Interaction Theory

2010年3月14日
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As an illustration we discuss the electromagnetic interaction. From the principle of minimum substitution, the Lagrangian density is of the form,

$$\mathcal{L} = \bar\Psi(x)\,\gamma^\mu(i\partial_\mu - eA_\mu)\psi(x) - m\bar\psi(x)\,\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Equation of motion

$$\begin{cases} (i\gamma^\mu\partial_\mu - m)\,\psi(x) = eA_\mu\gamma^\mu\psi \\ \partial_\nu F^{\mu\nu} = e\,\bar\psi\gamma^\mu\psi \end{cases}$$

non-linear coupled equations

Quantization

Write $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

$$\mathcal{L}_0 = \bar\psi\,(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\mathcal{L}_{int} = -e\,\bar\psi\gamma^\mu\psi\,A_\mu$$

Conjugate momenta $\quad \dfrac{\partial\mathcal{L}}{\partial(\partial_0\psi_\alpha)} = i\psi_\alpha^+(x)$

For em fields, we choose $\vec\nabla\cdot\vec A = 0$. $\quad \pi^i = \dfrac{\partial\mathcal{L}}{\partial(\partial_0 A^i)} = F^{0i} = -E^i$

From equation of motion $\quad \partial_\nu F^{0\nu} = e\,\psi^+\psi \Rightarrow -\nabla^2 A^0 = e\,\psi^+\psi$

Thus A^0 is not zero but it is not an independent dynamical variable,

$$A^0 = e\int d^3x'\,\frac{\psi^+(x',t)\,\psi(x',t)}{4\pi\,|\vec x' - \vec x|} = e\int \frac{d^3x'\,\rho(x',t)}{|\vec x - \vec x'|}$$

Commutation relation

$$\{\psi_\alpha(\vec x,t),\ \psi_\beta^+(\vec x',t)\} = \delta_{\alpha\beta}\,\delta^3(\vec x - \vec x') \qquad \{\psi_\alpha(\vec x,t),\ \psi_\beta(\vec x',t)\} = \cdots = 0$$

$$[\dot A_i(\vec x,t),\ A_j(\vec x',t)\} = -i\,\delta_{ij}^{tr}(\vec x - \vec x')$$

Commutators involving A_0 can be worked out as follows,

$$[A_0(\vec x,t),\ \psi_\alpha(\vec x',t)] = e\int\frac{d^3x''}{|\vec x - \vec x''|\,4\pi}\,[\psi^+(\vec x'',t)\,\psi(\vec x'',t),\ \psi_\alpha(\vec x',t)] = -\frac{e}{4\pi}\,\frac{\psi_\alpha(\vec x',t)}{|\vec x - \vec x'|}$$

Hamiltonian

$$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_\alpha)}\,\dot\psi_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_0 A^k)}\,\dot A_k - \mathcal{L} = \psi^+(-i\vec\alpha\cdot\vec\nabla + m)\psi + \frac{1}{2}(\vec E^2 + \vec B^2) + \vec E\cdot\vec\nabla A_0 + e\,\bar\psi\gamma_\mu\psi A^\mu$$

and $\quad H = \int d^3x\,\mathcal{H} = \int d^3x\left\{\psi^+[\vec\alpha\cdot(-i\vec\nabla - e\vec A) + \beta m]\,\psi + \frac{1}{2}(\vec E^2 + \vec B^2)\right\}$

A_0 does not appear in the interaction,

But if we write $\quad \vec E = \vec E_\ell + \vec E_t \quad$ where $\vec E_\ell = -\vec\nabla A_0$, $\quad \vec E_t = -\dfrac{\partial\vec A}{\partial t}$

Then $\quad \dfrac{1}{2}\displaystyle\int d^3x\,(\vec E^2 + \vec B^2) = \frac{1}{2}\int d^3x\,\vec E_\ell^{\,2} + \frac{1}{2}\int(\vec E_t^{\,2} + \vec B^2)\,d^3x$

the longitudinal part is

$$\frac{1}{2}\int\vec E_\ell^{\,2}\,d^3x = \frac{e}{4\pi}\int d^3x\,d^3y\,\frac{\rho(\vec x,t)\,\rho(\vec y,t)}{|\vec x - \vec y|}$$

Coulomb interaction

Physical States

In high energy physics, we study the interactions by scattering processes. We assume that the interactions of interest are all short-range in nature. Then far away from the interaction region, particles behave like free particles.

Choose the physical states to be eigenstates of energy momentum operators.

$$P_\mu\,|\Psi\rangle = p_\mu\,|\Psi\rangle$$

They are required to satisfy following requirements

$I_\mu |\Psi\rangle = p_\mu |\Psi\rangle$

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1) The eigenvalues p_μ all lie within forward light cone,

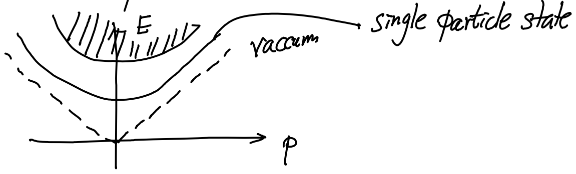
$$p^2 = p_\mu p^\mu \geq 0, \qquad p_0 \geq 0$$

2) there exists a non-degenerate Lorentz invariant ground state $|0\rangle$ with lowest energy

choose $\qquad p^0|0\rangle = 0 \quad \Rightarrow \quad \vec{p}\,|0\rangle = 0$

3) The exists stable single particle states $|\vec{P}_i\rangle$ with $P_i^2 = m_i^2$ for each stable particle

4) The vaccum and one particle states form discrete spectrum in p^μ



We associate a field $\phi(x)$ for each discrete state appearing in the spectrum of P_μ and assume that interactions do not violently the spectrum of states.

<u>In-fields and in-states</u> — asymptotic conditions

For simplicity, consider

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu_0^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

Equation of motion $\qquad (\Box + \mu^2)\phi = j(x) = \frac{\lambda}{3!}\phi^3$

conjugate momenta $\qquad \pi(x) = \frac{\partial \mathcal{L}}{\partial_0 \phi} = \partial_0 \phi$

<u>Commutation relations</u>

$$[\pi(\vec{x},t), \phi(\vec{y},t)] = -i\delta^3(x-y) \qquad [\pi(x,t), \pi(y,t)] = [\phi(x,t), \phi(y,t)] = 0$$

In scattering problem, at $t = -\infty$, particles progate freely. Let $\varphi_{in}(x)$ be the operator which creates free particle propagating with physical mass.

$$(\Box + \mu^2)\varphi_{in}(x) = 0$$

We will assume that $\varphi_{in}(x)$ transforms under coordinate displacements and Lorentz transformation in the same way as $\phi(x)$. In particular,

$$[P_\mu, \varphi_{in}(x)] = -i\partial_\mu \varphi_{in}(x)$$

$\Rightarrow$ $\varphi_{in}(x)$ creates one particle state from vacuum

Proof: consider states with definite momentum, $\quad P^\mu|n\rangle = P_n^\mu |n\rangle$

Then

$$-i\partial_\mu \langle n|\varphi_{in}(x)|0\rangle = \langle n|[P_\mu, \varphi_{in}(x)]|0\rangle = P_{n\mu}\langle n|\varphi_{in}|0\rangle$$

$$\Rightarrow \quad (\Box + \mu^2)\langle n|\varphi_{in}(x)|0\rangle = (\mu^2 - P_n^2)\langle n|\varphi_{in}(x)|0\rangle = 0 \quad \Rightarrow \quad P_n^2 = \mu^2$$

i.e. $\quad \varphi_{in}(x)|0\rangle$ is an <u>one-particle</u> state with mass μ^2

