Homework set 7, Optional

1. As discussed in class a 2×2 unitary matrix U will induce a rotation in R_3 through the connection,

$$h = \vec{\sigma} \cdot \vec{r}$$

   (a) Show that if U is diagonal the corresponding rotation about $z-$axis.

   (b) Show that if U is real

2. The quarks have spin $\frac{1}{2}$. Construct 3 quarks states with spin $\frac{3}{2}$ and $\frac{1}{2}$ respectively.

3. Consider a spin $\frac{1}{2}$ wave function

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

   which transforms under the rotation as

$$\psi \to \psi' = \exp\left(-i\frac{\vec{\sigma} \cdot \vec{\alpha}}{2}\right)\psi$$

   where $\vec{\sigma}$ are Pauli matrices and $\vec{\alpha}$ are arbitray real parameters. Let $\vec{\phi}$ be a spin 1 wave function which transforms under infinitemal rotation as

$$\vec{\phi} \to \vec{\phi}' = \vec{\phi} + i\vec{\alpha} \times \vec{\phi}$$

   (a) Show that the combination

$$\psi^\dagger \left(\vec{\sigma} \cdot \vec{\phi}\right)\psi$$

   is invariant under rotation.

   (b) Show that for finite rotation we can write

$$\vec{\sigma} \cdot \vec{\phi}' = U\left(\vec{\sigma} \cdot \vec{\phi}\right)U^\dagger$$

   where U is an arbitrary 2×2 unitary matrix.

4. The generators of 3-dimensional rotation group $O(3)$ can be taken as

$$J_i = \frac{1}{2}\varepsilon_{ijk}M_{jk} \qquad \text{with} \quad M_{jk} = -i\left(x_j\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_j}\right), \qquad i,j,k = 1,2,3$$

   (a) Show that

$$[J_i, J_j] = i\varepsilon_{ijk}J_k$$

   (b) Extension to rotation group in 4-dimension $O(4)$ can be achieved by extending the indices in M_{ij} to $i,j = 1,2,3,4$, i,e.

$$M_{jk} = -i\left(x_j\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_j}\right), \qquad j,k = 1,2,3,4$$

   Again write

$$J_i = \frac{1}{2}\varepsilon_{ijk}M_{jk} \qquad i,j,k = 1,2,3$$

   and define

$$K_i = M_{i4}$$

   Compute the commutators

$$[J_i, J_j] \qquad [J_i, K_j] \qquad [K_i, K_j]$$

   (c) Define

$$A_i = J_i + K_i, \qquad B_i = J_i - K_i$$

   Show that $A_i's$ and $B_i's$ each form a $SU(2)$ algebra and

$$[A_i, B_j] = 0$$

5. In Yang-Mills theory with isospin $SU(2)$ symmetry, consider a $SU(2)$ doublet of the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

which transform as

$$\psi \to \psi' = U(\theta)\,\psi, \qquad \text{with} \quad U(\theta) = \exp\left(-i\frac{\vec{\tau}\cdot\vec{\theta}(x)}{2}\right)$$

The covariant derivative is of the form

$$D_\mu \psi = \left(\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\right)\psi$$

Under the gauge transformation we have

$$\frac{\vec{\tau}\cdot\vec{A'}_\mu}{2} = U\left(\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\right)U^{-1} - \frac{i}{g}\left(\partial_\mu U\right)U^{-1}$$

Show that $\vec{A}_\mu$transform as $I = 1$ triplet.

6. Suppose the set of scalar fields $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ transform as a vector in $O(3)$. The self interaction of $\vec{\phi}$ is of the form,

$$L = \frac{1}{2}\left(\partial_\mu \vec{\phi}\right)^2 - V\left(\vec{\phi}\right)$$

where

$$V\left(\vec{\phi}\right) = -\frac{\mu^2}{2}\left(\vec{\phi}\cdot\vec{\phi}\right) + \frac{\lambda}{4}\left(\vec{\phi}\cdot\vec{\phi}\right)^2$$

(a) Find the minimum of $V\left(\vec{\phi}\right)$.

(b) Find the combinations of $\phi_i' s$, which are Goldstone bosons.