3. $\psi(0,\vec{x}) = \frac{1}{(\pi d^2)^{3/2}} \exp\left(-\frac{r^2}{2d^2}\right) w \qquad w = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$$\psi(0,\vec{x}) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left[b(p,s)\, e^{i\vec{p}\cdot\vec{x}} u(p,s) + d^+(p,s)\, e^{-i\vec{p}\cdot\vec{x}} v(p,s) \right]$$

$$\int \psi(0,\vec{x})\, e^{-i\vec{p}\cdot\vec{x}}\, d^3x = \sum_{s'} \frac{(2\pi)^{3/2}}{2E_p} \left[b(p,s)\, u(p,s) + d^+(-\vec{p},s)\, v(-\vec{p},s) \right]$$

note that $\quad v(-\vec{p},s) = \sqrt{\ } \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix} \chi_s$ and $u^+(p,s)\, v(-\vec{p},s) = 0$

$$u^+(p,s) \int \psi(0,\vec{x})\, e^{-i\vec{p}\cdot\vec{x}}\, d^3x = \sum_s \frac{(2\pi)^{3/2}}{2E_p} b(p,s)\, u^+(p,s)\, u(p,s')$$

useful relation

$(\not{p}-m)\, u(p,s) = 0 \qquad \bar{u}(p,s)(\not{p}-m) = 0$

$\bar{u}(p,s)\gamma^\mu(\not{p}-m)\, u(p,s) = 0 \qquad \bar{u}(p,s)(\not{p}-m)\gamma^\mu\, u(p,s) = 0$

Add these 2 equations

$$\bar{u}(p,s)(\gamma^\mu \not{p} + \not{p}\gamma^\mu)\, u(p) = 2m\, \bar{u}\gamma^\mu u.$$

$$(\gamma^\mu \not{p} + \not{p}\gamma^\mu) = (\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) p_\nu = 2g^{\mu\nu} p_\nu = 2p^\mu$$

or $\quad \boxed{\bar{u}(p,s)\gamma^\mu u(p,s) = \left(\frac{p_\mu}{m}\right) \bar{u}(p,s)\, u(p,s)}$

Using this we get

$$u^+(p,s)\, u(p,s) = \bar{u}(p,s)\gamma^0 u(p,s) = \left(\frac{E_p}{m}\right) \bar{u}(p,s)\, u(p,s') = 2m\,\delta_{ss'} \frac{E}{m} = 2E_p\,\delta_{ss'}$$

$$\int u^+(p,s)\, \psi(0,\vec{x})\, e^{-ip\cdot x}\, d^3x = (2\pi)^{3/2}\, b(p,s)$$

Similarly,

$$\int v^+(-\vec{p},s)\, e^{-i\vec{p}\cdot x}\, \psi(0,\vec{x})\, d^3x = \sum_{s'} \frac{(2\pi)^{3/2}}{2E_p} d^+(-\vec{p},s')\, v^+(-\vec{p},s')\, v(-\vec{p},s)$$

$$= (2\pi)^{3/2}\, d^+(-\vec{p},s)$$

$$\int u^+(p.s)\,\psi(0,x)\,e^{-ip\cdot x}\frac{d^3x}{(2\pi)^{3/2}} = \int\frac{d^3x}{(2\pi)^{3/2}}\,e^{-ip\cdot x}\,e^{-r^2/2d^2}\,\frac{1}{(\pi d^2)^{3/4}}\,u^+(p.s)\,\omega$$

Gaussian integral

$$\int_{-\infty}^{+\infty} e^{-bx^2+ax}\,dx = e^{+a^2/4b}\sqrt{\frac{\pi}{b}}$$

$$\int e^{ipx}\,e^{-x^2/2d^2}\,dx = \left[\exp\left(-\frac{p^2d^2}{2}\right)\right]\sqrt{\pi 2d^2}$$

$$\int\frac{d^3x}{(2\pi)^{3/2}}\,\frac{e^{-i\vec{p}\cdot\vec{x}}\,e^{-r^2/2d^2}}{(\pi d^2)^{3/4}} = \exp\left(-\frac{\vec{p}^{\,2}d^2}{2}\right)(2\pi)^{3/2}d^3\cdot\frac{1}{(2\pi)^{3/2}}\,\frac{1}{(\pi d^2)^{3/4}}$$

$$= \exp\left(-\frac{p^2d^2}{2}\right)\frac{d^{3/2}}{\pi^{3/4}}$$

$$u^+(p.s)\,\omega = \sqrt{E+m}$$

$$v^+(-p.s).\omega = \chi_s^+\left(-\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\begin{pmatrix}1\\0\end{pmatrix} = -\frac{p_z}{E+m}\sqrt{E+m} \qquad \text{for } s=1$$

