Quantum Field Theory I

Ling-Fong Li

May 4, 2010

Homework set 5, Due Tue May 25

1. In the $\lambda\phi^4$ theory, the interaction is given by
$$H_I = \frac{\lambda}{4!}\phi^4$$
   (a) Show that to lowest order in λ, the differential cross section for 2 particles elastic scattering in the center of mass frame is given by
$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{128\pi^2 S}$$
   where $S = (q_1 + q_2)^2$, q_1, q_2 are the momenta of the particles in the initial state.

   (b) Show that the 2-point Green's function satisifies the relation,
$$\left(\Box_x + \mu^2\right)\langle 0\left|T\left(\phi\left(x\right)\phi\left(y\right)\right)\right|0\rangle = \lambda\left\langle 0\left|T\left(\phi^3\left(x\right)\phi\left(y\right)\right)\right|0\right\rangle - i\delta^4\left(x-y\right)$$
   Verify this relation diagramatically to first order in λ.

2. Given a sequence of numbers $g_0, g_1, g_3, \cdots$, the most economical way to encompass them is through the generating function,
$$Z\left(j\right) = \sum_{m=0}^{\infty}\frac{g_m j^m}{m!}$$
   so that
$$g_m = \left.\frac{\partial^m Z\left(j\right)}{\partial j^m}\right|_{j=0}$$
   The generating functional for the Green's functions of scalar fields is defined by,
$$Z\left[J\right] = \sum_{m=0}^{\infty}\frac{i^m}{m!}\int d^4x_1\cdots d^4x_m J\left(x_1\right)\cdots J\left(x_m\right)G^{(m)}\left(x_1,\cdots x_m\right)$$
   where
$$G^{(m)}\left(x_1,\cdots x_m\right) = \langle 0\left|T\left(\phi\left(x_1\right)\cdots\phi\left(x_m\right)\right)\right|0\rangle$$
   and can be expressed in terms of generating functional as
$$G^{(m)}\left(x_1,\cdots x_m\right) = \left(-i\right)^m\left.\frac{\delta^m Z\left[J\right]}{\delta J\left(x_1\right)\cdots\delta J\left(x_m\right)}\right|_{J=0}$$
   (a) Show that $Z\left[J\right]$ can be written as
$$Z\left[J\right] = \left\langle 0\left|T\exp\left(i\int\phi\left(x\right)J\left(x\right)d^4x\right)\right|0\right\rangle$$
   (b) Show that for the case of free scalar field $Z\left[J\right]$ can be written as
$$Z\left[J\right] = \exp\left[-\frac{i}{2}\int d^4x d^4y J\left(x\right)\Delta_F\left(x-y\right)J\left(y\right)\right]$$
   where
$$i\Delta_F\left(x-y\right) = \langle 0\left|T\left(\phi\left(x\right)\phi\left(y\right)\right)\right|0\rangle$$

3. Transistion amplitude for a free particle with mass m moving in one dimension is given by

$$\langle q', t'|q, t\rangle = \left\langle q'|e^{-iH(t'-t)}|q\right\rangle$$

and

$$H = \frac{p^2}{2m}$$

   (a) Compute this amplitude directly from the Hamiltonian given above.

   (b) Compute this amplitude by using path integral method.

4. The Lagrangian for a simple harmonic oscillator is of the form

$$L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2$$

   (a) Show that the transistion amplitude has the form,

$$\langle q_f, t_f|q_i, t_i\rangle = \left[\frac{m\omega}{2\pi i \sin\omega\left(t_f - t_i\right)}\right]^{1/2} \exp\left\{\frac{im\omega}{2\sin\omega\left(t_f - t_i\right)}\left[\left(q_f^2 + q_i^2\right)\cos\omega\left(t_f - t_i\right) - 2q_f q_i\right]\right\}$$

   (b) Show that for an initial wave packet of the Gaussian form

$$\psi_a(q, 0) = \left(\frac{m\omega}{\pi}\right)^{1/4} \exp\left[-\frac{m\omega}{2}(q-a)^2\right]$$

   the wave function for $t \neq 0$ satisfies

$$|\psi_a(q, 0)|^2 = \left(\frac{m\omega}{\pi}\right)^{1/2} \exp\left[-\frac{m\omega}{2}(q - a\cos\omega t)^2\right]$$

   Namely, there is no spread of wave packet.