Klein-Gordon Eq

2010年2月8日
下午 03:02

Classically, we have

$$\frac{}{2m} + V(\vec{r})$$

Quantization, $E \to i\frac{\partial}{\partial t}$, $\vec{p} \to -i\vec{\nabla}$

$$i\frac{\partial \psi}{\partial t} = \left[-\frac{1}{2m}\vec{\nabla}^2 + V(\vec{r})\right]\psi \qquad \text{Schrodiger equation}$$

This equation does not work for relativistic system beacuse spatial coordinate $\vec{x}$ and time t are not on equal footing.

For relativistic free particle, we have

$$E^2 = \vec{p}^2 + m^2$$

The corresponding wave equation is then

$$(-\vec{\nabla}^2 + m^2)\psi = -\partial_0^2 \psi$$

or $\quad (\Box + m^2)\psi = 0 \qquad$ where $\Box \equiv \partial_0^2 - \vec{\nabla}^2 = \partial^\mu \partial_\mu = \partial^2$

This is known as Klein-Gordon equation

<u>Probability interpretation</u>

$$(\partial_0^2 - \vec{\nabla}^2 + m^2)\psi = 0 \quad \Longrightarrow \quad (\partial_0^2 - \vec{\nabla}^2 + m^2)\psi^* = 0$$

We can derive the continuity equation,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \qquad \text{where } \rho = i(\psi^*\partial_0\psi - \psi\partial_0\psi^*), \quad \vec{j} = -i(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)$$

Then $\quad P = \int \rho\, d^3x \quad$ is conserved i.e. $\quad \frac{dP}{dt} = \int_V \frac{\partial \rho}{\partial t} d^3x = -\int_V \vec{\nabla}\cdot\vec{j}\, d^3x = -\oint_S \vec{j}\cdot d\vec{S} = 0 \quad$ if $\vec{j} = 0$ on S

P = probability ? $\quad$ but ρ is not positive

e.g if $\quad \psi \sim e^{+iEt}\phi(x) \quad \Rightarrow \quad \rho = -2E|\phi|^2 < 0$.

If we take $\rho = \psi^*\psi > 0$ as in the case of Schrodinger equation, then

$$\frac{d}{dt}\int \psi^*\psi\, d^3x \neq 0 \qquad \text{not conserved.}$$

$\Rightarrow$ It is impossible to have probability interpretation.

<u>Solutions to Klein-Gordon Eq</u>

$$(\Box + m^2)\phi(x) = (\partial_0^2 - \vec{\nabla}^2 + m^2)\phi(x) = 0$$

plane wave solution $\qquad \phi(x) = e^{-ipx} \qquad$ if $\; p_0^2 - \vec{p}^2 - m^2 = 0 \;$ or $\; p_0 = \pm\sqrt{\vec{p}^2 + m^2}$

$P_0 = \omega_p = \sqrt{\vec{p}^2 + m^2}$ $\qquad$ or $\qquad P_0 = -\omega_p = -\sqrt{\vec{p}^2 + m^2}$ $\qquad\qquad \vec{p}$: arbitrary

positive energy $\qquad\qquad\qquad\qquad$ negative energy

General solution

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a(k)e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + a^\dagger(k)\, e^{-i\vec{k}\cdot\vec{x} + i\omega_k t}\right]$$

$$= \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right] \qquad \text{where } k\cdot x = \omega_k t - \vec{k}\cdot\vec{x}$$

Dirac Eq

2010年2月11日
下午 09:56

Dirac (1928): relativic wave equation 1st order in time derivative

assume an ansatz $E = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta m = \vec{\alpha}\cdot\vec{p} + \beta m$

then $E^2 = (\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta m)^2 = \frac{1}{2}(\alpha_i\alpha_j + \alpha_j\alpha_i) p_i p_j + \beta^2 m^2 + (\alpha_i\beta + \beta\alpha_i) m p_i$

To get relativistic energy momentum relation, we require

$$\begin{cases} \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \\ \alpha_i\beta + \beta\alpha_i = 0 \\ \beta^2 = 1 \end{cases}$$

these properties imply that α_i, β all have eigenvalues ± 1 and traceless $\Rightarrow$ even dimension

Bjorken and Drell notation

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac eq $\quad (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi = i\frac{\partial\psi}{\partial t}$

or $\quad (-i\beta\vec{\alpha}\cdot\vec{\nabla} - i\beta\partial_t + m)\psi = 0$

$$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha_i \quad \Rightarrow \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

then $\quad (-i\gamma^i\partial_i - i\gamma^0\partial_0 + m)\psi = 0$

or $\quad (i\gamma^\mu\partial_\mu - m)\psi = 0 \qquad$ Dirac equation in coveriant form

