\left(a^2\right)^{n-m}} \frac{\Gamma(m)\,\Gamma(n-m)}{\Gamma(n)}$$

we get

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{\left(k^2 + a^2 - i\varepsilon\right)^3} = \frac{1}{32\pi^2 \left(a^2 - i\varepsilon\right)}$$

and

$$\widetilde{\Gamma}\left(p^2\right) = \frac{-i\lambda^2}{32\pi^2}\int_0^1 \frac{d\alpha\,(1-\alpha)\,(2\alpha-1)\,p^2}{\left[\mu^2-\alpha\,(1-\alpha)\,p^2-i\varepsilon\right]}$$

It is straightforward to carry out the integration to compute $\widetilde{\Gamma}\left(p^2\right)$ to get

$$\begin{aligned}
\tilde{\Gamma}(p^2) &= \tilde{\Gamma}(s) = \frac{i\lambda^2}{32\pi^2}\left\{2+\left(\frac{4\mu^2-s}{|s|}\right)^{\frac{1}{2}}\ln\left[\frac{(4\mu^2-s)^{\frac{1}{2}}-(|s|)^{\frac{1}{2}}\}}{\{(4\mu^2-s)^{\frac{1}{2}}+(|s|)^{\frac{1}{2}}\}}\right]\right\} && \text{for } s<0\\
&= \frac{i\lambda^2}{32\pi^2}\left\{2-2\left(\frac{4\mu^2-s}{s}\right)^{\frac{1}{2}}\tan^{-1}\left(\frac{s}{4\mu^2-s}\right)^{\frac{1}{2}}\right\} && \text{for } 0<s<4\mu^2\\
&= \frac{i\lambda^2}{32\pi^2}\left\{2+(\frac{s-4\mu^2}{s})^{\frac{1}{2}}\,ln\left[\frac{s^{\frac{1}{2}}-(s-4\mu^2)^{\frac{1}{2}}}{s^{\frac{1}{2}}+(s-4\mu^2)^{\frac{1}{2}}}\right]+i\pi\right\} && \text{for } s>4\mu^2
\end{aligned}$$

Dimensional regularization

The basic idea here is that since the divergences come from integration of internal momentum in 4-dimensional space, the integral can be made finite in lower dimensional space. We can define the Feynman integrals as functions of space-time n and carry out the renormalization for lower values of n before taking the limit $n \longrightarrow 4$. Consider the integral

$$I = \int \frac{d^4 k}{(2\pi)^4}\left(\frac{1}{k^2-\mu^2}\right)\left[\frac{1}{(k-p)^2-\mu^2}\right]$$

which is divergent in 4-dimension. If we define this as integration over $n-$dimension

$$I(n) = \int \frac{d^n k}{(2\pi)^4}\frac{1}{(k^2-\mu^2)}\left[\frac{1}{(k-p)^2-\mu^2}\right]$$

then the integral is convergent for $n<4$.To define this integral for non-integer values of n, we first combine the denominators using Feynman parameters and make the Wick rotation,

$$\begin{aligned} I(n) &= \int_0^1 d\alpha \int \frac{d^n k}{\left[(k-\alpha p)^2 - a^2 + i\varepsilon\right]^2} \\ &= i\int_0^1 d\alpha \int \frac{d^n k}{\left[k^2+a^2-i\varepsilon\right]^2} \qquad \text{with} \quad a^2 = \mu^2 - \alpha(1-\alpha)p^2 \end{aligned}$$

Now introduce the spherical coordinates

$$\begin{aligned} \int d^n k &= \int_0^\infty k^{n-1} dk \int_0^{2\pi} d\theta_1 \int_0^\pi \sin\theta_2 d\theta_2 \int \cdots \int_0^\pi \sin^{n-2}\theta_{n-1} d\theta_{n-1} \\ &= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int_0^\infty k^{n-1} dk \end{aligned}$$

where we have used the formula,

$$\int_0^{\pi} \sin^m \theta d\theta = \frac{\sqrt{\pi}\Gamma\left(\dfrac{m+1}{2}\right)}{\Gamma\left(\dfrac{m+2}{2}\right)}$$

