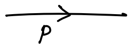
Renormalization

Consider a simple example of $\lambda\phi^4$ theory

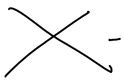
$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

$$\mathcal{L}_0 = \frac{1}{2}\left[(\partial_\mu\phi_0)^2 - \mu_0^2\phi_0^2\right], \quad \mathcal{L}_I = -\frac{\lambda_0}{4!}\phi_0^4$$

Feynman rule



$\frac{i}{p^2-\mu_0^2+i\varepsilon}$



$-i\lambda_0$

* 4-momentum conservation at each vertex

* integrate over internal momenta which are not fixed by momentum conservation

no propagator for external line

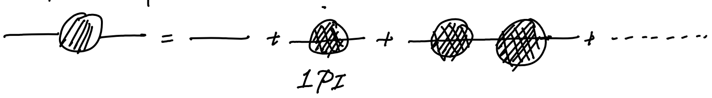
Simple example

2-point function

1PI : one-partide irreducible graphs.

graphs which can not be disconnected by cutting any one line

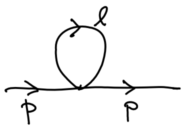
Complete 2 point function



$$\frac{i}{p^2-\mu_0^2+i\varepsilon} + \frac{i}{p^2-\mu_0^2+i\varepsilon}\left(-i\Sigma(p^2)\right)\frac{i}{p^2-\mu_0^2+i\varepsilon} + \cdots\cdots$$

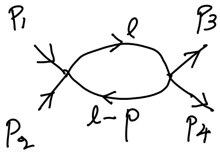
$$= \frac{i}{(p^2-\mu_0^2+i\varepsilon)}\left[\frac{1}{1+i\Sigma(p^2)\frac{i}{p^2-\mu_0^2+i\varepsilon}}\right] = \frac{i}{p^2-\mu_0^2-\Sigma(p^2)+i\varepsilon}$$

1- loop diagrams



$$-i\Sigma(p) = -\frac{i\lambda_0}{2}\int\frac{d^4l}{(2\pi)^4}\frac{i}{l^2-\mu_0^2+i\varepsilon}$$

quadratically divergent



$$\Gamma(p^2) = \frac{(-i\lambda_0)^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{i}{(l-p)^2-\mu_0^2+i\varepsilon}\frac{i}{l^2-\mu_0^2+i\varepsilon}$$

$p = p_1 + p_2$

log divergent

In $\Gamma(p)$, the dependence on the external momentum p is in the combination $(p-l)$ in the denominator.

$\Rightarrow$ if we differentiate $\Gamma(p)$ w.r.t. p, power of l will increase in denominator $\Rightarrow$ improve the convergence

$$\frac{\partial}{\partial p^2}\Gamma(p^2) = \frac{1}{2p^2}p_\mu\frac{\partial}{\partial p_\mu}\Gamma(p^2) = \frac{\lambda_0^2}{p^2}\int\frac{d^4l}{(2\pi)^4}\frac{(l-p)\cdot p}{[(l-p)^2-\mu_0^2+i\varepsilon]^2}\frac{1}{l^2-\mu_0^2+i\varepsilon}$$

Convergent

Thus if we expand $\Gamma(p^2)$ in Taylor series,

$\Gamma(p^2) = a_0 + a_1 p^2 + \cdots$

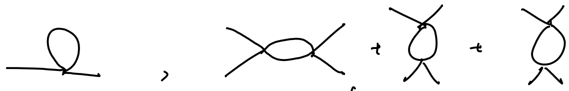
$\Gamma(p^2) = a_0 + a_1 p^2 + \cdots$

the divergences are contained in first few terms

In our simple case,

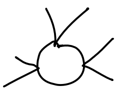
$\Gamma(p^2) = \Gamma(0) + \tilde{\Gamma}(p^2)$ then $\tilde{\Gamma}(p^2)$ is finite.

