Path Integral method

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1) Quantum Mechanics in 1-dim

Consider the transistion matrix element of the form,

$$\langle q' t' | q t \rangle = \langle q' | e^{-iH(t'-t)} | q \rangle$$

where $|q\rangle$'s are eigenstates of the position operator Q in the Schrodinger picture,

$$Q|q\rangle = q|q\rangle$$

and $|q,t\rangle$ denotes the corresponding state in Heisenberg picture,

$$|q,t\rangle = e^{iHt}|q\rangle$$

In the path integral formalism, this transistion matrix element can be written as

$$\langle q' t' | q t \rangle = N \int [dq] \exp\left\{ i \int_t^{t'} d\tau \, L(q,\dot{q}) \right\}$$

We now explain how this formula come about and what this formula means.

First divide the interval (t', t) into n intervals with space $\delta t = \frac{t'-t}{n}$ and write the transistion matrix element as,

$$\langle q' | e^{-iH(t'-t)} | q \rangle = \int dq_1 \cdots dq_{n-1} \langle q' | e^{-iH\delta t} | q_{n-1} \rangle \langle q_{n-1} | e^{-iH\delta t} | q_{n-2} \rangle \cdots \langle q_1 | e^{-iH\delta t} | q \rangle$$

For δt small enough, we can approximate each of matrix element as

$$\langle q' | e^{-iH\delta t} | q \rangle = \langle q' | (1 - iH(P,Q)\delta t) | q \rangle + O((\delta t)^2) + \cdots$$

If we take the Hamiltonian as

$$H(P,Q) = \frac{P^2}{2m} + V(Q)$$

then

$$\langle q' | H | q \rangle = \langle q' | \frac{P^2}{2m} | q \rangle + V\left(\frac{q+q'}{2}\right) \delta(q-q')$$

$$= \int \langle q' | \frac{P^2}{2m} | p \rangle \langle p | q \rangle \left(\frac{dp}{2\pi}\right) + V\left(\frac{q+q'}{2}\right) \int \frac{dp}{2\pi} e^{ip(q'-q)}$$

$$= \int \frac{dp}{2\pi} e^{ip(q'-q)} \left[\frac{p^2}{2m} + V\left(\frac{q+q'}{2}\right) \right]$$

and

$$\langle q' | e^{-iH\delta t} | q \rangle = \int \frac{dp}{2\pi} e^{ip(q'-q)} \left\{ 1 - i\delta t \left[\frac{p^2}{2m} + V\left(\frac{q+q'}{2}\right) \right] \right\}$$

$$\simeq \int \frac{dp}{2\pi} e^{ip(q'-q)} e^{-i\delta t [p^2/2m + V(q+q'/2)]}$$

The whole transistion matrix element can then be writen as

$$\langle q' | e^{-iH(t'-t)} | q \rangle \cong \int \left(\frac{dp_1}{2\pi}\right) \cdots \left(\frac{dp_n}{2\pi}\right) \int dq_1 \cdots dq_{n-1} \exp\left\{ i \sum_{i=1}^{n} p_i (q_i - q_{i-1}) - (\delta t) H\left(p_i, \frac{q_i + q_{i-1}}{2}\right) \right\}$$

This can be written formally as

$$\langle q' | e^{-iH(t'-t)} | q \rangle = \int \left[\frac{dp\, dq}{2\pi} \right] \exp\left\{ i \int_t^{t'} dt \left[p\dot{q} - H(p,q) \right] \right\}$$

$$\equiv \lim_{n\to\infty} \int \left(\frac{dp_1}{2\pi}\right) \cdots \left(\frac{dp_n}{2\pi}\right) \int dq_1 \cdots dq_{n-1} \exp\left\{ i \sum_{i=1} \delta t \left[p_i \left(\frac{q_i - q_{i-1}}{2} \right) - H\left(p_i, \frac{q_i + q_{i-1}}{2}\right) \right] \right\}$$