Then the $n-$dimensional integral is

$$I\left(n\right) = \frac{2i\pi^{n/2}}{\Gamma\left(\dfrac{n}{2}\right)} \int_0^1 d\alpha \int_0^{\infty} \frac{k^{n-1}dk}{\left[k^2 + a^2 - i\varepsilon\right]^2}$$

The dependence on n is now explicit and the integral is well-defined for $0 < \mathrm{Re}(n) < 4$. We can extend this domain of analyticity by integration by parts

$$\frac{1}{\Gamma(\frac{n}{2})} \int_0^{\infty} \frac{k^{n-1}dk}{[k^2 + a^2 - i\varepsilon]^2} = \frac{-2}{\Gamma(\frac{n}{2}+1)} \int_0^{\infty} k^n dk \frac{d}{dk}\left(\frac{1}{[k^2 + a^2 - i\varepsilon]^2}\right)$$

where we have used

$$z\Gamma(z) = \Gamma(z+1)$$

The integral is now well defined for $2 < \mathrm{Re}(n) < 4$. Repeat this procedure m times, the analyticity domain is extended to $-2m < \mathrm{Re}(n) < 4$ and eventually to $\mathrm{Re}(n) \to -\infty$. To see what happens as $n \longrightarrow 4$, we can integrate over k to get

$$I(n) = i\pi^{n/2}\Gamma\left(2 - \frac{n}{2}\right) \int_0^1 \frac{d\alpha}{[a^2 - i\varepsilon]^{2-n/2}}$$

Using the formula,

$$\Gamma\left(2 - \frac{n}{2}\right) = \frac{\Gamma\left(3 - \frac{n}{2}\right)}{2 - \frac{n}{2}} \to \frac{2}{4-n} \quad \textit{as } n \longrightarrow 4$$

we see that the singularity at $n = 4$ is a simple pole. Expand everything around $n = 4$,

$$\Gamma\left(2 - \frac{n}{2}\right) = \frac{2}{4-n} + A + (n-4)B + \cdots$$

$$a^{n-4} = 1 + (n-4)\ln a + \cdots$$

where A and B are some constants, we obtain the limit, as $n \longrightarrow 4$

$$I(n) \longrightarrow \frac{2i\pi^2}{4-n} - i\pi^2 \int_0^1 d\alpha \ln[\mu^2 - \alpha(1-\alpha)p^2] + i\pi^2 A$$

and the 1-loop contribution to 4-point function is,

$$\Gamma(p^2) = \frac{\lambda^2}{32\pi^2}\left\{\frac{2i}{4-n} - i\int_0^1 d\alpha \ln[\mu^2 - \alpha(1-\alpha)p^2] + iA\right\}$$

Taylor expansion around $p^2 = 0$ gives

$$\Gamma(p^2) = \Gamma(0) - \tilde{\Gamma}(p^2)$$

$$\Gamma(0) = \frac{\lambda^2}{32\pi^2}\left(\frac{2i}{4-\pi} - i\ln\mu^2 + iA\right) \simeq \frac{i\lambda^2}{16\pi^2(4-n)}$$

and

$$\begin{aligned}\tilde{\Gamma}(p^2) &= \frac{-i\lambda^2}{32\pi^2}\int_0^1 d\alpha \ln\left[\frac{\mu^2 - \alpha(1-\alpha)p^2}{\mu^2}\right] \\ &= \frac{-i\lambda^2}{32\pi^2}\int_0^1 \frac{d\alpha(1-\alpha)(2\alpha-1)p^2}{[\mu^2 - \alpha(1-\alpha)p^2]}\end{aligned}$$

Clearly this finite part is exactly the same as that given by the method of covariant regulariztion. The 1-loop self energy in dimensional-regularization scheme becomes

$$-i\Sigma(p^2) = \frac{\lambda}{2}\int \frac{d^n k}{(2\pi)^4}\frac{1}{k^2-\mu^2+i\varepsilon} = \frac{-i\lambda\pi^{n/2}\Gamma\left(1-\frac{n}{2}\right)}{32\pi^4(\mu^2)^{1-n/2}}$$