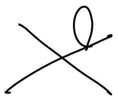
In 1-loop, the divergent graphs are (1PI)

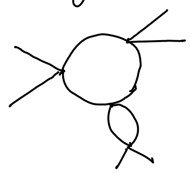


Other 1-loop graphs are either finite or contain the above graphs as subgraphs

e.g







finite

contains self-energy

mass and wavefunction renormalization

1PI self energy

$$\Sigma(p^2) = \Sigma(\mu^2) + (p^2-\mu^2)\Sigma'(\mu^2) + \tilde{\Sigma}(p^2) \quad , \mu^2 : \text{arbitrary}$$

$\tilde{\Sigma}(p^2)$ is finite and $\tilde{\Sigma}(\mu^2)=0$, $\tilde{\Sigma}'(\mu^2)=0$

Complete propagator

$$i\Delta(p) = \frac{i}{p^2-\mu_0^2 - \Sigma(\mu^2) - (p^2-\mu^2)\Sigma'(\mu^2) - \tilde{\Sigma}(p^2)}$$

Suppose we choose μ^2 such that

$$\mu_0^2 - \Sigma(\mu^2) = \mu^2 \qquad \text{mass renormalization}$$

then $\Delta(p^2)$ will have a pole at $p^2=\mu^2 \Rightarrow \mu^2$: physical mass , μ_0^2 : bare mass

$$i\Delta(p^2) = \frac{i}{(p^2-\mu^2)[1-\Sigma'(\mu^2)] - \tilde{\Sigma}(p^2)}$$

Since $\Sigma'(\mu^2)$ and $\tilde{\Sigma}(p^2)$ are both of order λ_0 or higher., we can approximate

$$\tilde{\Sigma}(p^2) \longrightarrow (1-\Sigma'(\mu^2))\,\tilde{\Sigma}(p^2)$$

Then

$$i\Delta(p^2) = \frac{iZ_\phi}{p^2-\mu^2-\tilde{\Sigma}(p^2)+i\varepsilon} \qquad Z_\phi = \frac{1}{1-\Sigma'(\mu^2)} \approx 1+\Sigma'(\mu^2)$$

We can get rid of Z_ϕ by defining the renormalized field

$$\phi = \frac{1}{\sqrt{Z_\phi}}\phi_0$$

So that the propagator for ϕ is

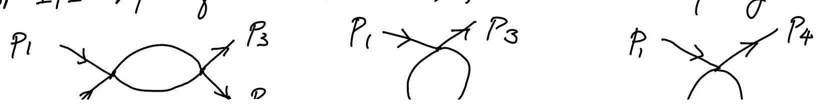
$$i\Delta_R(p) = \int d^4x\, e^{-ipx}\langle 0|T(\phi(x)\phi(0))|0\rangle = \frac{i}{p^2-\mu^2-\tilde{\Sigma}(p^2)+i\varepsilon} \qquad \text{finite}$$

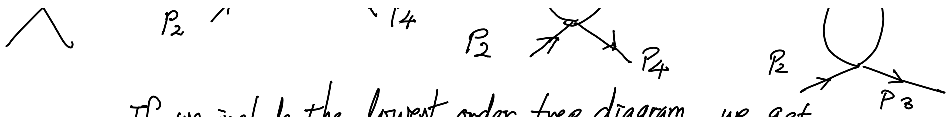
For more general Green's functions, we have

$$G_R^{(n)}(x_1\cdots x_n) = \langle 0|T(\phi(x_1)\cdots\phi(x_n))|0\rangle = Z_\phi^{-n/2}\langle 0|T(\phi_0(x_1)\cdots\phi_0(x_n))|0\rangle$$
$$= Z_\phi^{-n/2}\, G_0^{(n)}(x_1\cdots x_n)$$

Coupling constant renormalization

For 1PI 4-point functions $\Gamma^{(4)}(p_1\cdots p_4)$, there are 3 1-loop diagrams,





If we include the lowest order tree diagram, we get

$$\Gamma_0^{(4)}(s,t,u) = -i\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u)$$

$$s=(p_1+p_2)^2,\quad t=(p_1-p_3)^2,\quad u=(p_1-p_4)^2 \qquad , \quad s+t+u=4\mu^2$$