Note that $\quad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$

From the Dirac equation in the hermition form we get

$$-i\frac{\partial\psi^\dagger}{\partial t} = [(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi]^\dagger$$

$$\Rightarrow \quad +i\left(\frac{\partial\psi^\dagger}{\partial t}\psi + \psi^\dagger\frac{\partial\psi}{\partial t}\right) = +\psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi - [(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi]^\dagger\psi$$

$$i\frac{d}{dt}\int d^3x\,(\psi^\dagger\psi) = -i\int\left[\psi^\dagger(\vec{\alpha}\cdot\vec{\nabla})\psi - (-i)[(\vec{\alpha}\cdot\vec{\nabla})\psi]^\dagger\psi\right]d^3x = 0$$

$\downarrow$

conserved and positive!

Solution

Write $\quad \psi(x) = e^{-ip\cdot x}\begin{pmatrix} u \\ \ell \end{pmatrix} \qquad u, \ell$. 2 component column vector

Dirac eq $\Rightarrow \quad (\not{p} - m)\begin{pmatrix} u \\ \ell \end{pmatrix} = 0 \qquad$ where $\not{p} = \gamma^\mu p_\mu$

or $\quad \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} u \\ \ell \end{pmatrix} = p_0\begin{pmatrix} u \\ \ell \end{pmatrix} \quad \Rightarrow \quad \begin{cases} (p_0 - m)u - (\vec{\sigma}\cdot\vec{p})\ell = 0 \\ -(\vec{\sigma}\cdot\vec{p})u + (p_0 + m)\ell = 0 \end{cases}$ homogeneous equations

Non-trivial solution exits if $\quad \begin{vmatrix} p_0 - m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & p_0 + m \end{vmatrix} = 0 \quad$ or $\quad p_0^2 = \vec{p}^2 + m^2$

i) $p_0 = E = \sqrt{\vec{p}^2 + m^2}$, positive energy sol

$$\psi = e^{-ip\cdot x}\begin{pmatrix} u \\ \ell \end{pmatrix} = e^{-ip\cdot x} N\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi$$

χ: arbitrary 2 component vector
N: normalization constant.

ii) $p_0 = -E = -\sqrt{\vec{p}^2 + m^2}$, negative energy sol

$$\psi = e^{-ip\cdot x} N\begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi$$

Notation

$$u(p,s) = N\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s \quad , \quad v(p,s) = N\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_s \qquad N = \sqrt{E+m}$$

Dirac conjugate

momentum space $\quad (\not{p} - m)\psi(p) = 0$

$(E+m)$ [illegible] $(\;,\;)$

Dirac conjugate

Dirac eq in momentum space $\quad (\not{p}-m)\psi(p)=0$

Hermitian conjugate $\quad \psi^{+}(p)\,(\not{p}^{+}-m)=0$

Note that γ_μ not hermitian

$$\gamma_0^+=\gamma_0 \qquad \gamma_i^+=-\gamma_i \qquad \Rightarrow \quad \gamma_\mu^+=\gamma_0\gamma_\mu\gamma_0$$

Then

$$\psi^+(p)\,(\gamma_0\gamma_\mu\gamma_0 P^\mu - m)=0 \qquad \text{or} \qquad \psi^+(p)\gamma_0\,(\gamma_\mu P^\mu - m)=0$$

$$\Rightarrow \quad \boxed{\bar{\psi}(p)\,(\not{p}-m)=0}$$

where $\quad \bar{\psi}(p)=\psi^+\gamma_0 \qquad$ Dirac conjugate

Dirac equation under Lorentz transformation

$$(i\gamma^\mu\partial_\mu-m)\psi(x)=0$$

Lorentz transformation $\quad x^\mu \to x'^\mu=\Lambda^\mu{}_\nu x^\nu$

In the new coordinate system, the Dirac eq is of the form

$$(i\gamma^\mu\partial'_\mu-m)\psi'(x')=0$$

Note that we have used the same γ matrices (In general, different sets of γ-matrices are related by similarity transformation — Pauli's theorem)

Assume that $\psi'(x')$ and $\psi(x)$ are related by

$$\psi'(x')=S\,\psi(x)$$

Invert the Lorentz transformation,

$$x^\gamma=\Lambda^\gamma{}_\mu x'^\mu \qquad \Rightarrow \qquad \frac{\partial}{\partial x'^\mu}=\frac{\partial}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x'^\mu}=\Lambda^\gamma{}_\mu\frac{\partial}{\partial x^\gamma}$$