From the relation,

$$\Gamma\left(1-\frac{n}{2}\right) = \frac{\Gamma\left(3-\frac{n}{2}\right)}{\left(1-\frac{n}{2}\right)\left(2-\frac{n}{2}\right)}$$

we see that the quadratic divergnece has pole at $n=4$ and also at $n=2$. For $n\to 4$ we have,

$$-i\Sigma(0) = \frac{i\lambda\mu^2}{16\pi^2}\left(\frac{1}{4-n}\right)$$

Composite operator

In some cases, we need to consider Green's function of composite operator, an operator with more than one fields at same space time.

Consider a simple composite operator of the form $\Omega(x) = \frac{1}{2}\phi^2(x)$ in $\lambda\phi^4$ theory. Green's function with one insertion of Ω is of the form,

$$G_\Omega^{(n)}(x; x_1, x_2, x_3, ..., x_n) = \left\langle 0 \middle| T(\frac{1}{2}\phi^2(x)\phi(x_1)\phi(x_2)...\phi(x_n)) \middle| 0 \right\rangle$$

In momentum space we have

$$(2\pi)^4\delta^4(p + p_1 + p_2 + ... + p_n)G_{\phi^2}^{(n)}(p; p_1, p_2, p_3, ..., p_n) = \int d^4x\, e^{-ipx} \int \prod_{i=1}^{n} d^4x_i e^{-ip_i x_i}\, G_\Omega^{(n)}(x; x_1, x_2, x_3, ..., x_n)$$

In perturbation theory, we can use Wick's theorem to work out these Green's functions in terms of Feynman diagram.

Example, to lowest order in λ the 2-point function with one composite operator $\Omega(x) = \frac{1}{2}\phi^2(x)$ is, after using the Wick's theorem,

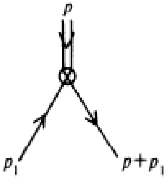
$$G_{\phi^2}^{(2)}(x; x_1, x_2) = \frac{1}{2}\left\langle 0 \middle| T\{\phi^2(x)\phi(x_1)\phi(x_2)\} \middle| 0 \right\rangle = i\Delta(x - x_1)i\Delta(x - x_2)$$

or in momentum space

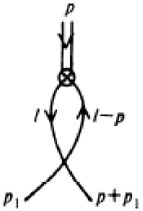
$$G_{\phi^2}^{(2)}(p; p_1, p_2) = i\Delta(p_1)i\Delta(p + p_1)$$

If we truncate the external propagators, we get

$$\Gamma_{\phi^2}^{(2)}(p, p_1, -p_1 - p) = 1$$



(a)



(b)

To first order in λ, we have

$$G^{(2)}_{\phi^2}(x, x_1, x_2) = \int \left\langle 0 \right| T\{\frac{1}{2}\phi^2(x)\phi(x_1)\phi(x_2)\frac{(-i\lambda)}{4!}\phi^4(y)\}\left|0\right\rangle d^4y$$

$$= \int d^4y \frac{-i\lambda}{2}[i\Delta(x-y)]^2 i\Delta(x_1-y)i\Delta(x_2-y)$$

The amputated 1PI momentum space Green's function is

$$\Gamma^{(2)}_{\phi^2}(p; p_1, -p-p_1) = \frac{-i\lambda}{2}\int \frac{d^4l}{(2\pi)^4}\frac{i}{l^2-\mu^2+i\epsilon}\frac{i}{(l-p)^2-\mu^2+i\epsilon}$$

To calculate this type of Green's functions systematically, we can add a term $\chi(x)\Omega(x)$ to $\mathcal{L}$

$$\mathcal{L}[\chi] = \mathcal{L}[0] + \chi(x)\Omega(x)$$

where $\chi(x)$ is a c-number source function. We can construct the generating functional $W[\chi]$ in the presence of this external source. We obtain the connected Green's function by differentiating $\ln W[\chi]$ with respect to χ and then setting χ. to zero.