We need one substraction to make this finite

"choose" a symmetric point $\quad s_0=t_0=u_0=\frac{4\mu^2}{3}$

$$\Gamma_0^{(4)}(s,t,u) = \underbrace{-i\lambda_0 + 3\Gamma(s_0)}_{-iZ_\lambda^{-1}\lambda_0} + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \qquad \text{where } \tilde{\Gamma}(s) = \Gamma(s) - \Gamma(s_0)$$

$$\Rightarrow \quad \Gamma_0^{(4)}(s_0,t_0,u_0) = -iZ_\lambda^{-1}\lambda_0$$

Renormalized 1PI 4 point function

$$\Gamma_R^{(4)} = \prod_{j=1}^{4}[i\Delta_R(p_j)]^{-1}\, G_R^{(4)}$$

$$\Rightarrow \qquad \Gamma_R^{(4)}(s,t,u) = Z_\phi^2\,\Gamma_0^{(4)}(s,t,u)$$

Define the renormalized coupling constant λ by

$$\lambda = Z_\phi^2 Z_\lambda^{-1}\lambda_0$$

then

$$\Gamma_R^{(4)}(\ ) = Z_\phi^2\,\Gamma_0^{(4)} = -iZ_\lambda^{-1}Z_\phi^2\lambda_0 + Z_\phi^2[\tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)]$$

$$= -i\lambda + Z_\phi^2[\tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)]$$

Since $Z_\phi = 1 + O(\lambda_0)$, $\tilde{\Gamma} = O(\lambda_0^2)$ $\quad \lambda = \lambda_0 + O(\lambda_0^2)$, we can approximate

$$\Gamma_R^{(4)}(\quad) = \underbrace{-i\lambda + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)}_{\text{finite}} + O(\lambda^3)$$

From the original Lagrangian (unrenormalized Lagrangian)

$$\mathcal{L}_0 = \frac{1}{2}[(\partial_\mu\phi_0)^2 - \mu_0^2\phi_0^2] - \frac{\lambda_0}{4!}\phi_0^4$$

We can write

$$\mathcal{L}_0 = \mathcal{L} + \Delta\mathcal{L}$$

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\phi)^2 - \mu^2\phi^2] - \frac{\lambda}{4!}\phi^4 \qquad \text{renormalized Lagrangian}$$

$$\Delta\mathcal{L} = \mathcal{L}_0 - \mathcal{L} = \frac{1}{2}(Z_\phi - 1)[(\partial_\mu\phi)^2 - \mu^2\phi^2] + \frac{\delta\mu^2}{2}\phi^2 - \frac{\lambda(Z_\lambda - 1)}{4!}\phi^4 \qquad \text{counter terms.}$$

where $\quad \mu^2 = \delta\mu^2 + \mu_0^2, \quad \phi = Z_\phi^{-1/2}\phi_0, \quad \lambda = Z_\lambda^{-1}Z_\phi^2\lambda_0$

<u>BPH renormalization</u>

(1) Starts with renormalized Lagangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

to generate free propagator and vertices

(2) The divergent parts of one-loop 1PI diagrams are isolated by Taylor expansion. Construct a set of counter terms $\Delta\mathcal{L}^{(1)}$ to cancel these divergences.

(3) A new Lagrangian $\mathcal{L}^{(1)} = \mathcal{L} + \Delta\mathcal{L}^{(1)}$ is used to generate 2-loop diagrams. and to counterterms $\Delta\mathcal{L}^{(2)}$ to cancel 2-loop divergences. This sequence of operations is iteratively applied

iteratively applied.

Regularization

In carrying out the renormalization, we need first to make divergent integral finite before we can do any manipulation about the integral.

