Scalar field

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下午 09:42

Consider a scalar field ϕ which satisfies the Klein-Gordon

$$(\partial^\mu \partial_\mu + \mu^2)\phi = 0$$

The corresponding Lagrangian density is of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{\mu^2}{2}\phi^2$$

because the Euler-Lagrange equation for this $\mathcal{L}$

$$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial^\mu \phi}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

gives

$$\partial^\mu \partial_\mu \phi + \mu^2 \phi = 0$$

which is exactly the Klein-Gordon equation.

Canonical quantization

Conjugat momentum $\quad \pi(\vec{x},t) = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = (\partial_0 \phi)$

Impose commutation relations,

$$[\phi(\vec{x},t), \pi(\vec{y},t)] = -i\,\delta^3(\vec{x}-\vec{y}), \qquad [\phi(\vec{x},t), \phi(\vec{y},t)] = 0, \quad [\pi(\vec{x},t), \pi(\vec{y},t)] = 0$$

Mode expansion

Recall that solutions to Klein-Gordon equation are of the form,

$$e^{ik\cdot x} \qquad \text{with} \qquad k_0^2 = \vec{k}^2 + \mu^2$$

General solution

$$\phi(\vec{x},t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\left[a(\vec{k})e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right], \quad k_0 = \sqrt{\vec{k}^2 + \mu^2}$$

Solve for $a(k)$ and $a^\dagger(k)$

$$a(k) = i\int d^3x \frac{e^{ik\cdot x}}{\sqrt{(2\pi)^3 2\omega_k}} \overleftrightarrow{\partial_0}\phi(x), \quad a^\dagger(k) = -i\int d^3x \frac{e^{-ik\cdot x}}{\sqrt{(2\pi)^3 2\omega_k}} \overleftrightarrow{\partial_0}\phi(x)$$

$a(k)$ and $a^\dagger(k)$ are field operators in momentum space

It is straightforward to compute their commutators,

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \delta^3(\vec{k}-\vec{k}')$$

$$[a(\vec{k}), a(\vec{k}')] = 0. \qquad [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$$

Hamiltonian

$$H = \int d^3x\, \mathcal{H} = \frac{1}{2}\int d^3x \left[\dot{\varphi}^2 + |\vec{\nabla}\phi|^2 + \mu^2\phi^2\right]$$

$$= \frac{1}{2}\int d^3k\, \omega_k \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] = \int d^3k\, \mathcal{H}_k$$

with $\mathcal{H}_k = \frac{\omega_k}{2}\left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right]$ Harmonic oscillator with frq ω_k

The momentum operator can be written as

$$\vec{P} = \frac{1}{2}\int d^3k\; \vec{k}\left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] = \int d^3k\, \vec{P}_k$$

with $\quad \vec{P}_k = \frac{\vec{k}}{2}\left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right]$

Note that in the usual harmonic oscillator $\quad aa^\dagger = a^\dagger a + 1.$

But here $\qquad a(k)a^\dagger(k) = a^\dagger(k)a(k) + \delta^3(0)$

We can interpret $\delta^3(0)$ as follows.

$$\delta^3(\vec{k}) = \int d^3x\; e^{i\vec{k}\cdot\vec{x}} \quad \rightarrow \quad \delta^3(0) = (2\pi)^{-3}\int d^3x = \frac{V}{(2\pi)^3}, \quad V\text{: total volume}$$

$\delta(k) = \int \frac{\ldots}{(2\pi)^3} e \ldots \quad \rightarrow \quad \delta(0) \ldots \int \ldots \frac{}{(2\pi)^3}$

Then $\mathcal{H}_k = \int d^3k\, \omega_k \left[a^+(k)a(k) + (2\pi)^3 V \right] = \int d^3k\, \omega_k\, a^+(k)a(k) + \frac{1}{2}\int \frac{V d^3k}{(2\pi)^3}$

The last term is just a constant and will be dropped.

To achieve this more formally, we introduce the normal ordering device.