Renormalization of composite operators

Superficial drgrees of divergence for Green 's function with one composite operator is,

$$D_\Omega = D + \delta_\Omega = D + (d_\Omega - 4)$$

where d_Ω is the canonical dimension of Ω. For the case of $\Omega(x) = \frac{1}{2}\phi^2(x)$, $\quad d_{\phi^2} = 2$ and $D_{\phi^2} = 2 - n \Rightarrow$ only $\Gamma^{(2)}_{\phi^2}$ is divergent. Taylor expansion takes the form,

$$\Gamma^{(2)}_{\phi^2}(p; p_1) = \Gamma^{(2)}_{\phi^2}(0,0) + \Gamma^{(2)}_{\phi^2 R}(p, p_1)$$

We can combine the counter term

$$\frac{-i}{2}\Gamma^{(2)}\phi^2(0,0)\chi(x)\phi^2(x)$$

with the original term to write

$$\frac{-i}{2}\chi\phi - \frac{i}{2}\Gamma^2_{\phi^2}(0,0)\chi\phi^2 = -\frac{i}{2}Z_{\phi^2}\chi\phi^2$$

In general, we need to insert counterterm $\Delta\Omega$ into the original addition

$$L \to L + \chi(\Omega + \Delta\Omega)$$

If $\Delta\Omega = C\Omega$, as in the case of $\Omega = \frac{1}{2}\phi^2$,we have

$$L[\chi] = L[0] + \chi Z_\Omega \Omega = L[0] + \chi\Omega_0$$

with

$$\Omega_0 = Z_\Omega \Omega = (1 + C)\Omega$$

Such composite operators are said to be mutiplicative renormalizable and Green's functions of unrenormalized operator Ω_0 is related to that of renormalized operator Ω by

$$G_{\Omega_0}^{(n)}(x;x_1,x_2,...x_n) = \langle 0|T\{\Omega_0(x)\phi(x_1)\phi(x_2)...\phi(x_n)\}|0\rangle$$

$$= Z_\Omega Z_\phi^{n/2} G_{IR}^{(n)}(x;x_1,...x_n)$$

For more general cases,$\Delta\Omega \neq c\Omega$ and the renormalization of a composite operator may require counterterm proportional to other composite operators.
Example: Conside 2 composite operators A and B. Denote the counterterms by ΔA and ΔB. Including the counter terms we can write,

$$L[\chi] = L[0] + \chi_A(A + \Delta A) + \chi_B(B + \Delta B)$$

Very often with counterterms ΔA and ΔB are linear combinations of A and B

$$\Delta A = C_{AA}A + C_{AB}B$$

$$\Delta B = C_{BA}A + C_{BB}B$$

We can write

$$L[\chi] = L[0] + (\chi_A\,\chi_B)\{C\}\begin{pmatrix} A \\ B \end{pmatrix} \quad where \;\; \{C\} = \begin{pmatrix} 1 + C_{AA} & C_{AB} \\ C_{BA} & 1 + C_{BB} \end{pmatrix}$$

Diagonalized $\{C\}$ by bi-unitary transformation

$$U\{C\}V^+ = \begin{pmatrix} Z_{A'} & 0 \\ 0 & Z_{B'} \end{pmatrix}$$

Then

$$L[\chi] = L[0] + Z_{A'} \chi_{A'} A' + Z_{B'} \chi_{B'} B'$$

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = V \begin{pmatrix} A \\ B \end{pmatrix} \quad \begin{pmatrix} \chi_{A'} & \chi_{B'} \end{pmatrix} = (\chi_A \;\; \chi_B)\, U$$

A' , B' are multiplicatively renormalizable.

Renormalization group

Discussion will be brief. Renormalization scheme requires specification of substraction points which introduce new mass scales. As we will see this introduces the concept of energy dependent "coupling constants",

$$e.g \quad \lambda = \lambda(s)$$

even though the coupling constants in the original Lagranggian are independent of energies.