We can expand $\varphi_{in}(x)$ as

$$\varphi_{in}(x) = \int d^3k\, [a_{in}(k) f_k(x) + a_{in}^+(k) f_k^*(x)] \qquad f_k = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{-ikx}$$

Invert this expansion $\qquad a_{in}(k) = \int d^3k\, f_k^*(x) \overleftrightarrow{\partial_0} \phi_{in}(x)$

also we have $\qquad [P^\mu, a_{in}(k)] = -k^\mu a_{in}(k), \qquad [P^\mu, a_{in}^+(k)] = k^\mu a_{in}^+(k)$

states are defined by $\qquad |k_1, in\rangle = \sqrt{(2\pi)^3 2\omega_k}\, a_{in}^+(k_1)|0\rangle$

$$|k_1 k_2 \cdots k_n, in\rangle = [\prod_i \sqrt{(2\pi)^3 2\omega_{k_i}}\, a_{in}^+(k_i)]|0\rangle$$

normalization $\qquad \langle k_2\, in|k_1, in\rangle = (2\pi)^3 2\omega_1 \delta^3(\vec{k}_1 - \vec{k}_2)$

$\langle p_1 \cdots p_m\, in | k_1 \cdots k_n\, in\rangle = 0$ unless $m = n$ and $(P_1 \cdots P_n)$ conicides with $(k_1, k_2 \cdots k_n)$

$\langle p_1, p_2 \cdots p_m, in \,|\, k_1 \cdots k_n, in \rangle = 0$ unless $m=n$ and $(p_1 \cdots p_n)$ conicides with $(k_1, k_2 \cdots k_n)$

Relation between $\varphi_{in}(x)$ and $\phi(x)$

$$(\Box + \mu_0^2)\phi(x) = j(x) \qquad \text{or} \qquad (\Box + m^2)\phi(x) = j(x) + \delta\mu^2\phi(x) = \tilde{j}(x) \qquad \delta\mu^2 = \mu^2 - \mu_0^2$$

Solve this equation formally,

$$\sqrt{Z}\,\varphi_{in}(x) = \varphi(x) - \int d^4y\, \Delta_{ret}(x-y, \mu^2)\,\tilde{j}(y)$$

where $\quad (\Box_x^2 + \mu^2)\Delta_{ret}(x-y, \mu^2) = \delta^4(x-y) \quad$ and $\Delta_{ret}(x-y, \mu^2) = 0$ for $x_0 < y_0$

This suggests that as $x_0 \to -\infty$, $\varphi(x) \longrightarrow \sqrt{Z}\,\varphi_{in}$. But this leads to contradiction

Correct asymptotic condition (LSZ) : Let $|\alpha\rangle$, $|\beta\rangle$ be any two normalizable states, $\varphi^f(t)$ is defined by smearing $\varphi(x)$ over space-like region

$$\varphi^f(t) = i\int d^3x\, f^*(\vec{x}, t)\overleftrightarrow{\partial_0}\varphi(\vec{x}, t) \qquad \text{with} \qquad (\Box + \mu^2)f = 0$$

Then the correct asymptotic condition is

$$\lim_{x_0 \to -\infty} \langle \alpha | \varphi^f(t) | \beta \rangle = \sqrt{Z}\,\langle \alpha | \varphi^f_{in} | \beta \rangle$$

where $\quad \varphi^f_{in} = i\int d^3x\, f^*(\vec{x}, t)\overleftrightarrow{\partial_0}\varphi_{in}(\vec{x}, t) \qquad$ time-independent.

Out fields and out states

Just like the case of in-field and in states, we can also reduce the dynamics to that of free particles for $t \to +\infty$ by defining

$$(\Box + \mu^2)\varphi_{out}(x) = 0$$

$$\varphi_{out}(x) = \int d^3k\, [a_{out}(k) f_k(x) + a^\dagger_{out}(k) f^*_k(x)], \qquad [P^\mu, a_{out}(k)] = -k^\mu a_{out}(k)$$

Asymptotic condition

$$\lim_{t \to \infty} \langle \alpha | \varphi^f(t) | \beta \rangle = \sqrt{Z}\,\langle \alpha | \varphi^f_{out} | \beta \rangle$$

S-matrix

Description of scattering : Start with state with n non-interacting particles. They interact when they are close to each other. After interaction, m particles separate

initial state $\qquad |p_1, p_2 \cdots p_n, in\rangle = |\alpha, in\rangle$

out state $\qquad |p_1', p_2' \cdots p_n', out\rangle = |\beta, out\rangle$

S-matrix $\qquad S_{\beta\alpha} \equiv \langle \beta\, out | \alpha\, in \rangle$

S-operator $\qquad \langle \beta\, out | \equiv \langle \beta\, in | S \qquad \langle \beta\, out | S^{-1} = \langle \beta\, in |$

then $\qquad S_{\beta\alpha} = \langle \beta\, out | \alpha\, in \rangle = \langle \beta\, in | S | \alpha\, in \rangle$

Properties of S-matrix

1) From the stability of vacuum $\quad |S_{00}| = 1$

$$\langle 0\, in | S = \langle 0\, out | = e^{-i\varphi}\langle 0\, in |.$$

2) Stability of the one-particle state requires

$$\langle p\ in | S | p\ in\rangle = \langle p\ out | p\ in\rangle = 1 \qquad \because\ |p\ in\rangle = |p\ out\rangle$$

3) $\varphi_{in}(x) = S\varphi_{out}S^{-1}$

Consider $\langle \beta, out | \varphi_{out}(x) | \alpha\ in\rangle = \langle \beta, in | S\varphi_{out}(x) | \alpha, in\rangle$

$\langle \beta\ out | \varphi_{out}$ is an out state $\Rightarrow$ $\langle \beta\ out | \varphi_{out} = \langle \beta\ in | \varphi_{in}(x) S$

Then $\langle \beta\ in | S\varphi_{out}(x) | \alpha\ in\rangle = \langle \beta\ in | \varphi_{in} S | \alpha\ in\rangle$.

or $S\varphi_{out} = \varphi_{in} S$ or $\varphi_{in}(x) = S\varphi_{out}S^{-1}$

4) Unitarity

Since $\langle \alpha\ in | S = \langle \alpha\ out |$ $\qquad S^{+}|\alpha\ in\rangle = |\alpha, out\rangle$

$\Rightarrow$ $\langle \beta\ in | SS^{+} | \alpha\ in\rangle = \langle \beta\ out | \alpha\ out\rangle = \delta_{\beta\alpha}$

as operator $SS^{+} = 1$, similar argument $\Rightarrow$ $S^{+}S = 1$

5) S is translational and Lorentz invariance

Under Lorentz transformation $x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + b^{\mu}$

then $U(\Lambda, b) S U^{-1}(\Lambda, b) = S$ where $U(\Lambda, b)\varphi(x)U^{-1}(\Lambda, b) = \varphi(\Lambda x + b)$

Proof.

$$\varphi_{in}(\Lambda x + b) = U(\Lambda, b)\varphi_{in}(x)U^{-1}(\Lambda, b) = US\varphi_{out}(x)S^{-1}U^{-1}$$

$$= (USU^{-1})\varphi_{out}(\Lambda x + b)(US^{-1}U)$$

But $\varphi_{in}(\Lambda x + b) = S\varphi_{out}(\Lambda x + b)S^{-1}$

$\Rightarrow$ $U(\Lambda, b) S U^{-1}(\Lambda, b) = S$

LSZ reduction

2010年3月17日
上午 11:12

We now work to set up the framework to compute the transistion matrix element $S_{\beta\alpha}$.