Normal ordering : move all creation operators $a^+(k)$ to the left of annihilation operators $a(k)$.

e.g.

$$:a(k)a^+(k): \; = a^+(k)a(k)$$
$$:a^+(k)a(k): \; = a^+(k)a(k)$$

Let the vaccum be defined by $\quad a(k)|0\rangle = 0 \quad \forall\, \vec{k} \quad \Rightarrow \quad \langle 0|a^+(k) = 0$

Then we will have the general property

$$\langle 0| :f(a,a^+): |0\rangle = 0$$

Thus if we define the Hamiltonian by normal ordering, then we can remove the constant term,

$$H = \frac{1}{2}\int d^3k\, \omega_k :\left[a^+(k)a(k) + a(k)a^+(k) \right]: \; = \int d^3k\, \omega_k\, a_k^+ a_k$$

Similarly, $\quad \vec{P} = \frac{1}{2}\int d^3k\, \vec{P}_k :\left[a^+(k)a(k) + a(k)a^+(k) \right]: \; = \int d^3k\, \vec{P}_k\, a_k^+ a_k$

It is then easy to write down the eigenstates and eigenvalues of H and $\vec{P}$.

For example, the state defined by

$$|\vec{k}\rangle = \sqrt{(2\pi)^3 2\omega_k}\; a^+(k)|0\rangle$$

will have the property,

$$H|\vec{k}\rangle = \omega_k |\vec{k}\rangle$$
$$\vec{P}|\vec{k}\rangle = \vec{k}\, |\vec{k}\rangle \qquad \text{where } \omega_k = \sqrt{\vec{k}^2 + \mu^2}$$

We can interpret this as one-particle state because it has definite energy ω_k and definite momentum $\vec{k}$, with relation

$$\omega_k^2 - \vec{k}^2 = \mu^2$$

Similarly, we can define 2 particle state by

$$|\vec{k}_1, \vec{k}_2\rangle = \sqrt{(2\pi)^3 2\omega_{k_1}}\sqrt{(2\pi)^3 2\omega_{k_2}}\; a^+(\vec{k}_1)a^+(\vec{k}_2)|0\rangle$$

Bose statistics

Any arbitrary state can be expand in terms of states with definite number of particles,

$$|\Phi\rangle = \left[C_0 + \sum_{n=1}^{\infty} \int d^3k_1 d^3k_2 \cdots d^3k_n\, C_n(k_1,k_2\cdots k_n)\, a^+(k_1)a^+(k_2)\cdots a^+(k_n) |0\rangle \right]$$

where $C_n(k_1, \cdots k_n)$ can be interpreted as the momentum space wavefunction

Since $\quad [a^+(k_i), a^+(k_j)] = 0$,

we see that

$$C_n(k_1 \ldots k_i, \cdots k_j \cdots k_n) = C_n(k_1 \cdots k_j \cdots k_i \cdots k_n)$$

This means the wavefunction $C_n(k_1 \ldots k_n)$ satisfies Bose statistics

Fermion fields

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Start with Dirac equation for free particles

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \qquad \text{or} \qquad \bar\psi(-i\gamma^\mu\overleftarrow{\partial}_\mu - m) = 0$$

The Lagrangian density for this equation is of the form

$$\mathcal{L} = \bar\psi_\alpha(i\not\partial - m)_{\alpha\beta}\psi_\beta$$

Conjugate momentum density is $\pi_\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_\alpha)} = i\psi_\alpha^\dagger$

If we impose the commutation relation like the boson case, we will get Dirac particles satisfy Bose statistics which is not we want.

It turns out that we need to impose anticommutation relations.

$$\{\pi_\alpha(\vec{x},t), \psi_\beta(\vec{y},t)\} = i\delta^3(\vec{x}-\vec{y}) \qquad \{\psi_\alpha(\vec{x},t), \psi_\beta(\vec{y},t)\} = 0, \qquad \{\pi_\alpha(\vec{x},t), \pi_\beta(\vec{y},t)\} = 0$$

Hamiltonian density $\mathcal{H} = \sum_\alpha \pi_\alpha\dot\psi_\alpha - \mathcal{L} = i\psi^\dagger\gamma_0\dot\psi - \bar\psi(i\not\partial - m)\psi = \bar\psi(i\vec{\gamma}\cdot\vec{\nabla} + m)\psi$