Renormalization group equation

In general, there is arbitrariness in choosing the renormalization schemes (or the substraction points). Nevertheless, the physical results should be the same, i.e. independent of renormalization schemes. In essence this is the physical content of the renormalization group equation. Suppose we have different renormalizarion scheme R and R'. From the point of view of BPH renormalization, we can write

$$\mathcal{L} = \mathcal{L}_R(R - quantities) = \mathcal{L}_{R'}(R' - quantities)$$

Recall that

$$\phi_R = Z_{\varphi R}^{-\frac{1}{2}}\phi_0, \qquad \lambda_R = Z_{\lambda R}^{-1} Z_{\phi R}^2 \lambda_0 \qquad \mu_R^2 = \mu_0^2 + \delta\mu_R^2$$

Similarly,

$$\phi_{R'} = Z_{\varphi R'}^{-\frac{1}{2}}\phi_0, \qquad \lambda_{R'} = Z_{\lambda R'}^{-1} Z_{\phi R'}^2 \lambda_0 \qquad \mu_{R'}^2 = \mu_0^2 + \delta\mu_{R'}^2$$

Since ϕ_0, λ_0 and μ_0 are the same, we can finite relations between $R-$ and R' quantities

Callan-Symanzik equation

This particular derivation of RG equation is conceptually simple. Start with the fact that for the bare propagator, we have

$$\frac{\partial}{\partial \mu_0^2}\left(\frac{i}{p^2 - \mu_0^2 + i\varepsilon}\right) = \frac{i}{p^2 - \mu_0^2 + i\varepsilon}(-i)\frac{i}{p^2 - \mu_0^2 + i\varepsilon}$$

This corresponds to insertion of composite operator $\Omega = \frac{1}{2}\phi_0^2$with zero momentum. Thus

$$\frac{\partial \Gamma^{(n)}(P_i)}{\partial \mu_0^2} = -i\Gamma^{(n)}_{\phi^2}(0; P_i)$$

In terms of renormalized (1PI) Green's functions, the relations are

$$\Gamma_R^{(n)}(P_i, \lambda, \mu) = Z_\phi^{\frac{n}{2}} \Gamma^{(n)}(P_i, \lambda_0, \mu_0^2)$$

$$\Gamma^{(n)}_{\phi^2 R}(P, P_i, \lambda, \mu) = Z_{\phi^2}^{-1} Z_\phi^{\frac{1}{2}} \Gamma^{(n)}_{\phi^2}(P, P_i, \lambda_0, \mu_0^2)$$

We now differentiate with respect to μ_0^2,

$$\frac{\partial}{\partial \mu_0^2}\Gamma_R^{(n)}(P_i, \lambda, \mu) = (\frac{\partial \mu^2}{\partial \mu_0^2}\frac{\partial}{\partial \mu^2} + \frac{\partial \lambda}{\partial \mu_0^2}\frac{\partial}{\partial \lambda})\Gamma_R^{(n)}(P_i, \lambda, \mu)$$

We can write this as

$$[\mu\frac{\partial}{\partial \mu} + \beta\frac{\partial}{\partial \lambda} + n\gamma]\Gamma_R^{(n)}(P_i, \lambda, \mu) = -i\mu^2\alpha\Gamma^{(n)}_{\phi^2 R}(0, P_i, \lambda, \mu)$$

$$\text{where} \quad \beta = 2\mu^2\frac{\frac{\partial \lambda}{\partial \mu_0^2}}{\frac{\partial \mu^2}{\partial \mu_0^2}}, \qquad \gamma = \mu^2\frac{\frac{\partial \ln Z_\phi}{\partial \mu_0^2}}{\frac{\partial \mu^2}{\partial \mu_0^2}}, \qquad \alpha = \frac{\frac{\partial Z_{\phi^2}}{\partial \mu_0^2}}{\frac{\partial \mu^2}{\partial \mu_0^2}}$$

This is usually referred to as **Callan-Symanzik** equation.