Consider

$$S_{\beta,\alpha p} = \langle \beta\, out | \alpha, p\, in \rangle$$

Using the creation operator we get

$$S_{\beta,\alpha p} = \langle \beta\, out|\alpha, p\, in\rangle = \sqrt{(2\pi)^3 2\omega_p}\,\langle \beta\, out| a_{in}^{+}(p)|\alpha\, in\rangle$$

$$= \sqrt{(2\pi)^3 2\omega_p}\,\left[\langle \beta\, out| a_{out}^{+}(p)|\alpha\, in\rangle + \langle \beta\, out|[a_{in}^{+}(p) - a_{out}^{+}(p)]|\alpha\, in\rangle\right]$$

$$= \sqrt{(2\pi)^3 2\omega_p}\,\left[\langle \beta - p, out|\alpha\, in\rangle - i\,\langle \beta\, out|\int d^3x\, f_p(x)\overleftrightarrow{\partial_0}[\phi_{in}(x) - \phi_{out}(x)]|\alpha\, in\rangle\right]$$

$\langle \beta - p, out|$ is the state obtained from $\langle \beta\, out|$ by removing a particle with momentum $\vec{p}$.

Asymptotic conditions

$$\langle\alpha|\phi_{in}|\beta\rangle = \frac{1}{\sqrt{Z}}\lim_{t\to-\infty}\langle\alpha|\phi(x)|\beta\rangle, \qquad \langle\alpha|\phi_{out}|\beta\rangle = \frac{1}{\sqrt{Z}}\lim_{t\to+\infty}\langle\alpha|\phi(x)|\beta\rangle$$

Identity

$$\left(\lim_{x_0\to+\infty} - \lim_{x_0\to-\infty}\right)\int d^3x\, g_1(x)\overleftrightarrow{\partial_0} g_2(x) = \int_{-\infty}^{+\infty} d^4x\, \partial_0\left(g_1(x)\overleftrightarrow{\partial_0} g_2(x)\right)$$

$$= \int_{-\infty}^{+\infty} d^4x\,\left[g_1(x)\partial_0^2 g_2(x) - \partial_0^2 g_1(x)\, g_2(x)\right]$$

Then

$$\int d^3x\, f_p(x)\overleftrightarrow{\partial_0}[\phi_{in} - \phi_{out}] = \int d^4x\,[\partial_0^2 f_p(x)\,\phi(x) - f_p(x)\,\partial_0^2\phi(x)] = \int d^4x\, f_p(x)(\Box + \mu^2)\phi(x)$$

where we have used $\partial_0^2 f_p(x) = (\partial_i^2 - \mu^2) f_p$ and integration by parts.

Put these together, we get the $\boxed{\text{reduction formula}}$

$$\boxed{\langle \beta\, out|\alpha\, p\, in\rangle = \sqrt{(2\pi)^3 2\omega_p}\,\langle \beta - p\, out|\alpha\, in\rangle + \frac{i}{\sqrt{Z}}\int e^{-ip\cdot x} d^4x\,(\Box + \mu^2)\langle \beta\, out|\alpha\, in\rangle}$$

To remove a particle from β, write $\beta = \gamma p'$

$$\langle \beta\, out|\phi(x)|\alpha\, in\rangle = \langle \gamma p'\, out|\phi(x)|\alpha\, in\rangle = \langle \gamma\, out|a_{out}(p')\phi(x)|\alpha\, in\rangle \sqrt{(2\pi)^3 2\omega_{p'}}$$

$$= \sqrt{}\left[\langle \gamma\, out|\phi(x) a_{in}(p')|\alpha\, in\rangle - \langle \gamma\, out|(a_{out}(p')\phi(x) - \phi(x)a_{in}(p'))|\alpha\, in\rangle\right]$$

$$= \sqrt{}\left[\langle \gamma\, out|\phi(x)|\alpha - p'\, in\rangle - i\int d^3y\,\langle \gamma\, out|(\phi_{out}(y)\phi(x) - \phi(x)\phi_{in}(y))|\alpha\, in\rangle \overleftrightarrow{\partial_0} f_{p'}^{*}(y)\right]$$

$$= \sqrt{}\left[\;'' - i\int d^3y\,\left(\lim_{y_0\to+\infty} - \lim_{y_0\to-\infty}\right)\langle \gamma\, out|T(\phi(y)\phi(x))|\alpha\, in\rangle \overleftrightarrow{\partial_0} f_{p'}^{*}(y)\;\right]$$

Following the same procedure as before, we can get

$$\langle \beta\, out|\phi(x)|\alpha\, in\rangle = \sqrt{(2\pi)^3 2\omega_{p'}}\left\{\langle \gamma\, out|\phi(x)|\alpha - p', in\rangle + \frac{i}{\sqrt{Z}}\int d^4y\,\langle \gamma\, out|T(\phi(x)\phi(y))|\alpha\, in\rangle(\overleftarrow{\Box_y^2 + \mu^2})\, e^{ip\cdot x}\right\}$$

It is clear how to remove all particles from "in" and "out" state

$$\boxed{\begin{aligned}\langle p_1\cdots p_n, out|q_1\cdots q_m\, in\rangle = \left(\frac{i}{\sqrt{Z}}\right)^{m+n}\prod_{i=1}^{m}\prod_{j=1}^{n}\int d^4x_i\, d^4y_j\, e^{-iq_i\cdot x_i}(\overrightarrow{\Box_{x_i} + \mu^2})\\ \langle 0|T(\phi(y_1)\cdots\phi(y_n)\,\phi(x_1)\cdots\phi(x_m))|0\rangle(\overleftarrow{\Box_{y_j} + \mu^2})\, e^{ip_j\cdot y_j}\end{aligned}}$$

for all $p_i \neq q_j$

<u>In and Out fields for Fermions</u>

$$\psi_{in}(x) = \int d^3p\sum_s\left[b_{in}(p,s)U_{ps}(x) + d_{in}^{+}(p,s)V_{ps}(x)\right]$$

where $U_{ps}(x) = \frac{1}{\sqrt{(2\pi)^3 2E_p}}\, u(p,s)\, e^{-ip\cdot x}$ $\qquad V_{ps}(x) = \frac{1}{\sqrt{(2\pi)^3 2E_p}}\, v(p,s)\, e^{ip\cdot x}$

where $U_{ps}(x) = \frac{1}{\sqrt{(2\pi)^3 2E_p}} u(p,s) e^{-ip\cdot x}$ $V_{ps}(x) = \frac{1}{\sqrt{(2\pi)^3 2E_p}} v(p,s) e^{ip\cdot x}$

Inversion

$b_{in}(p,s) = \int d^3x \, U_{ps}^{+}(x) \psi_{in}(x)$. $d_{in}(p,s) = \int d^3x \, \psi_{in}^{+}(x) V_{ps}(x)$.

$b_{in}^{+}(p,s) = \int d^3x \, \psi_{in}^{+}(x) U_{ps}(x)$. $d_{in}^{+}(p,s) = \int d^3x \, V_{ps}(x) \psi_{in}(x)$.