In most cases, Hamiltonian depends quadractically on p. Using the formula

$$\int_{-\infty}^{+\infty} \frac{dx}{2\pi} e^{-ax^2 + bx} = \frac{1}{\sqrt{4\pi a}} e^{b^2/4a}$$

we do the integration over momentum

$$\int \frac{dp_i}{2\pi} \left[\frac{-i\delta t}{2m} p_i^2 + i p_i (q_i - q_{i-1}) \right] = \left(\frac{m}{2\pi i \delta t} \right)^{1/2} \exp\left[\frac{im(q_i - q_{i-1})^2}{2\delta t} \right]$$

Then

$$\langle q'|e^{-iH(t'-t)}|q\rangle = \lim_{n\to\infty}\left(\frac{m}{2\pi i\delta t}\right)^{n/2}\int\prod_{i=1}^{n-1}dq_i\,\exp\left\{i\sum_{i=1}^{n}\delta t\left[\frac{m}{2}\left(\frac{q_i-q_{i-1}}{\delta t}\right)^2-V\right]\right\}$$

or

$$\langle q't'|qt\rangle = \langle q'|e^{-iH(t'-t)}|q\rangle = N\int[dq]\,\exp\left\{i\int_t^{t'}dt\left[\frac{m}{2}\dot{q}^2-V(q)\right]\right\}$$

<u>Green's functions</u>

Consider the time-ordered product between ground state,

$$G(t_1,t_2)=\langle 0|T(Q^H(t_1)Q^H(t_2))|0\rangle$$

Inserting complete set of states, we get

$$G(t_1,t_2)=\int dq\,dq'\,\langle 0|q',t'\rangle\langle q't'|T(Q^H(t_1)Q^H(t_2))|q,t\rangle\langle q,t|0\rangle$$

The matrix element $\langle 0|q,t\rangle = \phi_0(q)e^{-iE_0t}=\phi_0(q,t)$ is the wavefunction for ground state. Consider the case $t'>t_1>t_2>t$, We can write

$$\langle q't'|T(Q^H(t_1)Q^H(t_2))|qt\rangle = \langle q'|e^{-iH(t'-t_1)}Q^S e^{-iH(t_1-t_2)}Q^S e^{-iH(t_2-t)}|q\rangle$$

$$=\int\langle q'|e^{-iH(t'-t_1)}|q_1\rangle q_1\langle q_1|e^{-iH(t_1-t_2)}|q_2\rangle q_2\langle q_2|e^{-iH(t_2-t)}|q\rangle$$

$$=\int\left[\frac{dp\,dq}{2\pi}\right]q_1(t_1)q_2(t_2)\exp\left\{i\int_t^{t'}d\tau\left[p\dot{q}-H(p,q)\right]\right\}$$

It is not hard to see that for the time sequence $t'>t_2>t_1>t$, we get the same formula. because the path integral orders the time sequence automatically.

The Green's function is then

$$G(t_1,t_2)=\int dq\,dq'\,\phi_0(q',t')\phi_0^*(q,t)\int\left[\frac{dq\,dp}{2\pi}\right]q(t_1)q(t_2)\,\exp\left\{i\int_t^{t'}d\tau\left[p\dot{q}-H(p,q)\right]\right\}$$

We can remove the ground state wavefunction $\phi_0(q,t)$ as follows.

Write

$$\langle q't'|\mathcal{O}(t_1,t_2)|q,t\rangle=\int dQ\,dQ'\,\langle q't'|Q'T'\rangle\langle Q'T'|\mathcal{O}(t_1,t_2)|QT\rangle\langle QT|qt\rangle$$

where $\mathcal{O}(t_1,t_2)=T(Q^H(t_1)Q^H(t_2))$

Let $|n\rangle$ be the energy eigenstate

$$H|n\rangle=E_n|n\rangle\quad,\quad\langle q|n\rangle=\phi_n^*(q)$$

Then

$$\langle q't'|Q'T'\rangle=\langle q'|e^{-iH(t'-T')}|Q'\rangle=\sum_n\langle q'|n\rangle e^{-iE_n(t'-T')}\langle n|Q'\rangle$$

$$=\sum_n\phi_n^*(q')\phi_n(Q')e^{-iE_n(t'-T')}$$

To isolate the ground state wavefunction, we take an "unusual limit"

$$\lim_{t'\to -i\infty}\langle q't'|Q'T'\rangle=\phi_0^*(q')\phi_0(Q')e^{-E_0|t'|}e^{iE_0T'}$$