Then

$$(i\gamma^\mu\Lambda^\alpha{}_\mu-m)\,S\psi(x)=0 \qquad \text{or} \qquad [\,i(S^{-1}\gamma^\mu S)\Lambda^\alpha{}_\mu\partial_\alpha-m\,]\,\psi(x)=0$$

In order for this equation to be equivalent to the original Dirac equation, we require

$$(S^{-1}\gamma^\mu S)\,\Lambda^\alpha{}_\mu=\gamma^\alpha$$

or

$$\boxed{(S^{-1}\gamma^\mu S)=\Lambda^\mu{}_\alpha\gamma^\alpha}$$

To construct S, we consider infinitesmal transformation

$$\Lambda^\mu{}_\nu=g^\mu{}_\nu+\epsilon^\mu{}_\nu+\cdots \qquad |\epsilon^\mu{}_\nu|\ll 1.$$

Pseudo-orthogonality implies

$$\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta g_{\mu\nu}=g_{\alpha\beta} \qquad \Longrightarrow \qquad g_{\mu\nu}(g^\mu{}_\alpha+\epsilon^\mu{}_\alpha)(g^\nu{}_\beta+\epsilon^\nu{}_\beta)=g_{\alpha\beta}$$

$$\text{or} \quad \epsilon_{\alpha\beta}+\epsilon_{\beta\alpha}=0 \qquad\qquad \epsilon_{\alpha\beta}:\ \text{antisymmetric}$$

Write S as

$$S=1-\frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}+O(\epsilon^2) \qquad \text{then} \quad S^{-1}=1+\frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu} \qquad\qquad \sigma_{\mu\nu}:\ 4\times4 \text{ matrices.}$$

Then

$$(1+\tfrac{i}{4}\sigma_{\alpha\beta}\epsilon^{\alpha\beta})\,\gamma^\mu\,(1-\tfrac{i}{4}\sigma_{\alpha\beta}\epsilon^{\alpha\beta})=(g^\mu{}_\alpha+\epsilon^\mu{}_\alpha)\gamma^\alpha$$

$$\Rightarrow \qquad \epsilon^{\alpha\beta}\,\frac{i}{4}[\sigma_{\alpha\beta},\gamma^\mu]=\epsilon^\mu{}_\alpha\gamma^\alpha=\frac{1}{2}\epsilon^{\alpha\beta}\,(g_\alpha{}^\mu\gamma_\beta-g_\beta{}^\mu\gamma_\alpha)$$

$$\Rightarrow \qquad [\sigma_{\alpha\beta},\gamma_\mu]=2i\,(g_{\beta\mu}\gamma_\alpha-g_{\alpha\mu}\gamma_\beta)$$

It is straightforward to check that $\sigma_{\alpha\beta}$ given by

$$\sigma_{\alpha\beta}=\frac{i}{2}[\gamma_\alpha,\gamma_\beta]$$

will have the right property.

For the finite Lorentz transformation, we have

$$S=\exp\left[-\frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}\right]$$

Note that [illegible] $\Rightarrow$ S is not unitary

Note that $\sigma_{\mu\nu}^{+} = \gamma_0 \sigma_{\mu\nu} \gamma_0$ & $S^{+} = \gamma^0 S^{-1} \gamma^0 \Rightarrow S$ is not unitary

From $\psi'(x') = S\psi(x)$ we get $\psi'^{+}(x') = \psi^{+} S^{+} = \psi^{+} \gamma^0 S^{-1} \gamma^0$

or $\bar{\psi}'(x') = \bar{\psi}(x) S^{-1}$

Fermion bilinears

$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x) S^{-1} S \psi(x) = \bar{\psi}(x)\psi(x)$ Lorentz invariant

Similarly, $\bar{\psi}\gamma_\mu\psi$ 4-vector

$\bar{\psi}\gamma_\mu\gamma_5\psi$ axial vector

$\bar{\psi}\sigma_{\mu\nu}\psi$ 2nd rank antisymm tensor

$\bar{\psi}\gamma_5\psi$ pseudo scalar

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

Hole theory (Dirac 1930)

To solve the problem with negative energy states, Dirac proposed that the vacuum is the one in which $E<0$ states are filled and $E>0$ states are empty. Then Pauli exclusion principle will prevent an electron from moving into $E<0$ states.

