Quantum Field Theory
Homework set 1, Due Tu March 16

1. The fine structure constant α which characterizes the electromagnetic interaction is given by

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$$

   Compute the magnitude of α.

2. Show that the combination

$$\frac{d^3p}{2E}, \qquad \text{with} \;\; E = \sqrt{\vec{p}^{\,2} + m^2}$$

   which occurs frequently in phase space calculation integration is invariant under Lorentz transformation.

3. Consider the combination of the form,

$$(\phi_1, \phi_2) \equiv \int d^3x \left[\phi_1^* \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1^*\right]$$

   (a) Show that this combination is time independent

   (b) Write the plane wave solutions in the form,

$$f_p^{(\pm)}(x) = e^{\mp ip\cdot x} \frac{1}{\sqrt{(2\pi)^3 \, 2\omega_p}}, \qquad \text{where } p_0 = \omega_p = \sqrt{\vec{p}^{\,2} + \mu^2} \geq 0$$

   Show that

$$\int d^3x \left[f_{p'}^{(\pm)*}(x)\, i\overleftrightarrow{\partial_0} f_p^{(\pm)}(x)\right] = \pm\delta^3\left(\vec{p} - \vec{p}\,'\right)$$

   and

$$\int d^3x \left[f_{p'}^{(\pm)*}(x)\, i\overleftrightarrow{\partial_0} f_p^{(\mp)}(x)\right] = 0$$

   (c) If we write the general solution $\phi(x)$ in the form,

$$\phi(x) = \int d^3p \left[a_p f_p^{(+)}(x) + a_p^* f_p^{(-)}(x)\right]$$

   compute the coefficients, a_p and a_p^*.

4. Consider a system where 2 particles interacting with eac other through potential energy $V\left(\vec{x}_1 - \vec{x}_2\right)$ so that the Lagrangian is of the form,

$$L = \frac{m_1}{2}\left(\frac{d\vec{x}_1}{dt}\right)^2 + \frac{m_2}{2}\left(\frac{d\vec{x}_2}{dt}\right)^2 - V\left(\vec{x}_1 - \vec{x}_2\right)$$

   (a) Show that this Lagrangian is invariant under the spatial translation given by

$$\vec{x}_1 \to \vec{x}_1' = \vec{x}_1 + \vec{a}, \qquad \vec{x}_2 \to \vec{x}_2' = \vec{x}_2 + \vec{a},$$

   where $\vec{a}$ is an arbitrary vector.

   (b) Use Noether's theorem to construct the conserved quantity corresponding to this symmetry.