Similarly,

$$\lim_{t\to i\infty}\langle QT|qt\rangle=\phi_0(q)\phi_0^*(Q)e^{-E_0|t|}e^{-iE_0T}$$

With these we can write

$$\lim_{\substack{t'\to -i\infty\\ t\to i\infty}}\langle q't'|\mathcal{O}(t_1,t_2)|qt\rangle=\int dQ\,dQ'\,\phi_0^*(q')\phi_0(Q')\langle Q'T'|\mathcal{O}(t_1,t_2)|QT\rangle\phi_0^*(Q)\phi_0(q)\,e^{-E_0|t'|}e^{iE_0T}e^{-iE_0T}e^{-E_0|t|}$$

$$=\phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|}G(t_1,t_2)$$

It is easy to see that

$$\lim_{\substack{t'\to -i\infty\\ t\to i\infty}}\langle q't'|qt\rangle=\phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|}$$

Finally,

$$G(t_1,t_2) = \lim_{\substack{t'\to -i\infty \\ t\to i\infty}} \left[\frac{\langle q't'|T(Q^H(t_1)Q^H(t_2))|qt\rangle}{\langle q't'|qt\rangle}\right]$$

$$= \lim_{\substack{t'\to -i\infty \\ t\to i\infty}} \frac{1}{\langle q't'|qt\rangle}\int\left[\frac{dp\,dq}{2\pi}\right] q(t_1)q(t_2)\exp\left\{i\int d\tau\,[p\dot{q}-H(p,q)]\right\}$$

This can generalized to n-point Green's function

$$G(t_1,t_2\cdots t_n) = \langle 0|T(q(t_1)q(t_2)\cdots q(t_n))|0\rangle$$

$$= \lim_{\substack{t'\to -i\infty \\ t\to i\infty}} \frac{1}{\langle q't'|qt\rangle}\int\left[\frac{dp\,dq}{2\pi}\right] q(t_1)q(t_2)\cdots q(t_n)\exp\left\{i\int_t^{t'} d\tau\,[p\dot{q}-H(p,q)]\right\}$$

It is very useful to introduce generating functional for these n-point functions

$$W[J] = \lim_{\substack{t'\to -i\infty \\ t\to i\infty}} \frac{1}{\langle q't'|qt\rangle}\int\left[\frac{dp\,dq}{2\pi}\right]\exp\left\{i\int_t^{t'} d\tau\,[p\dot{q}-H(p,q)+J(\tau)q(\tau)]\right\}$$

Then $\quad G(t_1,t_2\cdots t_n) = (-i)^n \left.\dfrac{\delta^n W[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right|_{J=0}$

The unphysical limit, $t'\to -i\infty$, $t\to i\infty$, should be interpreted in term of Euclidean Green's functions defined by

$$S^{(n)}(\tau_1,\tau_2,\cdots\tau_n) = i^n G^{(n)}(-i\tau_1,-i\tau_2\cdots,-i\tau_n)$$

Generating functional for $S^{(n)}$ is then

$$W_E[J] = \lim_{\substack{\tau'\to\infty \\ \tau\to -\infty}} \int[dq]\exp\left\{\int_\tau^{\tau'} d\tau''\left[-\left(\frac{dq}{d\tau''}\right)^2 - V(q) + J(\tau'')q(\tau'')\right]\right\}$$

Since we can adjust the zero point of $V(q)$ such that

$$\frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q) > 0$$

This provides the damping which makes the Gaussian integral converging

Field Theory

We can extend the treatment for quantum mechanics to field theory with following replacements,

$$\prod_{i=1}^{\infty}[dq_i\,dp_i] \longrightarrow [d\phi(x)\,d\pi(x)]$$

$$L(q\dot{q}) \longrightarrow \int\mathcal{L}(\phi,\partial_\mu\phi)\,d^3x \qquad\qquad H(p,q)\longrightarrow\int\mathcal{H}(\phi,\pi)\,d^3x$$