Mode expansion

$$\psi(\vec{x},t) = \sum_s\int\frac{d^3p}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E_p}}\left[b(p,s)u(p,s)e^{-ip\cdot x} + d^\dagger(p,s)v(p,s)e^{ip\cdot x}\right]$$

$$\psi^\dagger(\vec{x},t) = \sum_s\int\frac{d^3p}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E_p}}\left[b^\dagger(p,s)u^\dagger(p,s)e^{ip\cdot x} + d(p,s)v^\dagger(p,s)e^{-ip\cdot x}\right]$$

Invert this relation

$$b(p,s) = \int\frac{d^3x\, e^{+ip\cdot x}}{(2\pi)^{3/2}\sqrt{2E_p}}\, u^\dagger(p,s)\psi(\vec{x},t) \qquad b^\dagger(p,s) = \int\frac{d^3x\, e^{-ip\cdot x}}{\sqrt{(2\pi)^3 E_p}}\,\psi^\dagger(\vec{x},t)u(p,s)$$

$$d^\dagger(p,s) = \int\frac{d^3x}{\sqrt{(2\pi)^3 2E_p}}\, v^\dagger(p,s)\psi(\vec{x},t) \qquad d(p,s) = \int\frac{d^3x\, e^{ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}}\,\psi^\dagger(\vec{x},t)v(p,s)$$

From these we can compute the anti-commutation relations,

$$\{b(p,s), b^\dagger(p',s)\} = \delta_{ss'}\delta^3(\vec{p}-\vec{p}'), \qquad \{d(p,s), d^\dagger(p',s')\} = \delta_{ss'}\delta^3(\vec{p}-\vec{p}')$$

all other anticommutators vanish.

The Hamiltonian is of the form,

$$H = \int d^3x\,\mathcal{H} = \int\bar\psi(i\vec{\gamma}\cdot\vec{\nabla} + m)\psi\, d^3x = i\int\psi^\dagger\partial_0\psi\, d^3x$$

$$= \sum_s\int d^3p\, E_p\left[b^\dagger(p,s)b(p,s) - d(p,s)d^\dagger(p,s)\right] = \sum_s\int d^3p\,\mathcal{H}_{ps}$$

where $\mathcal{H}_{ps} = E_p\left[b^\dagger(p,s)b(p,s) - d(p,s)d^\dagger(p,s)\right]$

Similarly, $\vec{P} = \sum_s\int d^3p\,\vec{P}_p$

$$\vec{P}_p = \vec{p}\left[b^\dagger(p,s)b(p,s) - d(p,s)d^\dagger(p,s)\right]$$

We can compute the commutators with $b^\dagger(p,s)$ to get

$$[H, b^\dagger(p,s)] = \sum_{s'}\int d^3p'\,[b^\dagger(p',s')b(p',s'), b^\dagger(p,s)]$$

$$= \int d^3p'\sum_{s'}\left(b^\dagger(p',s')\{b(p',s'), b^\dagger(p,s)\} - \{b^\dagger(p,s), b(p,s)\}b(p',s')\right)$$

$$= b^\dagger(p,s)E_p$$

This means

$b^\dagger(p,s)$ is an operator which increases the energy eigenvalue by E_p

Similarly

Similarly. $[\vec{P}, b^+(p.s)] = \vec{p}\, b^+(p.s)$

and $b^+(p.s)$ increases the momonetum eigenvalue by $\vec{p}$. Combine these two, we can say that $b^+(p.s)$ creats a particle with E_p and $\vec{p}$ ($E_p = \sqrt{\vec{p}^2 + m^2}$)

In the same way, we can show that, $d^+(p.s)$ also creates a particle

Number operator

Define $N^+_{ps} = b^+(p.s) b(p.s)$. $\qquad N^-_{ps} = d^+(p.s) d(p.s)$

These are number operators.

Furthermore, it is simple to show that

$$(N^+_{ps})^2 = N^+_{ps} \qquad (N^-_{ps})^2 = N^-_{ps}$$

Thus eigenvalues of $N^\pm_{ps}$ are either 0 or 1. $\Rightarrow$ satisfy Pauli exclusion principle

Symmetry

$$\mathcal{L} = \bar{\Psi}(i\vec{\gamma}\cdot\vec{\nabla} + m)\psi$$

It is clear that $\mathcal{L}$ is invariant under the phase transformation,

$$\psi(x) \longrightarrow e^{i\alpha}\psi(x) \quad \Rightarrow \quad \psi^+(x) \to \psi^+(x)\, e^{-i\alpha} \qquad \alpha: \text{some real constant.}$$

From Noether's theorem, the conserved current for this symmetry is

$$j_\mu = \bar{\Psi}\gamma_\mu\psi$$

The corresponding conserved charge is

$$Q = \int j_0(x)\, d^3x = \sum_s \int d^3p\, [N^+(p.s) - N^-(p.s)]$$

Thus particle and anti-particle have opposite "charge".