Weinberg's Theorem:(simplified version)
Write the external momenta as $P_i = \sigma R_i$, and take $\sigma \to \infty$ limit, the asymptotic behaviors are

$$\Gamma_R^{(n)} \sim \sigma^{4-n}, \qquad \Gamma_{\phi^2 R}^{(n)} \sim \sigma^{2-n}$$

So in the large momenta region, we can neglect $\Gamma_{\phi^2 R}^{(n)}$,

$$[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} - n\gamma(\lambda)]\Gamma_{as}^{(n)}(P_i, \lambda, \mu) = 0$$

Define a dimensionless quantity $\bar{\Gamma}$ by

$$\Gamma_{as}^{(n)}(P_i, \lambda, \mu) = \mu^{4-n}\bar{\Gamma}_R^{(n)}(\frac{P_i}{\mu}, \lambda)$$

Since $\bar{\Gamma}$ is dimensionless, as we scale up the momenta we can write

$$(\mu\frac{\partial}{\partial\mu} + \sigma\frac{\partial}{\partial\sigma})\bar{\Gamma}_R^{(n)}(\frac{\sigma p_i}{\mu}, \lambda) = 0$$

and

$$[\mu\frac{\partial}{\partial\mu} + \sigma\frac{\partial}{\partial\sigma} + (n-4)]\Gamma_{as}^{(n)}(\frac{\sigma P_i}{\mu}, \lambda) = 0$$

From Callan-Symanzik equation we get

$$[\sigma\frac{\partial}{\partial\sigma} - \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda) + (n-4)]\Gamma_{as}^{(n)}(\sigma p_i, \lambda, \mu) = 0$$

To solve this equation, we remove the non-derivative terms by the transformation

$$\Gamma_{as}^{(n)}(\sigma p_i, \lambda, \mu) = \sigma^{4-n} \exp[n \int_0^\lambda \frac{\gamma(x)}{\beta(x)} dx] \Gamma^{(n)}(\sigma p_i, \lambda, \mu)$$

Then $F^{(n)}$ satisfies the equation

$$[\sigma \frac{\partial}{\partial \sigma} - \beta(\lambda) \frac{\partial}{\partial \lambda}] \Gamma^{(n)}(\sigma p_i, \lambda, \mu) = 0$$

or

$$[\frac{\partial}{\partial t} - \beta(\lambda) \frac{\partial}{\partial \lambda}] \Gamma^{(n)}(e^t p_i, \lambda, \mu) = 0 \quad \text{where} \quad t = \ln \sigma$$

Introduce the effective, or running constant $\bar{\lambda}$ as solution to the equation

$$\frac{d\bar{\lambda}(t, \lambda)}{dt} = \beta(\bar{\lambda}) \qquad \text{with initial condition} \quad \bar{\lambda}(0, \lambda) = \lambda$$

This equation has the solution

$$t = \int_\lambda^{d\bar{\lambda}(t,\lambda)} \frac{dx}{\beta(x)}$$

It is straightforward to show that

$$\frac{1}{\beta(\bar{\lambda})} \frac{d\bar{\lambda}}{d\lambda} = \beta(\lambda) \quad and \quad [\frac{\partial}{\partial t} - \beta(\lambda) \frac{\partial}{\partial \lambda}] \bar{\lambda}(t, \lambda) = 0$$

In other words, $F^{(n)}$ depends on t and λ only through the combination $\bar{\lambda}(t, \lambda)$

$$F^{(n)} = F^{(n)}(p_i, \bar{\lambda}(t, \lambda), \mu)$$

Also

$$\begin{aligned}\exp[n\int_0^{\lambda}\frac{\gamma(\lambda)}{\beta(\lambda)}d\lambda] &\sim \exp[n\int_0^{\bar\lambda}\frac{\gamma(x)}{\beta(x)}dx + n\int_{\bar\lambda}^{\lambda}\frac{\gamma(x)}{\beta(x)}dx] \\ &= H(\bar\lambda)\exp[-n\int_{\lambda}^{\bar\lambda}\frac{\gamma(x)}{\beta(x)}dx]\end{aligned}$$

where

$$H(\bar\lambda) = exp[n\int_0^{\bar\lambda}\frac{\gamma(x)}{\beta(x)}dx]$$

The solution is then

$$\Gamma_{as}^{(n)}(\sigma p_i,\lambda,\mu) = \sigma^{4-n}exp[-n\int_0^t \gamma(\bar\lambda(x',\lambda))dx']H(\bar\lambda)F^{(n)}(p_i,\bar\lambda(t,\lambda),\mu)$$

