Quantum Field Theory
Homework set 2, Due Tu March 30

1. The Dirac Hamiltonian for free particle is given by

$$H = \vec{\alpha} \cdot \vec{p} + \beta m$$

The angular momentum operator is of the form,

$$\vec{L} = \vec{r} \times \vec{p}$$

(a) Compute the commutators,

$$\left[\vec{L}, H\right]$$

Is $\vec{L}$ conserved?

(b) Define $\vec{S} = -\frac{i}{4}\left(\vec{\alpha} \times \vec{\alpha}\right)$ and show that

$$\left[\vec{L} + \vec{S}, H\right] = 0$$

(c) Show that $\vec{S}$ satisfy the angular momentum algebra, i.e.

$$[S_i, S_j] = i\varepsilon_{ijk} S_k$$

and

$$\vec{S}^2 = \frac{3}{4}.$$

2. The Dirac spinors are of the form,

$$u\left(p,s\right) = \sqrt{E+m}\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s, \qquad v\left(p,s\right) = \sqrt{E+m}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_s \qquad s = 1, 2$$

where

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a) Show that

$$\bar{u}\left(p,s\right)u\left(p,s'\right) = 2m\delta_{ss'}, \qquad \bar{v}\left(p,s\right)v\left(p,s'\right) = 2m\delta_{ss'}$$
$$\bar{v}\left(p,s\right)u\left(p,s'\right) = 0, \qquad \bar{u}\left(p,s\right)v\left(p,s'\right) = 0$$

(b) Show that

$$\sum_s u_\alpha\left(p,s\right)\bar{u}_\beta\left(p,s\right) = \left(\not{p} + m\right)_{\alpha\beta}$$
$$\sum_s v_\alpha\left(p,s\right)\bar{v}_\beta\left(p,s\right) = \left(\not{p} - m\right)_{\alpha\beta}$$

3. Suppose a free Dirac particle at t=0, is described by a wavefunction,

$$\psi\left(0, \vec{x}\right) = \frac{1}{\left(\pi d^2\right)^{3/4}} \exp\left(-\frac{r^2}{2d^2}\right)\omega$$

where d is some constant and

$$\omega = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Compute the wavefunction for $t \neq 0$. What happens when d is very small.

4. One-dimensional Schrodinger equation is given by

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V\left(x\right)\right]\psi\left(x,t\right)=i\hbar\frac{\partial\psi}{\partial t}$$

(a) Find the Lagrangian density which will give Schrodinger equation as the equation of motion.

(b) From the Lagrangian density find the conjugate momenta and impose the quantization conditions. Also find the Hamiltonian.

(c) For the case of linear harmonic oscillators the potential is given by

$$V\left(x\right)=\frac{1}{2}m\omega^2x^2$$

(d) Expand the field operator $\psi\left(x,t\right)$ in terms of the eigenfunctions of the linear harmonic oscillator,

$$\psi\left(x,t\right)=\sum_n a_n\phi_n\left(x\right)e^{iE_nt/\hbar}$$

where $E_n=\left(n+\frac{1}{2}\right)\hbar\omega$ and $\phi_n\left(x\right)$ is the normalized eigenfunction of the harmonic oscillator.Compute the commutors

$$\left[a_n,a_m\right],\qquad\left[a_n,a_m^\dagger\right]$$

(e) Find the eigenvalues of the Hamiltonian.