Electromagnetic fields

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Start with free Maxwell's equations,

$$\vec{\nabla}\cdot\vec{E}=0,\quad \vec{\nabla}\cdot\vec{B}=0,\quad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0,\quad \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}-\epsilon_0\frac{\partial\vec{E}}{\partial t}=0$$

Introduce vector and scalar potentials, $\vec{A},\phi$. so that

$$\vec{B}=\vec{\nabla}\times\vec{A},\qquad \vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}$$

It is convenient to write these as

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu \qquad \text{with}\quad F^{0i}=\partial^0A^i-\partial^iA^0=E^i,\quad F^{ij}=\partial^iA^j-\partial^jA^i=\epsilon_{ijk}B_k$$

The other 2 equations can be written as

$$\partial_\nu F^{\mu\nu}=0 \qquad \mu=0,1,2,3$$

For example, for $\mu=0$, $\partial_i F^{0i}=0 \Rightarrow \vec{\nabla}\cdot\vec{E}=0$

$\mu=i$ $\quad \partial_\nu F^{i\nu}=0 \Rightarrow \quad \vec{\nabla}\times\vec{B}-\frac{\partial\vec{E}}{\partial t}=0$

It is easy to see that $F^{\mu\nu}$ which is an antisymmetric derivative of A_μ is invariant under the transformation,

$$A^\mu\longrightarrow A^\mu+\partial^\mu\alpha \qquad \alpha=\alpha(x)\ \text{an arbitrary function}$$

This is the gauge transformation.

It turns out that the Lagrangian density is of the form,

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}=\frac{1}{2}(\vec{E}^2-\vec{B}^2)$$

Conjugate momenta $\quad \pi_0=\frac{\partial\mathcal{L}}{\partial(\partial_0A_0)}=0 \qquad \pi^i(x)=\frac{\partial\mathcal{L}}{\partial(\partial_0A_i)}=F^{0i}=E^i$

Hamiltonian density

$$\mathcal{H}=\pi^k\dot{A}_k-\mathcal{L}=(\partial^kA^0-\partial^0A^k)\partial_0A_k+\frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu-\partial^\nu A^\mu)=\frac{1}{2}(\vec{E}^2+\vec{B}^2)+(\vec{E}\cdot\vec{\nabla})A_0$$

Hamiltonian can be written as

$$H=\int d^3x\,\mathcal{H}=\frac{1}{2}\int d^3x\,(\vec{E}^2+\vec{B}^2) \qquad \text{where we have used } \vec{\nabla}\cdot\vec{E}=0$$

Quantization

$$[\pi^i(\vec{x},t),A^j(\vec{y},t)]=-i\delta_{ij}\delta^3(\vec{x}-\vec{y}),\ \ldots\ldots$$

$\downarrow$ not consistent with $\vec{\nabla}\cdot\vec{E}=0$ because $[\vec{\nabla}\cdot\vec{E}(\vec{x},t),A_j(\vec{y},t)]=-i\partial_j\delta^3(\vec{x}-\vec{y})\neq0$

In momentum space

$$\partial_j\delta^3(\vec{x}-\vec{y})=i\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}k_j$$

In order to get zero for the commutator of $\vec{\nabla}\cdot\vec{E}$, we can do the following replacement,

$$\delta_{ij}\delta^3(\vec{x}-\vec{y})\longrightarrow \quad \delta^{tr}_{ij}(\vec{x}-\vec{y})\equiv\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left(\delta_{ij}-\frac{k_ik_j}{\vec{k}^2}\right)$$

then $\quad \partial_i\delta^{tr}_{ij}(\vec{x}-\vec{y})=i\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}k_i\left(\delta_{ij}-\frac{k_ik_j}{\vec{k}^2}\right)=0$