For example, the generating functional for scalar field is of the form

$$W[J] \sim \int[d\phi][d\pi]\exp\left\{i\int d^4x\,[\pi(x)\partial_0\phi - \mathcal{H}(\pi,\phi) + J(x)\phi(x)]\right\}$$

$$\sim \int[d\phi]\exp\left\{i\int d^4x\,[\mathcal{L}(x) + J(x)\phi(x)]\right\}$$

Consider the example of $\lambda\phi^4$ theory

$$\mathcal{L}(\phi) = \mathcal{L}_0(\varphi) + \mathcal{L}_1(\phi)$$

$$\mathcal{L}_0(\phi) = \frac{1}{2}(\partial_\lambda\phi)^2 - \frac{\mu^2}{2}\phi^2, \qquad \mathcal{L}_1(\varphi) = -\frac{\lambda}{4!}\phi^4$$

Generating functional

$$W[J] = \int[d\phi]\exp\left\{-\int d^4x\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}(\vec\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 + J\phi\right]\right\}$$

$$W[J]=\int[d\phi]\exp\left\{-\int d^4x\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2+\frac{1}{2}(\vec{\nabla}\phi)^2+\frac{1}{2}\mu^2\phi^2+\frac{\lambda}{4!}\phi^4+J\phi\right]\right\}$$

Can be written as

$$W[J]=\left[\exp\int d^4x\,\mathcal{L}_I\left(\frac{\delta}{\delta J}\right)\right]W_0[J]$$

where

$$W_0[J]=\int[d\phi]\exp\left[-\frac{1}{2}\int d^4x\,d^4y\;\phi(x)K(x,y)\phi(y)+\int d^4z\,J(z)\phi(z)\right]$$

and $\quad K(x,y)=\delta^4(x-y)\left(-\frac{\partial^2}{\partial\tau^2}-\vec{\nabla}^2+\mu^2\right)$

The Gaussian integral for many variables is

$$\int d\phi_1\cdots d\phi_N\,\exp\left[-\frac{1}{2}\sum_{ij}\phi_i K_{ij}\phi_j+\sum_k J_k\phi_k\right]\sim\frac{1}{\sqrt{\det K}}\exp\left[\frac{1}{2}\sum_{ij}J_i(K^{-1})_{ij}J_j\right]$$

Apply this to the case of scalar fields,

$$W_0[J]=\exp\left[\frac{1}{2}\int d^4x\,d^4y\,J(x)\Delta(x,y)J(y)\right]$$

where

$$\int d^4y\,K(x,y)\Delta(y,z)=\delta^4(x-z)$$

It is not difficult to see that

$$\Delta(x,y)=\int\frac{d^4k_E}{(2\pi)^4}\frac{e^{ik_E(x-y)}}{k_E^2+\mu^2}\qquad\text{where } k_E=(ik_0,\vec{k})\text{ , the Euclidean momentum}$$

Perturbative expansion in power of $\mathcal{L}_1$ gives

$$W[J]=W_0[J]\{1+\lambda\omega_1[J]+\lambda^2\omega_2[J]+\cdots\}$$

where $\quad\omega_1=-\frac{1}{4!}W_0^{-1}[J]\left\{\int d^4x\left[\frac{\delta}{\delta J(x)}\right]^4\right\}W_0[J]$

$$\omega_2=\frac{1}{2(4!)^2}W_0^{-1}[J]\left\{\int d^4x\left[\frac{\delta}{\delta J(x)}\right]^4\right\}^2W_0[J]$$

Use the explicit form for $W_0[J]$, we can compute ω_1 as follows

$$W_0[J]=1+\frac{1}{2}\int d^4x\,d^4y\,J(x)\Delta(x,y)J(y)+\left(\frac{1}{2}\right)^2\frac{1}{2!}\int d^4y_1d^4y_2\cdots d^4y_4\left[J(y_1)\Delta(y_1,y_2)J(y_2)\,J(y_3)\Delta(y_3,y_4)J(y_4)\right]+\cdots\cdots$$