If we set $t=0$ (or $\sigma=0$), we see that

$$\Gamma_{as}^{(n)}(p_i,\lambda,\mu) = H(\lambda)F^{(n)}(p_i,\lambda,\mu)$$

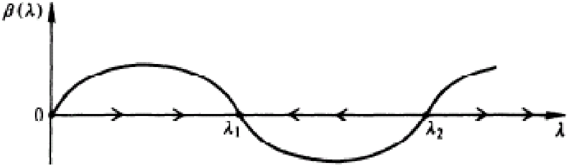
Thus the solution has the simple form

$$\Gamma_{as}^{(n)}(\sigma p_i,\lambda,\mu) = \sigma^{4-n}exp[-n\int_0^t \gamma(\bar\lambda(x',\lambda))dx'_{as}^{\prime(n)}(p_i,\bar\lambda(t,\lambda),\mu)$$

Effective coupling constant $\bar{\lambda}$

$$\frac{d\bar{\lambda}(t,\lambda)}{dt} = \beta(\bar{\lambda}) \qquad \text{initial condition} \quad \bar{\lambda}(0,\lambda) = \lambda$$

Suppose $\beta(\lambda)$ has the following simple behavior



Suppose $0 < \lambda < \lambda_1$,then at$t = 0$, $\frac{\bar{\lambda}}{dt} \mid_{t=0} > 0 \Rightarrow \bar{\lambda}$ increases as t increases
This increase will continue until $\bar{\lambda}$ reaches λ_1, where $\frac{\bar{\lambda}}{dt} = 0$
On the other hand, if initially $\lambda_1 < \lambda < \lambda_2$, then $\frac{\bar{\lambda}}{dt} \mid_{t=0} < 0$,$\bar{\lambda}$will decrease until it reaches λ_1. Thus as $t \to \infty$, we get

$$\lim_{t\to\infty} \bar{\lambda}(t,\lambda) = \lambda_1 \qquad \lambda_1 : \text{ultraviolet stable fixed point}$$

and

$$\Gamma_{as}^{(n)}(p_i, \bar{\lambda}(t,\lambda), \mu) \to_{t\to\infty} \Gamma_{as}^{(n)}(p_i, \lambda_1, \mu)$$

Example: Suppose $\beta(x)$ has a simple zero at $\lambda = \lambda_1$,

$$\beta(\lambda) \simeq a(\lambda_1 - \lambda) \qquad a > 0$$

Then

$$\frac{d\bar{\lambda}}{dt} = a(\lambda_1 - \lambda) \Rightarrow \bar{\lambda} = \lambda_1 + (\lambda - \lambda_1)e^{-at}$$

i.e. the approach to fixed point is exponential in t, or power in $t = \ln\sigma$. Also the prefactor can be simplified,

$$\int_0^t \gamma(\bar{\lambda}(x,\lambda))dx = \int_\lambda^{\bar{\lambda}} \frac{\gamma(y)dy}{\beta(y)} \approx \frac{-\gamma(\lambda_1)}{a}\int_\lambda^{\bar{\lambda}} \frac{d\lambda'}{\lambda' - \lambda_1} = \frac{-\gamma(\lambda_1)}{a}\ln(\frac{\bar{\lambda}-\lambda_1}{\lambda - \lambda_1})$$

$$= \gamma(\lambda_1)t = \gamma(\lambda_1)\ln\sigma$$

$$\lim_{\sigma\to\infty}\Gamma_{as}^{(n)}(\sigma p_i, \lambda, \mu) = \sigma^{4-n[1+\gamma(\lambda_1)]}\Gamma_{as}^{(n)}(p_i, \lambda_1, \mu)$$

Thus the asymptotic behavior in field theory is controlled by the fixed point λ_1 and $\gamma(\lambda_1)$ anomalous dimension.