Reduction formula for fermions

(a) remove electron from the in-state

$$\langle \beta\, out | \alpha; p s\, in \rangle = -\frac{i}{\sqrt{Z_2}} \int d^4x \langle \beta\, out | \bar{\psi}_\alpha(x) | \alpha\, in \rangle \overleftarrow{(-i\not{\partial} - m)}_{\alpha\beta} \, u_\beta(p,s) e^{-ip\cdot x}$$

(b) remove positron (anti-particle) from the in state

$$\langle \beta\, out | \alpha; \bar{p}\bar{s}\, in \rangle = \frac{i}{\sqrt{Z_2}} \int d^4x \, e^{-i\bar{p}\cdot x} \, \bar{v}_\alpha(\bar{p},\bar{s}) \overrightarrow{(i\not{\partial} - m)}_{\alpha\beta} \langle \beta\, out | \psi_\beta(x) | \alpha\, in \rangle$$

(c) remove electron from out state

$$\langle \beta, p's'\, out | \alpha\, in \rangle = -\frac{i}{\sqrt{Z_2}} \int d^4x \, \bar{u}_\alpha(p',s') e^{ip'\cdot x} \overrightarrow{(i\not{\partial} - m)}_{\alpha\beta} \langle \beta\, out | \psi_\beta(x) | \alpha\, in \rangle$$

(d) remove positron from out state

$$\langle \beta, \bar{p}', \bar{s}'\, out | \alpha\, in \rangle = \frac{i}{\sqrt{Z_2}} \int d^4x \langle \beta\, out | \bar{\psi}_\alpha(x) | \alpha\, in \rangle \overleftarrow{(-i\not{\partial} - m)}_{\alpha\beta} \, v(\bar{p}',\bar{s}') e^{i\bar{p}\cdot x}$$

U matrix

2010年3月18日
上午 09:32

Assume that $\phi(x)$ and $\pi(x)$ are related to ϕ_{in} and π_{in} by unitary U matrix,

$$\varphi(\vec{x},t)=U^{-1}(t)\,\phi_{in}(\vec{x},t)\,U(t)\quad,\qquad \pi(\vec{x},t)=U^{-1}(t)\,\pi_{in}(\vec{x},t)\,U(t)$$

in fields satisfy the free field equation of motion

$$\partial_0\phi_{in}(x)=i[H_{in}(\phi_{in},\pi_{in}),\phi_{in}]\;,\qquad \partial_0\pi_{in}(x)=i[H_{in}(\phi_{in},\pi_{in}),\pi_{in}]$$

where $H_{in}(\phi_{in},\pi_{in})$ is the free field Hamiltonian with mass μ.

On the other hand,

$$\partial_0\phi(x)=i[H(\varphi,\pi),\phi]\qquad \partial_0\pi(x)=i[H(\varphi,\pi),\pi]$$

Then from eq for ϕ_{in}, we get

$$\partial_0\phi_{in}=\frac{\partial}{\partial t}[U\phi U^{-1}]=\frac{\partial U}{\partial t}\phi U^{-1}+U\frac{\partial\phi}{\partial t}U^{-1}+U\phi\frac{\partial U^{-1}}{\partial t}$$

$$=\frac{\partial U}{\partial t}(U^{-1}\phi_{in}U)U^{-1}+U(i[H(\varphi,\pi),\varphi])U^{-1}-U\phi U^{-1}\frac{\partial U}{\partial t}U^{-1}$$

$$=\left(\frac{\partial U}{\partial t}U^{-1}\right)\phi_{in}+i[H(\phi_{in},\pi_{in}),\phi_{in}]-\phi_{in}\dot{U}U^{-1}$$

Use $\partial_0\phi_{in}=i[H_{in}(\phi_{in},\pi_{in}),\phi_{in}]$

we get $\left[\frac{\partial U}{\partial t}U^{-1}+iH_I(\phi_{in},\pi_{in}),\phi_{in}\right]=0$

where $H_I(\phi_{in},\pi_{in})=H(\phi_{in},\pi_{in})-H_{in}(\phi_{in},\pi_{in})$ contains all the interaction.

Similarly, we can show

$$\left[\frac{\partial U}{\partial t}U^{-1}+iH_I(\phi_{in},\pi_{in}),\pi_{in}\right]=0$$

This means the combination $\frac{\partial U}{\partial t}U^{-1}+iH_I$ commutes with all the operators

We can take this to be a c-number. For simplicity, we can take this c-number to be zero. Thus

$$\boxed{i\frac{\partial U(t)}{\partial t}=H_I(t)\,U(t)}$$

For conveience, we define

$$U(t,t')\equiv U(t)U^{-1}(t')\qquad \text{time evolution operator}$$

then

$$i\frac{\partial U(t,t')}{\partial t}=H_I(t)\,U(t,t')\qquad \text{with } U(t,t)=1$$

We can convert this to integral equation

$$U(t,t')=1-i\int_{t'}^{t}dt_1\,H_I(t_1)\,U(t_1,t')$$

Iterate this equation

$$U(t,t')=1-i\int_{t'}^{t}dt_1\,H_I(t_1)+(-i)^2\int_{t'}^{t}dt_1\,H_I(t_1)\int_{t'}^{t_1}dt_2\,H_I(t_2)+\cdots$$

$$+(-i)^n\int_{t'}^{t}dt_1\int_{t'}^{t_1}dt_2\cdots\int_{t'}^{t_{n-1}}dt_n\,H_I(t_1)H_I(t_2)\cdots H_I(t_n)+\cdots$$

This can be written as

$$U(t,t')=1+\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{t'}^{t}dt_1\int_{t'}^{t}dt_2\cdots\int_{t'}^{t}dt_n\,T(H_I(t_1)H_I(t_2)\cdots H_I(t_n))$$

$$=T\left(\exp\left[-i\int_{t'}^{t}d^4x\,\mathcal{H}_I(\phi_{in},\pi_{in})\right]\right)$$

Perturbation Expansion of Vaccum expectation value

$\tau(x_1, x_2 \cdots x_n) = \langle 0 | T(\phi(x_1)\phi(x_2)\cdots\phi(x_n)) | 0 \rangle$

Using U matrix we can write this in terms of ϕ_{in}

$$\tau = \langle 0 | T(U^{-1}(t_1)\phi_{in}(x_1) U(t_1,t_2)\phi_{in}(x_2) U(t_2,t_3) \cdots U(t_{n-1},t_n)\phi_{in}(x_n) U(t_n)) | 0 \rangle$$

$$= \langle 0 | T(U^{-1}(t) U(t,t_1)\phi_{in}(x_1) \cdots \phi_{in}(x_n) U(t_n,t') U(t') | 0 \rangle$$

let $t > t_1 \cdots t_n > t'$

Then $\tau = \langle 0 | U^{-1}(t)\, T(U(t,t_1)\phi_{in}(x_1) \cdots \phi_{in}(x_n) U(t_n, t'))\, U(t') | 0 \rangle$

$$= \langle 0 | U^{-1}(t)\, T(\phi_{in}(x_1) \cdots \phi_{in}(x_n) \exp[-i\int_{t'}^{t} H_I(t'')dt''])\, U(t') | 0 \rangle$$