So the non-zero commutator is of the form

$$[E^i(\vec{x},t),A_j(\vec{y},t)]=-i\delta^{tr}_{ij}(\vec{x}-\vec{y})$$

As a consequence, we also have

$$[E^i(\vec{x},t),\vec{\nabla}\cdot\vec{A}(\vec{y},t)]=0$$

Now that A_0 and $\vec{\nabla}\cdot\vec{A}$ commute with all operators, they must be a c-number.
In other words, A_0 and longitudinal part of $\vec{A}$ are not dynamical degree of freedom.
We can choose a gauge such that $A_0=0$, $\vec{\nabla}\cdot\vec{A}=0$ (radiation gauge)

We can choose a gauge such that $A_0=0$, $\vec\nabla\cdot\vec A=0$ (radiation gauge)

In this gauge $\pi^i = \partial^i A^0 - \partial^0 A^i = -\partial^0 A^i$

$$[\partial_0 A^i(\vec x,t),\ A^j(\vec y,t)] = -i\,\delta^{tr}_{ij}(\vec x-\vec y)$$

Mode expansion

Equation of motion $\partial_\nu F^{\mu\nu}=0$

or $\partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) = \Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0$

In the gauge we have chosen, $A_0=0$, $\vec\nabla\cdot\vec A=0$, we have $\partial_\nu A^\nu=0$.

Then the wave equation become

$$\Box\vec A = 0 \qquad \text{massless Klein-Gordon eq}$$

The general solution is

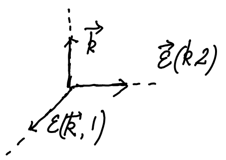
$$\vec A(\vec x,t) = \int\frac{d^3k}{\sqrt{2\omega(2\pi)^3}}\sum_\lambda \vec\varepsilon(\vec k,\lambda)\left[a(k,\lambda)e^{-ik\cdot x} + a^+(k,\lambda)e^{ik\cdot x}\right] \qquad \omega=k_0=|\vec k|$$

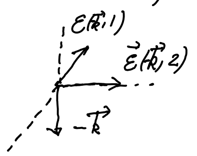
In the gauge $\vec\nabla\cdot\vec A=0$, there are only 2 independent degrees of freedom,

$$\vec\varepsilon(k,\lambda),\ \lambda=1,2 \qquad \text{with} \quad \vec k\cdot\vec\varepsilon(k,\lambda)=0$$

Standard choice

$$\vec\varepsilon(k,\lambda)\cdot\vec\varepsilon(k,\lambda') = \delta_{\lambda\lambda'}, \qquad \vec\varepsilon(-\vec k,1) = -\vec\varepsilon(k,1), \qquad \vec\varepsilon(-\vec k,2) = \vec\varepsilon(-\vec k,2)$$





We can solve for $a(k,\lambda)$ and $a^+(k,\lambda)$ to get

$$a(k,\lambda) = i\int\frac{d^3x}{\sqrt{(2\pi)^3 2\omega}}\left[e^{ik\cdot x}\overleftrightarrow{\partial_0}\,\vec\varepsilon(\vec k,\lambda)\cdot\vec A(x)\right]$$

$$a^+(k,\lambda) = -i\int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega}}\left[e^{-ik\cdot x}\overleftrightarrow{\partial_0}\,\vec\varepsilon(\vec k,\lambda)\cdot\vec A(x)\right]$$

The commutation relations are then of the form,

$$[a(k,\lambda),\ a^+(k',\lambda')] = \delta_{\lambda\lambda'}\delta^3(\vec k-\vec k') \qquad [a(k,\lambda),\ a(k',\lambda')]=0,\ \ [a^+(k',\lambda'),\ a(k,\lambda)]=0$$

The normal ordered form for the Hamiltonian and momentum operators are

$$H = \frac12\int d^3x : (\vec E^2+\vec B^2) : = \int d^3k\ \omega\sum_\lambda a^+(k,\lambda)\,a(k,\lambda)$$

$$\vec P = \int d^3x : \vec E\times\vec B : = \int d^3k\ \vec k\sum_\lambda a^+(k,\lambda)\,a(k,\lambda)$$

The vaccum is defined by $a(\vec k,\lambda)|0\rangle = 0 \quad \forall\ \vec k,\lambda$