1) Pauli-Villars

$$\frac{1}{k^2-\mu_0^2} \longrightarrow \left(\frac{1}{k^2-\mu_0^2} - \frac{1}{k^2-\Lambda^2}\right) = \frac{(\mu_0^2-\Lambda^2)}{(k^2-\mu_0^2)(k^2-\Lambda^2)} \longrightarrow \frac{1}{k^4} \quad \text{more convergent}$$

This has the advantage of being covariant

2) Dimensional regularization

$$\int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2-\mu^2}\right) \left[\frac{1}{(k-p)^2-\mu^2}\right]$$

4-dimension $\longrightarrow$ d-dimension

$$\int \frac{d^dk}{(2\pi)^4} \frac{1}{(k^2-\mu^2)} \left[\frac{1}{(k-p)^2-\mu^2}\right] \qquad \text{convergent for } d<4$$

Power counting

Superficial degree of divergence D:

$D = (\#$ of loop momenta in numerator$) - (\#$ of loop momenta in denominator$)$

B = number of external lines

IB = " " internal lines

n = number of vertices

Counting the lines in the graph, we get

$3n \rightarrow 4n = 2(IB) + B$

$\#$ of loops L

$L = IB - n + 1$

Then $D = 4L - 2(IB)$

Eliminating L, & (IB)

$D = 4 - B$

thus $D \geq 0$ for $B \leq 4$

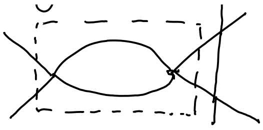
$\lambda\phi^4$ theory has the symmetry $\phi \rightarrow -\phi \Rightarrow$ only B = even are non-zero

$\Rightarrow B = 2, 4$ are superfically divergent.

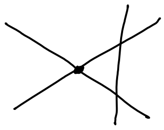
Comments on subgraph divergences

Convergence property of Feynman integrals (Weinberg's Theorem)

The general Feynman integral convergers if the superficial degree of divergence of the graph togather with the superficial degree of divergence of all subgraphs are negative.



$D=-2$ for overall graph
subgraph is log divergent $D_S=0$



subgraph divergence is cancelled by lower order counter term.

Renormalization Group

Discussion will be brief

Renormalization scheme requires specification of substraction points.

substraction points $\Rightarrow$ new mass scale

This introduces the concept of energy dependent "coupling constants"

eventhough coupling constants are dimensionless,

e.g $\lambda = \lambda(s)$.

Renormalization group equation

There is arbitrariness in choosing the renormalization schemes (or the substraction points). Nevertheles, the physical results should be the same, i.e. independent of renormalization schemes.

From the point of view of BPH renormalization, we can write

$$\mathcal{L} = \mathcal{L}_R(R\text{-quantities}) = \mathcal{L}_{R'}(R'\text{-quantities})$$

Recall that

$$\phi_R = Z_{\varphi R}^{-1/2}\phi_0, \quad \lambda_R = Z_{\lambda R}^{-1} Z_{R\phi}^2 \lambda_0 \quad \mu_R^2 = \mu_0^2 + \delta\mu_R^2$$

Similarly,

$$\phi_{R'} = Z_{\varphi' R}^{-1/2}\phi_0, \quad \lambda_{R'} = Z_{\lambda R'}^{-1} Z_{R'\phi}^2 \lambda_0 \quad \mu_{R'}^2 = \mu_0^2 + \delta\mu_{R'}^2$$

Since ϕ_0, λ_0 and μ_0 are the same, we can finite relations between $R-$ and R' quantities

Callan-Symanzik equation

This particular derivation of RG equation is conceptually simple

Start with the fact that for the bare propagator, we have

$$\frac{\partial}{\partial\mu_0^2}\left(\frac{i}{p^2-\mu_0^2+i\varepsilon}\right) = \frac{i}{p^2-\mu_0^2+i\varepsilon}(-i)\frac{i}{(p^2-\mu_0^2+i\varepsilon)}$$

or $\frac{\partial}{\partial\mu_0^2}\left(\xrightarrow[p]{}\right) = \xrightarrow[p]{}\times\xrightarrow[p]{}$ insertion of composite operator $\Omega = \frac{1}{2}\phi_0^2$ with zero momentum