Theorem : $|0\rangle$ is an eigenstate of $U(-t)$ as $t \to +\infty$

Consider $\langle p\alpha\, in | U(-t) | 0 \rangle = \sqrt{(2\pi)^3 2\omega_p}\, \langle \alpha\, in | a_{in}(p) U(-t) | 0 \rangle$

$$= -i\sqrt{(2\pi)^3 2\omega_p} \int d^3x\, f_p^*(\vec{x}, -t') \overleftrightarrow{\partial_0} \langle \alpha\, in | \phi_{in}(\vec{x}, -t) U(-t) | 0 \rangle$$

$$= \quad '' \quad \int d^3x\, f_p^*(\vec{x}, -t') \overleftrightarrow{\partial_0'} \langle \alpha\, in | U(-t')\phi(\vec{x}, -t') U^{-1}(-t') U(t) | 0 \rangle$$

Carry out the $\overleftrightarrow{\partial_0}$, we get

$$\partial_0' f_p^*(\vec{x}, -t)\, [U(-t')\phi(-t') U^{-1}(-t')] - f_p^* [\dot{U}(-t')\phi U^{-1}(-t') + U(-t')\dot{\phi}(-t') U^{-1}(-t')$$
$$+ U(-t')\phi(-t')\dot{U}^{-1}(-t')]\, U(-t)$$

Now we take the limit $t = t'$

$$\Rightarrow \partial_0' f_p^*\, U(-t)\phi(-t) - f_p^* [\dot{U}(-t)\phi + U(-t)\dot{\phi} + U(-t)\phi(-t)\dot{U}^{-1}(-t)\, U(-t)]$$

Then $\langle p\alpha\, in | U(-t) | 0 \rangle = \sqrt{\quad}\, \{ \langle \alpha\, in | U(-t) \underset{0}{\underset{\shortparallel}{a_{in}(p)}} | 0 \rangle + i\int d^3x\, f_p^*(x, -t) \langle \alpha | \dot{U}(-t)\phi + U\phi\dot{U}^{-1} U | 0 \rangle \}$

In the last term

$$\dot{U}\phi + U\phi\dot{U}^{-1}U = \dot{U}(U^{-1}\phi_{in} U) + U(U^{-1}\phi_{in} U)(-U^{-1}\dot{U}U^{-1})U = \dot{U}U^{-1}\phi_{in} U - \phi_{in}\dot{U}U^{-1}U$$

$$= [\dot{U}U^{-1}, \phi_{in}]\, U = -i[H_I(\pi_{in}, \phi_{in}), \phi_{in}]\, U = 0 \qquad \text{if there is no derivative coupling}$$

Then we get the result $\quad \langle p\alpha\, in | U(-t) | 0 \rangle \quad$ as $t \to \infty$ for all in-states

This means $\quad U(-t)|0\rangle = \lambda_- |0\rangle \quad \lambda_-$ some phase as $t \to \infty$

Similarly, $\quad U(t)|0\rangle = \lambda_+ |0\rangle \quad \lambda_+ \quad ''$ as $t \to \infty$

These phases can be written as

$$\lambda_- \lambda_+^* = \langle 0 | U^{-1}(t) | 0 \rangle \langle 0 | U(-t) | 0 \rangle = \langle 0 | U^{-1}(t) U(-t) | 0 \rangle = \langle 0 | T(\exp[+i\int_{-t}^{t} dt' H_I(t')]) | 0 \rangle$$

$$= [\langle 0 | T(-i\int_{-t}^{t} dt' H_I(t')) | 0 \rangle]^{-1}$$

Use this relation

$$\tau(x_1, x_2, x_3 \cdots x_n) = \langle 0 | U^{-1}(t)\, T(\phi_{in}(x_1)\phi_{in}(x_2) \cdots \phi_{in}(x_n)) \exp[-i\int_{-t}^{t} H_I(t')dt']) U(t) | 0 \rangle$$

$$= \lambda_- \lambda_+^* \langle 0 | T(\phi_{in}(x_1) \cdots \phi_{in}(x_n)) \exp[-i\int_{-t}^{t} H_I(t')dt']) | 0 \rangle$$

or
$$\boxed{\tau(x_1, x_2 \cdots x_n) = \frac{\langle 0 | T(\phi_{in}(x_1) \cdots \phi_{in}(x_n) \exp[-i\int_{-\infty}^{+\infty} dt' H_I(t')]) | 0 \rangle}{\langle 0 | T(\exp[-i\int_{-\infty}^{+\infty} dt' H_I(t')]) | 0 \rangle}}$$

More explicitly,

$$\tau(x_1, x_2 \cdots x_n) = \frac{\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{+\infty} dy_1 \cdots dy_m \langle 0 | T(\phi_{in}(x_1) \cdots \phi_{in}(x_n) \mathcal{H}_I(y_1)\mathcal{H}_I(y_2) \cdots \mathcal{H}_I(y_m)) | 0 \rangle}{\sum_{m=0} \frac{(-i)^m}{m!} \int_{-\infty}^{+\infty} dy_1 \cdots dy_m \langle 0 | T(\mathcal{H}_I(y_1) \cdots \mathcal{H}_I(y_m)) | 0 \rangle}$$

Wick's theorem : convert time-ordered product to normal ordering

Wick's theorem : convert time-ordered product to normal ordering

$$T(\phi_{in}(x_1)\cdots\phi_{in}(x_n)) = :\phi_{in}(x_1)\cdots\phi_{in}(x_n): + [\langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2))|0\rangle :\phi(x_3)\phi(x_4)\cdots\phi(x_n): + \text{permutations}]$$

$$+ [\langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2))|0\rangle\langle 0|T(\phi_{in}(x_3)\phi_{in}(x_4))|0\rangle :\phi_{in}(x_5)\cdots\phi_{in}(x_n): + \text{permutations}]$$

$$- - - - -$$

$$+ \begin{cases} [\langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2))|0\rangle\langle 0|T(\phi_{in}(x_3)\phi_{in}(x_4))|0\rangle\cdots\langle 0|T(\phi_{in}(x_{n-1})\phi_{in}(x_{n-2}))|0\rangle + \text{permutations}] & n \text{ even} \\ \langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2))|0\rangle\cdots\cdots\langle 0|T(\phi_{in}(x_{n-2})\phi_{in}(x_{n-1}))|0\rangle\phi_{in}(x_n) + \text{permutations}] & n \text{ odd}\end{cases}$$

This theorem can be proved by induction.

We will illustrate this for the simple case of $n=2$,

It is clear that

$$T(\phi_{in}(x_1)\phi_{in}(x_2)) = :\phi_{in}(x_1)\phi_{in}(x_2): + (c\text{-number})$$

We can take this equation between vaccum states to compute the c-number term,

$$\langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2))|0\rangle = (c-\text{number})$$

Then we get

$$T(\phi_{in}(x_1)\phi_{in}(x_2)) = :\phi_{in}(x_1)\phi_{in}(x_2): + \langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2))|0\rangle$$

Most useful application of Wick's theorem

$$\langle 0|T(\phi_{in}(x_1)\cdots\phi_{in}(x_n))|0\rangle = 0 \qquad n \text{ odd}$$

$$\langle 0|T(\phi_{in}(x_1)\cdots\phi_{in}(x_n))|0\rangle = \sum_{\text{permutation}}[\langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2))|0\rangle\langle 0|T(\phi_{in}(x_3)\phi_{in}(x_4))|0\rangle\cdots\cdots]$$

Notation

$$\phi_{in}(x_1)\phi_{in}(x_2) = \langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2))|0\rangle \qquad \text{Contraction}$$