In this picture,

hole in negative sea $\Leftrightarrow$ absence of an electron with charge $-|e|$ and energy $-E$

$\Leftrightarrow$ presence of $+|e|$ and $+E$ particle "positron"

charge conjugation: particle $\Longleftrightarrow$ anti-particle

Lorentz group

2010年2月12日
下午 09:50

The Dirac γ matrices seem to be ad hoc in Dirac eq. In fact, they are related to representations of Lorentz group.

Lorentz group : collection of linear transformations of space-time coordinates

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

which leaves the proper time

$$\tau^2 = (x^0)^2 - (\vec{x})^2 = x^\mu x^\nu g_{\mu\nu} = x^2$$

invariant. This requires the transformation matrix $\Lambda^\mu{}_\nu$ satisfies the pseudo-orthogonality relation,

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g_{\mu\nu} = g_{\alpha\beta}$$

<u>Generators</u>

For infinitesmal transformation.

$$\Lambda^\mu{}_\alpha = g^\mu{}_\alpha + \epsilon^\mu{}_\alpha \qquad \text{with } |\varepsilon^\mu{}_\alpha| \ll 1$$

As before, the pseudo-orthogonality relation implies,

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$$

Consider $f(x^\mu)$, an arbitrary function of x^μ, Under the infinitesmal Lorentz transformation,

$$f(x^\mu) \to f(x'^\mu) = f(x^\mu + \varepsilon^\mu{}_\alpha x_\alpha) = f(x^\mu) + \varepsilon_{\alpha\beta} x^\beta \partial_\alpha f$$

$$= f(x^\mu) + \tfrac{1}{2}\varepsilon_{\alpha\beta}[x^\beta \partial^\alpha - x^\alpha \partial^\beta] f(x) + \cdots$$

Introduce an operator $M_{\mu\nu}$ to represent this change,

$$f(x') = f(x) + \tfrac{i}{2}\varepsilon_{\alpha\beta} M^{\alpha\beta} f(x) + \cdots$$

then

$$M^{\alpha\beta} = i(x^\alpha \partial^\beta - x^\beta \partial^\alpha) \qquad \text{generators for Lorentz group}$$

Note that for $\alpha, \beta = 1, 2, 3$, these are just the angular momentum operator.

It is straightforward to work out the commutators of these generators,

$$[M_{\alpha\beta}, M_{\gamma\delta}] = i\{g_{\beta\gamma} M_{\alpha\delta} - g_{\alpha\gamma} M_{\beta\delta} - g_{\beta\delta} M_{\alpha\gamma} + g_{\alpha\delta} M_{\beta\gamma}\}$$

Define

$$M_{ij} = \epsilon_{ijk} J_k, \qquad M_{0i} = K_i$$

$$\Downarrow$$

$$J_i = \tfrac{1}{2}\epsilon_{ijk} M_{jk}$$

Then

$$[J_i, J_j] = \tfrac{1}{2}\varepsilon_{ik\ell}\varepsilon_{jmn}[M_{k\ell}, M_{mn}] = (\tfrac{1}{2})^2 \epsilon_{ik\ell}\varepsilon_{jmn}(g_{\ell m} M_{kn} - g_{km} M_{\ell n} - g_{\ell n} M_{km} + g_{kn} M_{\ell m})$$

$$= (\tfrac{1}{2})^2 i[-\epsilon_{ik\ell}\varepsilon_{j\ell n} M_{kn} + \epsilon_{ik\ell}\varepsilon_{jkn} M_{\ell n} + \varepsilon_{ik\ell}\varepsilon_{jm\ell} M_{km} - \epsilon_{ik\ell}\varepsilon_{jmk} M_{\ell m}]$$

Use identity

$$\epsilon_{abc}\varepsilon_{a\ell m} = (\delta_{b\ell}\delta_{cm} - \delta_{bm}\delta_{c\ell})$$

$$\boxed{[J_i, J_j] = i\epsilon_{ijk} J_k} \qquad \text{as expected.}$$

Similarly, we can get

$$\boxed{[K_i, K_j] = -i\epsilon_{ijk} J_k}$$

$$\boxed{[J_i, J_j] = i\,\epsilon_{ijk}\, J_k}$$ as expected.

