Quantum Field Theory I

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April 19, 2010

Homework set 4, Due Tue May 4

1. Dirac equation for electron moving in the electromagnetic field can be obtained from the free Dirac equation by the replacement $i\partial_\mu \longrightarrow i\partial_\mu - eA_\mu$,

$$\left[\gamma^\mu\left(i\partial_\mu - eA_\mu\right) - m\right]\psi\left(\vec{x}, t\right) = 0$$

Then the equation for the positron is

$$\left[\gamma^\mu\left(i\partial_\mu + eA_\mu\right) - m\right]\psi_c\left(\vec{x}, t\right) = 0$$

Assume that ψ_c is related to ψ by

$$\psi_c = \tilde{C}\psi^*$$

$\tilde{C}$ is called the charge conjugation matrix.

   (a) Find $\tilde{C}$ in terms of Dirac γ matrices.

   (b) For the $v-$spinor of the form,

$$v\left(p, s\right) = N\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_s$$

   Compute its charge conjugate $v_c\left(p, s\right) = \tilde{C}v^*\left(p, s\right)$

   (c) To implement the charge conjugation for the fermion field, we write

$$\psi_c = C\psi C^{-1} = \tilde{C}\psi^*$$

   where C is the charge conjugation operator. Find the relation between $\bar{\psi}_c\gamma^\mu\psi_c$ and $\bar{\psi}\gamma^\mu\psi$.

2. The parity operator P is defined as

$$\psi_p = P\psi P^{-1} = \gamma_0\psi\left(t, -\vec{x}\right)$$

   (a) Find the relation between $\bar{\psi}_p\psi_p$ and $\bar{\psi}\psi$.

   (b) Repeat part (a) for $\bar{\psi}_p\gamma_5\psi_p$, $\bar{\psi}_p\gamma_\mu\psi_p$,$\bar{\psi}_p\gamma_\mu\gamma_5\psi_p$,$\bar{\psi}_p\sigma_{\mu\nu}\psi_p$.

3. The left-handed and right-handed components of a Dirac particle are defined by,

$$\psi_L \equiv \frac{1}{2}\left(1-\gamma_5\right)\psi, \qquad \psi_R \equiv \frac{1}{2}\left(1+\gamma_5\right)\psi$$

   (a) Show that they are eigenstates of γ_5 matrix. What are the eigenvalues?

   (b) Are they eigenstates of parity operator?

   (c) Write the u spinor in the form,

$$u\left(p, s\right) = N\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s$$

   where N is some normalization constant and χ_s is an arbitrary 2 component spinor. Show that if we choose χ_s to be eigenstate of $\vec{\sigma}\cdot\vec{p}$,

$$\left(\vec{\sigma}\cdot\hat{p}\right)\chi_s = \frac{1}{2}\chi_s$$

then $u(p,s)$ is an eigenstate of the helicity operator $\lambda = \vec{S}\cdot\hat{p}$ where $\vec{S}$ is the spin operator given by

$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

(d) Show that in the limit $m \to 0$, the helicity eigenstate is also eigenstate of γ_5.

4. Consider a free scalar field $\phi(x)$ where the 4-momentum operator is of the form,

$$P^\mu = \int d^3k\; k^\mu a^\dagger(k)\, a(k)$$

(a) As a useful tool show that for two operators A and B, the following identity holds

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots$$

(b) Use this identity to show that

$$e^{iP\cdot x} a(k)\, e^{-iP\cdot x} = a(k)\, e^{-ik\cdot x}$$

and

$$[P^\mu, \phi(x)] = i\partial^\mu \phi(x)$$

(c) Let $|K\rangle$ be an eigenstate of P^μ, satisfying $P^\mu |K\rangle = K^\mu |K\rangle$. Show that

$$\langle K|\phi(x)\,\phi(y)|K\rangle = \langle K|\phi(x-y)\,\phi(0)|K\rangle$$