Example:

$$\langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2) :\phi^3_{in}(y_1): :\phi^3_{in}(y_2):)|0\rangle$$

$$= \langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2) :\phi_{in}(y_1)\phi_{in}(y_1)\phi_{in}(y_1): :\phi_{in}(y_2)\phi_{in}(y_2)\phi_{in}(y_2):)|0\rangle$$

$$+ \cdots\cdots$$

Feynman Propagators

real scalar field

$$\langle 0|T(\phi_{in}(x)\phi_{in}(y))|0\rangle = i\Delta_F(x-y,\mu^2) = i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2-\mu^2+i\varepsilon} = \int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\, i\Delta_F(k)$$

with $\quad i\Delta_F(k) = \dfrac{i}{k^2-\mu^2+i\varepsilon}$

Complex scalar field

$$\langle 0|T(\phi_{in}(x)\phi^*_{in}(y))|0\rangle = i\Delta_F(x-y,\mu^2) = i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2-\mu^2+i\varepsilon}$$

Fermion field

$$\langle 0|T(\psi^{in}_\alpha(x)\bar\psi^{in}_\beta(y))|0\rangle = iS_F(x-y,m)_{\alpha\beta} = i\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{(\not p+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon} = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\, iS_F(p)_{\alpha\beta}$$

photon field

$$\langle 0|T(A^{in}_\mu(x)A^{in}_\nu(y))|0\rangle = iD^{tr}_F(x-y) = i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2+i\varepsilon}\Big[-g_{\mu\nu} - \frac{k_\mu k_\nu}{(k\cdot\eta)^2-k^2} + \frac{(k\cdot\eta)(k_\mu\eta_\nu+k_\nu\eta_\mu)}{(k\cdot\eta)^2-k^2}$$

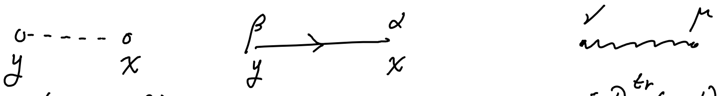
$$-\frac{k^2\eta_\mu\eta_\nu}{(k\cdot\eta)^2-k^2}\Big]$$

where $\eta_\mu = (1,0,0,0)$

where $\eta_\mu = (1,0,0,0)$

It can be shown that only term contributes is "$g_{\mu\nu}$" as a consequence of the gauge invariance.

<u>Graphic representation</u>



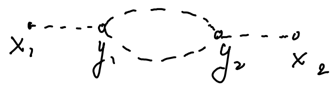
$i\Delta_F(x-y,\mu^2)$ $\quad$ $iS_F(x-y,m)_{\alpha\beta}$ $\quad$ $iD_F^{tr}(x-y)$

each line (propagator) represents a contraction in Wick's expansion.

e.g.

1. $\phi_{in}(x_1)\phi_{in}(x_2) :\phi_{in}(y_1)\phi_{in}(y_1)\phi_{in}(y_1): :\phi_{in}(y_2)\phi_{in}(y_2)\phi_{in}(y_2):$

2. $\phi_{in}(x_1)\phi_{in}(x_2) :\phi_{in}(y_1)\phi_{in}(y_1)\phi_{in}(y_1): :\phi_{in}(y_2)\phi_{in}(y_2)\phi_{in}(y_2):$



<u>Vaccum Amplitude</u>

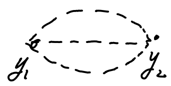
In the denominator of τ-function, there are no external lines

$$\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\infty}^{+\infty}d^4y_1\cdots d^4y_m \langle 0|T(\mathcal{H}_I(\phi_{in}(y_1))\cdots\mathcal{H}_I(\phi_{in}(y_m)))|0\rangle$$

e.g. 2nd order term for the case $\mathcal{H}_I = \frac{\lambda}{3!}:\phi^3:$

$$\langle 0|T(\mathcal{H}_I(\phi_{in}(y_1))\mathcal{H}_I(\phi_{in}(y_2)))|0\rangle = \left(\frac{\lambda}{3!}\right)^2 \langle 0|T(:\phi_{in}^3(y_1)::\phi_{in}^3(y_2):)|0\rangle$$

$$= \quad :\phi_{in}(y_1)\phi_{in}(y_1)\phi_{in}(y_1)::\phi_{in}(y_2)\phi_{in}(y_2)\phi_{in}(y_2): \quad 3\times 2$$



<u>closed loop diagram</u> : graphs with no external lines (lines with open end)

<u>disconned diagram</u> : a subgraph not connected to any external lines

<u>connected diagram</u> : graph not disconnected.

All graphs appearing in the numerator of the τ-function can be separated uniquely into connected and disconnected parts

$$\sum_m \frac{(-i)^m}{m!}\int d^4y_1\cdots d^4y_m \langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2)\cdots\phi_{in}(x_n)\mathcal{H}_I(\phi_{in}(y_1))\cdots\mathcal{H}_I(\phi_{in}(y_m)))|0\rangle$$

connected part

$$= \sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int d^4y_1\cdots d^4y_m \langle 0|T(\phi_{in}(x_1)\cdots\phi_{in}(x_n)\mathcal{H}_I(y_1)\cdots\mathcal{H}_I(y_s))|0\rangle_c$$

$$\times \frac{m!}{s!(m-s)!} \langle 0|T(\mathcal{H}_I(y_{s+1})\mathcal{H}\cdots\mathcal{H}_I(y_m))|0\rangle$$

$$= \sum_s \frac{(-i)^s}{s!}\int d^4y_1\cdots d^4y_s \langle 0|T(\phi_{in}(x_1)\cdots\phi_{in}(x_n)\mathcal{H}_I(y_1)\cdots\mathcal{H}_I(y_s))|0\rangle_c \sum_{r} \frac{(-i)^r}{r!}\int d^4z_1\cdots d^4z_r \langle 0|T(\mathcal{H}_I(z_1)\cdots\mathcal{H}_I(z_r))|0\rangle \quad r=m-s$$

It is not hard to see that

$$\tau(x_1,x_2\cdots x_n) = \frac{\sum_i G_i(x_1x_2\cdots x_n)}{\sum_k D_k} = \frac{\sum_i G_i^c(x_1\cdots x_n)\sum_k D_k}{\sum_k D_k} = \sum_i G_i^c(x_1\cdots x_n)$$

where $G_i^c(x_1\cdots x_n)$: connected diagrams with n external lines

D_k: closed loop diagrams

Hence in calculating τ-function with n external lines, we can ignore all disconnected graphs.

Example: $\mathcal{H}_I = \frac{\lambda}{3!}\phi^3$

$$\phi(q_1)+\phi(q_2) \longrightarrow \phi(p_1)+\phi(p_2)$$

$$S_{\beta\alpha} = \langle \beta\, out|\alpha\, in\rangle = \langle p_1,p_2\, out|q_1,q_2\, in\rangle$$

$$= \left(\frac{-i}{\sqrt{Z}}\right)^4 \int d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2\, e^{ip_1y_1} e^{ip_2y_2} (\Box_{y_1}+\mu^2)(\Box_{y_2}+\mu^2)\langle 0|T(\phi(y_1)\phi(y_2)\phi(x_1)\phi(x_2))|0\rangle \overleftarrow{(\Box_{x_1}+\mu^2)}\,\overleftarrow{(\Box_{x_2}+\mu^2)}\, e^{-iq_1x_1} e^{-iq_2x_2}$$

$$= \left(\frac{-i}{\sqrt{Z}}\right)^4 \int dx_1\, dx_2\, dy_1\, dy_2\, (\mu^2-p_1^2)(\mu^2-p_2^2)(\mu^2-q_1^2)(\mu^2-q_2^2)\, \tau(y_1,y_2,x_1,x_2)\, e^{+i(p_1y_1+p_2y_2)}\, e^{-i(q_1x_1+q_2x_2)}$$