Similarly, we can get

$$\boxed{\begin{aligned}[K_i, K_j] &= -i\,\epsilon_{ijk}\, J_k \\ [J_i, K_j] &= i\,\epsilon_{ijk}\, K_k\end{aligned}}$$

For convience, we define

$$A_i = \tfrac{1}{2}(J_i + i K_i), \qquad B_i = \tfrac{1}{2}(J_i - i K_i)$$

Then

$$[A_i, A_j] = i\,\epsilon_{ijk}\, A_k$$
$$[B_i, B_j] = i\,\epsilon_{ijk}\, B_k$$
$$[A_i, B_j] = 0$$

This means that the algebra of Lorentz generators factorizes into 2 independent $SU(2)$ algebra. The representation's are just the tensor products of $SU(2)$ alagebra. Thus we label the irreducible representation by (j_1, j_2) which transforms as $(2j_1+1)$-dim representation under A_i algebra and $(2j_2+1)$-dim representation under B_i algebra

<u>Simple representations</u>

1) $(1/2, 0)$ representation χ_α.

$$A_i \chi_a = \left(\frac{\sigma_i}{2}\right)_{ab} \chi_b \quad \Rightarrow \quad \tfrac{1}{2}(J_i + iK_i)\chi_a = \left(\frac{\sigma_i}{2}\right)_{ab}\chi_b$$
$$B_i \chi_a = 0 \qquad\qquad\qquad \tfrac{1}{2}(J_i - iK_i)\chi_a = 0$$

or
$$\begin{cases} \vec{J}\chi = \left(\frac{\vec{\sigma}}{2}\right)\chi \\ \vec{K}\chi = -i\left(\frac{\vec{\sigma}}{2}\right)\chi \end{cases}$$

2) $(0, 1/2)$ representation η_a

$$A_i \eta_a = 0 \qquad\qquad \Rightarrow \quad \tfrac{1}{2}(J_i + iK_i)\eta_a = 0$$
$$B_i \eta_a = \left(\frac{\sigma_i}{2}\right)_{ab}\eta_b \qquad \tfrac{1}{2}(J_i - iK_i)\eta_a = \left(\frac{\sigma_i}{2}\right)_{ab}\eta_b$$

or
$$\begin{cases} \vec{J}\eta = \frac{\vec{\sigma}}{2}\eta \\ \vec{K}\eta = \left(i\frac{\vec{\sigma}}{2}\right)\eta \end{cases}$$

If we define a 4-component ψ by

$$\psi = \begin{pmatrix}\chi \\ \eta\end{pmatrix} \Rightarrow \vec{J}\psi = \begin{pmatrix}\frac{\vec{\sigma}}{2} & \\ & \frac{\vec{\sigma}}{2}\end{pmatrix} \qquad \vec{K}\psi = \begin{pmatrix}-i\vec{\sigma}/2 & \\ & i\vec{\sigma}/2\end{pmatrix}\psi.$$

ψ are related to the 4-component Dirac field we studied before. This can be seen as follows.

Consider Dirac matrices in the following form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix} \qquad \text{where} \quad \sigma^\mu = (1, \vec{\sigma}), \quad \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

In other words,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0\end{pmatrix}$$

It is straightforward to check that in this case

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This means that in 4-component field $\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$. χ is right-handed and η is left-handed. In this representation, it is easy to check that

$$\sigma_{0i} = i\gamma_0\gamma_1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix}$$

$$\sigma_{ij} = i\gamma_i\gamma_j = i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

In the Lorentz transformation of Dirac field,

$$\psi'(x') = S\psi = \exp\left\{-\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}\right\} = \exp\left\{-\frac{i}{4}(2\sigma_{0i}\varepsilon^{0i} + \sigma_{ij}\varepsilon^{ij})\right\}$$

Write $\varepsilon^{0i} = \beta^i$, $\varepsilon^{ij} = \varepsilon^{ijk}\theta^k$

$$\sigma_{ij}\epsilon^{ij} = \varepsilon^{ijk}\theta^k\epsilon_{ijl}\begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix} = 2\begin{pmatrix} \vec{\sigma}\cdot\vec{\theta} & 0 \\ 0 & \vec{\sigma}\cdot\vec{\theta} \end{pmatrix}$$

$$\sigma_{0i}\varepsilon^{0i} = \begin{pmatrix} -i\vec{\sigma}\cdot\vec{\beta} & 0 \\ 0 & i\vec{\sigma}\cdot\vec{\beta} \end{pmatrix}$$

$$\Rightarrow \quad -\frac{i}{4}(2\sigma_{0i}\varepsilon^{0i} + \sigma_{ij}\varepsilon^{ij}) = -i\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{\theta}}{2} - \frac{i\vec{\sigma}\cdot\vec{\beta}}{2} & 0 \\ 0 & \frac{\vec{\sigma}\cdot\vec{\theta} + i\vec{\sigma}\cdot\vec{\beta}}{2} \end{pmatrix}$$

If we write the Lorentz transformations in terms of generators,

$$L = \exp(-iM_{\mu\nu}\varepsilon^{\mu\nu})$$