$$= -\frac{(p_x+ip_y)}{E+m}\sqrt{E+m} \qquad s=2$$

$$\Rightarrow \quad \frac{d^+(-\vec{p}.s)}{b(p.s)} \sim \frac{p}{E+m}$$

ratio is small for $|p| \ll m$ non-relativistic motion

ratio is of order 1 for $p \simeq E$ relativistic motion

4. Schrodinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x)$$

(a) Lagrangian density

$$L = \psi^+ i\partial_0\psi - \frac{1}{2m}(\partial_x\psi^+)(\partial_x\psi) - \psi^+\psi\,V(x)$$

$$\frac{\partial L}{\partial\psi^+(x)} = i\partial_0\psi - V(x)\psi \qquad \frac{\partial L}{\partial_0\psi^+} = 0 \qquad \frac{\partial L}{\partial_x\psi} = -\frac{1}{2m}\partial_x\psi$$

then $\partial_x\frac{\partial L}{\partial_x\psi^+} + \partial_0\frac{\partial L}{\partial_0\psi^+} = \frac{\partial L}{\partial\psi^+} \Rightarrow i\partial_0\psi - V\psi = -\frac{1}{2m}\partial_x^2\psi$ Schroding eq conjugate momenta

$$\pi(x,t) = \frac{\partial L}{\partial(\partial_0\psi)} = i\psi^+$$

Hamiltonian density

$$\mathcal{H} = \pi\partial_0\psi - \mathcal{L} = \cancel{\pi\partial_0\psi - [} = i\psi^+\partial_0\psi - [i\psi^+\partial_0\psi - \frac{1}{2m}(\partial_x\psi^+)(\partial_x\psi) - V(x)\psi^+(x)\psi]$$

$$= \frac{1}{2m}(\partial_x\psi^+)(\partial_x\psi) + V(x)\,\psi^+(x)\psi(x)$$

~~mode expansion~~ eigenstates

ψ. $\left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi_n(x) = E_n\psi_n(x)$ & $\int \psi_n^*(x)\psi_m(x)\,d^3x = \delta_{nm}$

solution to time dependent Schrodinger equation is $\psi_n(x)e^{-iE_nt}$

Quantization

$$[\psi(x,t), \pi(x',t)] = i\delta(x-x')$$

or $[\psi(x,t), \psi^+(x',t)] = \delta(x-x')$

Mode expansion

$$\psi(x,t) = \sum_n a_n\psi_n(x)e^{-iE_nt} \Rightarrow \psi^+ = \sum_n a_n\psi_n^* e^{+iE_nt}$$

$$a_n = \int e^{iE_nt}\psi_n(x)\psi(x,t)\,dx \qquad a_n^+ = \int e^{-iE_nt}\psi_n^*(x)\psi^+(x,t)\,dx$$

Note that a_n and a_n^+ are time-independent

$$[a_n, a_m^+] = e^{iE_nt}e^{-iE_mt}\int dx\,dx'\underbrace{[\psi(x,t), \psi^+(x',t)]}_{\delta(x-x')}\psi_n(x)\psi_m^*(x')$$

$$= \delta_{nm}$$

Similarly $[a_n, a_n] = 0$

Hamiltonian

$$H = \int dx\,\mathcal{H}(x) = \int\left[\frac{1}{2m}(\partial_x\psi^+)(\partial_x\psi) + V(x)\psi^+(x)\psi(x)\right]dx$$

$$= \int\sum_{n,m}\left\{\frac{1}{2m}\partial_x\psi_n^*(x)\,\partial_x\psi_m(x)\,a_n^+a_m + V(x)\,a_n^+a_m\,\psi_n^*(x)\psi_m(x)\right\}dx$$

$$= \sum_{n,m}\int\left\{-\frac{1}{2m}\psi_n^*\partial_x^2\psi_m\,a_n^+a_m + a_n^+a_m\,\psi_n^*(x)V(x)\psi_m\right\}dx$$

$$= \sum_{n,m}\int\left\{\psi_n^*\left(-\frac{1}{2m}\partial_x^2 + V(x)\right)\psi_m\,a_n^+a_m\right\}dx$$

$$H = \sum_{n,m} \int \{ E_m \psi_n^*(x) \psi_m(x) \, dx \; a_n^+ a_m \} = \sum_n (a_n^+ a_m)$$

Eigenstates of H

$$|0\rangle, \; a_n^+ |0\rangle, \; a_n^+ a_m^+ |0\rangle \cdots$$

where $\quad a_n |0\rangle = 0$ for all n

<u>eigenvalues</u>

$$E(n_1, n_2 \cdots) = \cancel{\sum_{i=1\ldots} n_i \epsilon_i} \sum_k n_k E_k \qquad n_k : \text{\# of particles in level } E_k$$

For the case of harmonic oscillator, we have

$$E_k = (k + \tfrac{1}{2}) \hbar \omega$$