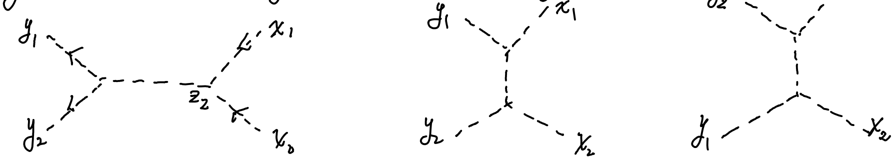
Perturbative expansion of τ-function.

$$\tau(y_1,y_2,x_1,x_2) = \sum_n \frac{(-i)^n}{n!}\int d^4z_1\cdots d^4z_n \langle 0|T(\phi(y_1)\phi(y_2)\phi(x_1)\phi(x_2)\mathcal{H}_I(z_1)\cdots\mathcal{H}_I(z_n))|0\rangle$$

lowest order contribution

$$\tau^{(2)}(y_1,y_2,x_1,x_2) = \frac{(-i)^2}{2!}\int d^4z_1\, d^4z_2 \langle 0|T\left(\phi(y_1)\phi(y_2)\phi(x_1)\phi(x_2)\left(\frac{\lambda}{3!}\phi^3(z_1)\right)\left(\frac{\lambda}{3!}\phi^3(z_2)\right)\right)|0\rangle$$

Using Wick's theorem, we get for the connected diagrams,



Their contribution to $\tau(x_1x_2y_1y_2)$ is

$$\tau^{(2)}(x_1x_2y_1y_2) = \frac{(-i\lambda)^2}{2!}\int d^4z_1\, d^4z_2\; i\Delta_F(y_1-z_1)\; i\Delta_F(y_2-z_1)\; i\Delta_F(x_1-z_2)\; i\Delta_F(x_2-z_2)\; i\Delta_F(z_1-z_2) + \cdots$$

Use the progator in momentum space

$$i\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4}\, \frac{i}{k^2-\mu^2+i\varepsilon}$$

Then

$$\tau^{(2)}(x_1,x_2,y_1,y_2) = \frac{(-i\lambda)^2}{2!}\int dz_1\, dz_2 \int \frac{d^4k_1}{(2\pi)^4}\,\frac{d^4k_2}{(2\pi)^4}\cdots\frac{d^4k_5}{(2\pi)^4}\, e^{-ik_1(y_1-z_1)}\; i\Delta_F(k_1)\; e^{-ik_2\cdot(z_1-x_1)}\; i\Delta_F(k_2)$$
$$e^{-ik_3(z_1-z_2)}\quad e^{-ik_4(y_2-z_2)}\quad e^{-ik_5(z_2-x_2)}$$

z_1 integration $\int d^4z_1\, e^{i(k_1-k_2-k_3)\cdot z_1} = (2\pi)^4\delta^4(k_1-k_2-k_3)$

z_2 integration $\int d^4z_2\, e^{i(k_3+k_4-k_5)\cdot z_2} = (2\pi)^4\delta^4(k_3+k_4-k_5)$

} energy-momentum conservation at each vertex

Then $\tau^{(2)}(x_1x_2y_1y_2) = \frac{(-i\lambda)^2}{2!}\int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_4}{(2\pi)^4}\, (2\pi)^4(k_1-k_2+k_4-k_5)\; i\Delta_F(k_1)\, i\Delta_F(k_2)\; i\Delta_F(k_4)\, i\Delta_F(k_5)$

$$\times\, e^{-ik_1y_1}\; e^{-ik_2x_1}\; e^{-ik_4y_2}\; e^{ik_5x_2}$$

$$\int \tau^{(2)}(x_1,x_2,y_1,y_2)\, d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2\; e^{i(p_1y_1+p_2y_2)}\; e^{-i(q_1x_1+q_2x_2)}$$

$$\Rightarrow\quad k_2=q_1 \quad k_5=q_2,\quad p_1=k_1,\quad p_2=k_4$$

We see that the external line propagators cancell out and

$$S_{\beta\alpha} = \left(\frac{1}{\sqrt{Z}}\right)^4 (2\pi)^4\delta^4(p_1+p_2-q_1-q_2)\; i\Delta_F(p_1-q_1) + \cdots$$

This is rather simple answer in momentum space.

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Perturbation Expansion

2010年3月19日
下午 09:56

Consider the vaccum expectation value of the form,

$$\tau(x_1 \cdots x_n) = \langle 0 | T(\phi(x_1) \cdots \phi(x_n)) | 0 \rangle$$

Coming from LSZ reduction.

Use U-matrix, we can write this as

$$\tau(x_1 \cdots x_n) = \langle 0 | T\left(U^{-1}(t_1) \phi_{in}(x_1) U(t_1, t_2) \phi_{in}(x_2) \cdots U(t_{n-1}, t_n) \phi_{in}(x_n) U(t_n) \right) | 0 \rangle$$

$=$

Cross section and Decay rate

2010年3月21日
下午 03:22

Write the S-matrix element as

$$S_{fi}=\delta_{fi}+i(2\pi)^4\delta^4(P_f-P_i)\,T_{fi}$$

T_{fi}: invariant amplitude for $i\to f$

For $i\neq f$, the transistion probability is

$$|S_{fi}|^2=(2\pi)^4\delta^4(0)\left[(2\pi)^4\delta^4(P_f-P_i)\,|T_{fi}|^2\right]$$

To interprete $\delta^4(0)$, we write

$$(2\pi)^4\delta^4(P_f-P_i)=\int d^4x\, e^{-i(P_f-P_i)\cdot x}$$

The integration is over some large but finite volume V and time interval T. Then

$$(2\pi)^4\delta^4(0)=VT$$

and $$|S_{fi}|^2=VT\,(2\pi)^4\delta^4(P_f-P_i)\,|T_{fi}|^2$$

The transistion rate (transistion probability per unit time) is then

$$w_{fi}=(2\pi)^4\delta^4(P_f-P_i)\,|T_{fi}|^2\,V$$

Decay rates

$$a(p)\longrightarrow c_1(k_1)+c_2(k_2)+\cdots+c_n(k_n)\qquad P_f=\sum_{i=1}^{n}k_i\qquad P_i=P$$

The number of states in the volume elements $d^3k_1\,d^3k_2\cdots d^3k_n$ in momentum space is

$$\prod_{\ell=1}^{n}\frac{d^3k_\ell}{(2\pi)^3 2\omega_{k_\ell}}$$

The transistion rate, summing over final states is

$$dw'=(2\pi)^4\delta^4\Big(P-\sum_{j=1}k_j\Big)\,|T_{fi}|^2\,V\prod_{\ell=1}^{n}\frac{d^3k_\ell}{(2\pi)^3 2\omega_{k_\ell}}$$

For the invariant normalization of the physical states we have been using,

$$\langle p|p'\rangle=(2\pi)^3\delta^3(p-p')\,2\omega_p\quad\Rightarrow\quad\langle p|p\rangle=(2\pi)^3\delta^3(0)\,2\omega_p=V2\omega$$

is the # of particle in the initial state.

