Quantum Field Theory I

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Homework set 6, Due Tue June 17

1. The Lagrangian for electromagnetic field is of the form,

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Suppose we add a mass term to the Lagrangian so that

$$L' = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + m^2 A_\mu A^\mu$$

   (a) Show that the equation of motion is of the form,

$$\left(\partial^\mu \partial_\mu + m^2\right) A^\nu - \partial^\nu \partial_\mu A^\mu = 0$$

   (b) Define the generating functional by

$$W[J] = \int [dA^\mu] \exp\left\{ i \int d^4x \left(L' + J^\mu A_\mu\right)\right\}$$

   Show that

$$W[J] = \exp\left\{ \frac{i}{2} \int d^4x d^4y J^\mu(x) G_{\mu\nu}(x-y) J^\nu(y)\right\}$$

   where Green's function by

$$\left[\left(\partial^\alpha \partial_\alpha + m^2\right) g_{\mu\nu} - \partial_\nu \partial_\mu\right] G^{\nu\beta}(x-y) = g_\mu^\beta \delta^4(x-y)$$

   Find the Green's function in momentum space.

2. Consider the unperturbed and perturbed parts of the scalar field theory of the form,

$$L_0(x) = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{\mu^2}{2}\phi^2, \qquad L_1(x) = -\frac{m^2}{2}\phi^2$$

In the perturbation theory, the two point Green's function is given by,

$$G^{(2)}(x_1, x_2) = \langle 0 | T(\phi(x_1)\phi(x_2)) | 0\rangle = \frac{\left\langle 0 \left| T\left(\phi(x_1)\phi(x_2)\exp\left[-i\int L_1(x()) d^4x\right]\right)\right| 0\right\rangle}{\left\langle 0\left| T\left(\exp\left[-i\int L_1(x()) d^4x\right]\right)\right| 0\right\rangle}$$

Use Wick's theorem to demonstrate explicitly the the respective disconnected graphs in the numeratro and denominator cancel.

3. Use the power-counting argument to construct counterterms and draw all the one-loop divergent 1PI graphs for the real scalar field with interactions of the form,

$$L_{int} = -\frac{\lambda_1}{3!}\phi^3 - \frac{\lambda_2}{4!}\phi^4$$

4. The QED Lagrangian is of the form,

$$L = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi - e\bar{\psi}\gamma^\mu \psi A_\mu - -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

(a) Show that this Lagrangian is invariant under the gauge transformation,

$$\psi(x) \longrightarrow \psi'(x) = e^{-i\alpha(x)}\psi(x)$$

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)$$

(b) Use the power-counting argument to construct counterterms. Note that the photon propagator can be taken to be of the form,

$$D_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon}$$