Then $\omega_1=-\frac{1}{4!}\left[\int\Delta(x,y_1)\Delta(x,y_2)\Delta(x,y_3)\Delta(x,y_4)J(y_1)J(y_2)J(y_3)J(y_4)+3!\,\Delta(x,y_1)\Delta(x,y_2)J(y_1)J(y_2)\Delta(x,x)\right]$

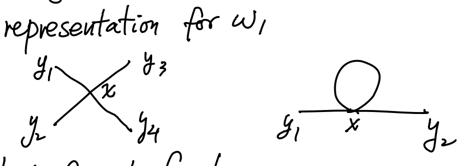
where we have dropped all J indep terms

All arguments (x_i,y_i) are integrated over

In this computation we have used the identity,

$$\frac{\delta}{\delta J(x)}\int d^4y_1\,J(y_1)f(y_1)=\int\delta^4(x-y_1)\,d^4y_1\,f(y_1)=f(x)$$

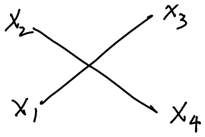
graphical representation for ω_1

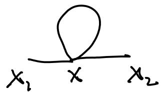


The connected Green's function is

$$G^{(n)}(x_1x_2\cdots x_n)=\left.\frac{\delta^n\ln W[J]}{\delta J(x_1)\,\delta J(x_2)\cdots\delta J(x_n)}\right|_{J=0}$$

Thus replacing y_i by external x_i, we get contributions for 4-point, 2 point functions,





Grassmann algebra

For the quantization of fermion field, using path integral, we need to integrate over anti-commuting c-number functions. This can be realized as elements of Grassmann algebra.

In an n-dimensional Grassmann algebra, the n generators $\theta_1, \theta_2, \theta_3 \cdots \theta_n$ satisfy

$$\{\theta_i, \theta_j\} = 0 \qquad i,j = 1,2\cdots n$$

and every element can be expanded in a finite series,

$$P(\theta) = P_0 + P^{(1)}_{i_1}\theta_{i_1} + P^{(2)}_{i_1 i_2}\theta_{i_1}\theta_{i_2} + \cdots + P^{(n)}_{i_1\cdots i_n}\theta_{i_1}\cdots\theta_{i_n}$$

Simplest case: $n=1$

$$\{\theta,\theta\} = 0 \quad \text{or} \quad \theta^2 = 0 \qquad P(\theta) = P_0 + \theta P_1$$

We can define the "differentiation" and "integration" as follows,

$$\frac{d}{d\theta}\theta = \theta\frac{\overleftarrow{d}}{d\theta} = 1 \qquad \Rightarrow \qquad \frac{d}{d\theta}P(\theta) = P_1$$

Integration is defined in such a way that it is invariant under translation,

$$\int d\theta\, P(\theta) = \int d\theta\, P(\theta + \alpha) \qquad \alpha \text{ is another Grassmann variable}$$

which implies

$$\int d\theta = 0$$

We can normalize the integral such that

$$\int d\theta\, \theta = 1$$

Then

$$\int d\theta\, P(\theta) = P_1 = \frac{d}{d\theta}P(\theta)$$

Consider a change of variable $\theta \to \tilde\theta = a + b\theta$

Since

$$\int d\tilde\theta\, P(\tilde\theta) = \frac{d}{d\tilde\theta}P(\tilde\theta) = P_1$$

&

$$\int d\theta\, P(\tilde\theta) = \int d\theta\,[P_0 + \tilde\theta P_1] = \int d\theta\,[P_0 + (a+b\theta)P_1] = bP_1$$

We get

$$\int d\tilde\theta\, P(\tilde\theta) = \int d\theta \left(\frac{d\tilde\theta}{d\theta}\right)^{-1} P(\tilde\theta(\theta))$$