The decay rate per particle is then

$$dw=\frac{dw'}{V2\omega_p}=(2\pi)^4\delta^4\Big(P-\sum_{j=1}^{n}k_j\Big)\,|T_{fi}|^2\,\frac{1}{2\omega_p}\prod_{\ell}\frac{d^3k_\ell}{(2\pi)^3 2\omega_{k_\ell}}$$

If there are "m" identical particles in the final state, divide this by $m!$

$$\boxed{dw=\frac{1}{2\omega_p}\,|T_{fi}|^2\,\frac{d^3k_1}{(2\pi)^3 2\omega_1}\cdots\frac{d^3k_n}{(2\pi)^3 2\omega_k}\,(2\pi)^4\delta^4\Big(P-\sum_{j=1}^{n}k_j\Big)\,S}\qquad S=\prod_j\frac{1}{(m_j)!}$$

Cross section

Scattering processes

$$a(p_1)+b(p_2)\longrightarrow c_1(k_1)+c_2(k_2)+\cdots+c_n(k_n)$$

The transistion rate is given by, after summing over final states,

$$dw'=(2\pi)^4\delta^4\Big(p_1+p_2-\sum_j k_j\Big)\,|T_{fi}|^2\,V\prod_{\ell=1}^{n}\frac{d^3k_\ell}{(2\pi)^3 2\omega_\ell}$$

We normalize this to one particle in the beam and one particle in the target and divide this by the flux ~ relative velocity divided by the volume, to get differential cross section

$$d\sigma=\frac{1}{2\omega_{p_1}V}\,\frac{1}{2\omega_{p_2}V}\,(2\pi)^4\delta^4\Big(p_1+p_2-\sum_{j=1}^{n}k_j\Big)\,|T_{fi}|^2\,V\Big(\prod_{\ell=1}^{n}\frac{d^3k_\ell}{(2\pi)^3 2\omega_{k_\ell}}\Big)\frac{V}{|\vec v_1-\vec v_2|}$$

Velocity factor can be written as

Velocity factor can be written as

$$I = |\vec{v}_1 - \vec{v}_2| = \left| \frac{\vec{P}_1}{E_1} - \frac{\vec{P}_2}{E_2} \right|$$

In the c.m frame $\quad \vec{P}_1 = -\vec{P}_2 = \vec{p} \qquad P_1 = (E_1, \vec{p}), \; P_2 = (E_2, -\vec{p})$

$$I = \frac{|\vec{p}|}{E_1 E_2}(E_1 + E_2)$$

$$(P_1 \cdot P_2)^2 = (E_1 E_2 + \vec{p}^{\,2})^2 = E_1^2 E_2^2 + 2E_1 E_2 \vec{p}^{\,2} + \vec{p}^{\,4}$$

$$(P_1 \cdot P_2)^2 - m_1^2 m_2^2 = (\vec{p}^{\,2} + m_1^2)(\vec{p}^{\,2} + m_2^2) + 2E_1 E_2 \vec{p}^{\,2} + \vec{p}^{\,4} - m_1^2 m_2^2$$

$$= \vec{p}^{\,2}\left[2\vec{p}^{\,2} + (m_1^2 + m_2^2) + 2E_1 E_2\right] = \vec{p}^{\,2}(E_1 + E_2)^2$$

$$\Rightarrow \quad I = \frac{1}{E_1 E_2}\sqrt{(P_1 \cdot P_2)^2 - m_1^2 m_2^2}$$

$$d\sigma = \frac{1}{I}\left(\frac{1}{2\omega_{P_1}}\right)\left(\frac{1}{2\omega_{P_2}}\right)(2\pi)^4 \delta^4\left(P_1 + P_2 - \sum_{j=1}^{n} k_j\right) |T_{fi}|^2 \frac{d^3 k_1}{(2\pi)^3 2\omega_1} \frac{d^3 k_2}{(2\pi)^3 2\omega_2} \cdots \frac{d^3 k_n}{(2\pi)^3 2\omega_n}$$

Feynman Rules

2010年3月22日
上午 11:34

Since the final forms for transistion matrix elements T_{fi} are quite simple, we can use simple rules to sidestep all those tedious intermediate steps.

Draw all connected Feynman graphs with appropriate external lines. Label each with momenta and impose momentum conservation for each vertex.

1. For each internal fermion line with momentum p, enter the propagator

$$iS_F(p) = \frac{i}{\not{p} - m + i\varepsilon}$$

2. For each internal boson line of spin 0, with momentum q, enter the propagator

$$i\Delta_F(q) = \frac{i}{q^2 - \mu^2 + i\varepsilon}$$

3. For each internal photon line with momentum k, enter the propagator

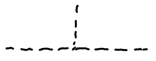
$$iD_F(k)_{\mu\nu} = \frac{-i g_{\mu\nu}}{k^2 + i\varepsilon}$$

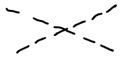
4. For each internal momentum l not fixed by momentum conservation, enter

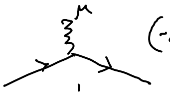
$$\int \frac{d^4 l}{(2\pi)^4}$$

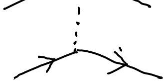
5. For each closed fermion loop, enter (-1). also the should be a factor of (-1) between graphs which differ only by an interchange of two external identical fermion lines.

At each vertex, the factors depend on the explicit form of interactions.

(a) $\frac{\lambda\phi^3}{3!}$  $(-i\lambda)$

(b) $\frac{\lambda\phi^4}{4!}$  $(-i\lambda)$

(c) $e\bar{\psi}\gamma_\mu\psi A^\mu$  $(-ie\gamma_\mu)$

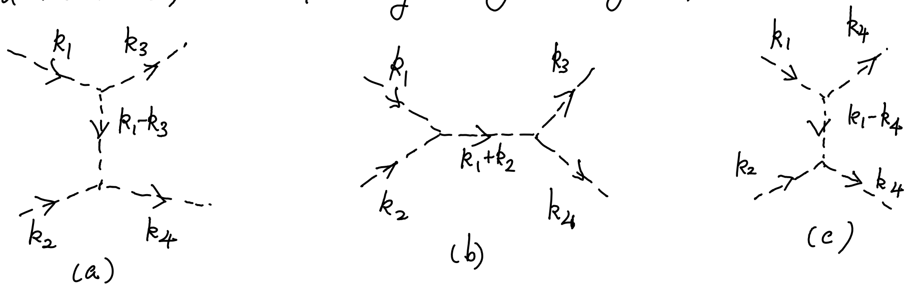
(d) $f\bar{\psi}\gamma_5\psi\phi$  $-if\gamma_5$

Example in $\lambda\phi^3$ theory

2010年3月22日
下午 12:24

Consider scattering processes $\phi(k_1)+\phi(k_2) \longrightarrow \phi(k_3)+\phi(k_4)$

To second order in λ, we have following 3 Feynman diagrams for this reaction



We can write down the matrix element for each graph,

$$T^{(a)} = (-i\lambda)^2 \frac{i}{(k_1-k_3)^2-\mu^2} \qquad T^{(b)} = (-i\lambda)^2 \frac{i}{(k_1+k_2)^2-\mu^2} \quad , \quad T^{(c)} = (-i\lambda)^2 \frac{i}{(k_1-k_4)^2-\mu^2}$$

Total amplitude

$$T = T^{(a)} + T^{(b)} + T^{(c)}$$

Mandelstam variables

$s = (k_1+k_2)^2$ total energy in c.m. frame

$t = (k_1-k_3)^2$ momentum transfer (scattering angle)

$u = (k_1-k_4)^2$

$$s+t+u = 4\mu^2$$