Thus the "Jacobian" is the inverse of that for c-number integration

It is easy to generalize to the case of n-dimensional Grassmann algebra,

$$\frac{d}{d\theta_i}(\theta_1\theta_2\cdots\theta_n) = \delta_{i1}\theta_2\cdots\theta_n - \delta_{i2}\theta_1\theta_3\cdots\theta_n + \cdots + (-1)^{n-1}\delta_{in}\theta_1\theta_2\cdots\theta_{n-1}$$

$$\{d\theta_i, d\theta_j\} = 0$$

$$\int d\theta_i = 0 \qquad \int d\theta_i\,\theta_j = \delta_{ij}$$

For a change of variables of the form $\tilde\theta_i = b_{ij}\theta_j$, we have

$$\int d\tilde\theta_n d\tilde\theta_{n-1}\cdots d\tilde\theta_1\, P(\tilde\theta) = \int d\theta_n\cdots d\theta_1 \left[\det\frac{d\tilde\theta}{d\theta}\right]^{-1} P(\tilde\theta(\theta))$$

Proof:

$$\tilde\theta_1\tilde\theta_2\cdots\tilde\theta_n = b_{1i_1}b_{2i_2}\cdots b_{ni_n}\theta_{i_1}\cdots\theta_{i_n}$$

RHS is non-zero only if $i_1, i_2\cdots i_n$ are all different and we can write

$$\tilde\theta_1\tilde\theta_2\cdots\tilde\theta_n = b_{1i_1}b_{2i_2}\cdots b_{ni_n}\,\epsilon_{i_1 i_2\cdots i_n}\theta_1\cdots\theta_n$$
$$= (\det b)\,\theta_1\theta_2\theta_3\cdots\theta_n$$

From the normalization condition,

From the normalization condition,

$$1=\int d\tilde{\theta}_n d\tilde{\theta}_{n-1}\cdots d\tilde{\theta}_1\ (\tilde{\theta}_1\tilde{\theta}_2\cdots\tilde{\theta}_n)=(\det b)\int d\tilde{\theta}_n\cdots d\tilde{\theta}_1(\theta_1\theta_2\cdots\theta_n)$$

We see that

$$d\tilde{\theta}_n d\tilde{\theta}_{n-1}\cdots d\tilde{\theta}_1=(\det b)^{-1}d\theta_1\cdots d\theta_n$$

In field theory, we need to make use of Gaussian integral,

$$G(A)=\int d\theta_n\cdots d\theta_1\exp\left(\tfrac{1}{2}(\theta,A\theta)\right) \qquad \text{where } (\theta,A\theta)=\theta_i A_{ij}\theta_j$$

First consider the simple case of $n=2$, where

$$A=\begin{pmatrix}0 & A_{12}\\ -A_{12} & 0\end{pmatrix}$$

Then

$$G(A)=\int d\theta_2 d\theta_1\exp(\theta_1\theta_2 A_{12})=\int d\theta_2 d\theta_1(1+\theta_1\theta_2 A_{12})=A_{12}=\sqrt{\det A}\,.$$

The generalization to arbitrary n is

$$G(A)=\int d\theta_n\cdots d\theta_1\ \exp\left(\tfrac{1}{2}(\theta,A\theta)\right)=\sqrt{\det A} \qquad n \text{ even}$$

and for "complex" Grassmann variables

$$\int d\theta_n d\bar{\theta}_n\, d\theta_{n-1}d\bar{\theta}_{n-1}\cdots d\theta_1 d\bar{\theta}_1\ \exp(\bar{\theta},A\theta)=\det A$$

For the Fermion fields, the generating functional is of the form,

$$W[\eta,\bar{\eta}]=\int[d\psi(x)][d\bar{\psi}(x)]\exp\left\{i\int d^4x\,[\mathcal{L}(\psi,\bar{\psi})+\bar{\psi}\eta+\bar{\eta}\psi]\right\}$$

It is not hard to see that if we $\mathcal{L}$ depends on $\psi,\bar{\psi}$ quadratically

$$\mathcal{L}=(\bar{\psi},A\psi)$$

then we have

$$W=\int[d\psi(x)][d\bar{\psi}(x)]\exp\left\{\int d^4x\ \bar{\psi}A\psi\right\}=\det A$$