Thus

$$\frac{\partial\Gamma^{(n)}(P_i)}{\partial\mu_0^2} = -i\,\Gamma^{(n)}_{\phi^2}(0;P_i)$$

In terms of renormalized (1PI) Green's function

$$\Gamma_R^{(n)}(P_i,\lambda,\mu) = Z_\varphi^{n/2}\,\Gamma^{(n)}(P_i,\lambda_0,\mu_0^2)$$

$$\Gamma^{(n)}_{\phi^2 R}(P,P_i,\lambda,\mu) = Z_{\phi^2}^{-1} Z_\phi^{n/2}\,\Gamma^{(n)}_{\phi^2}(P,P_i;\lambda_0,\mu_0^2)$$

Now $$\frac{\partial}{\partial\mu_0}\Gamma_R^{(n)}(P_i,\lambda,\mu) = \left(\frac{\partial\mu^2}{\partial\mu_0^2}\frac{\partial}{\partial\mu^2} + \frac{\partial\lambda}{\partial\mu_0^2}\frac{\partial}{\partial\lambda}\right)\Gamma_R^{(n)}(P_i,\lambda,\mu)$$

$$\Rightarrow \left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + n\gamma\right] \Gamma_R^{(n)}(p_i, \lambda, \mu) = -i\mu^2\alpha\, \Gamma_{\phi^2 R}^{(n)}(0, p_i, \lambda, \mu)$$

where

$$\beta = 2\mu^2 \frac{(\partial\lambda/\partial\mu_0^2)}{(\partial\mu^2/\partial\mu_0^2)} \quad , \quad \gamma = \mu^2 \frac{\partial \ln Z_\phi/\partial\mu_0^2}{\partial\mu^2/\partial\mu_0^2} \qquad \alpha = \frac{\partial Z_{\phi^2}/\partial\mu_0^2}{\partial\mu^2/\partial\mu_0^2}$$

<u>Weinberg's theorem</u> : (simplified version)

If we write the external momenta as $p_i = \sigma k_i$, and take $\sigma \to \infty$ limit,

$$\Gamma_R^{(n)} \sim \sigma^{4-n} \quad , \quad \Gamma_{\phi^2 R}^{(n)} \sim \sigma^{2-n}$$

So in the large momenta region, we can neglect $\Gamma_{\phi^2 R}^{(n)}$,

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} - n\gamma(\lambda)\right] \Gamma_{as}^{(n)}(p_i, \lambda, \mu) = 0$$

Define a dimensionless quantity $\bar{\Gamma}$ by

$$\Gamma_{as}^{(n)}(p_i, \lambda, \mu) = \mu^{4-n}\, \bar{\Gamma}_R^{(n)}(p_i/\mu, \lambda)$$

Since $\bar{\Gamma}$ is dimensionless, as we scale up the momenta we can write

$$\left(\mu\frac{\partial}{\partial\mu} + \sigma\frac{\partial}{\partial\sigma}\right) \bar{\Gamma}_R^{(n)}(\sigma p_i/\mu, \lambda) = 0$$

and $$\left[\mu\frac{\partial}{\partial\mu} + \sigma\frac{\partial}{\partial\sigma} + (n-4)\right] \Gamma_{as}^{(n)}(\sigma p_i/\mu, \lambda) = 0$$

From Callan-Symanzik equation we get

$$\left[\sigma\frac{\partial}{\partial\sigma} - \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda) + (n-4)\right] \Gamma_{as}^{(n)}(\sigma p_i, \lambda, \mu) = 0$$

To solve this equation, we remove the non-derivative terms by the transformation

$$\Gamma_{as}^{(n)}(\sigma p_i, \lambda, \mu) = \sigma^{4-n} \exp\left[n\int_0^\lambda \frac{\gamma(x)}{\beta(x)}dx\right] F^{(n)}(\sigma p_i, \lambda, \mu)$$

Then $F^{(n)}$ satisfies the equation

$$\left[\sigma\frac{\partial}{\partial\sigma} - \beta(\lambda)\frac{\partial}{\partial\lambda}\right] F^{(n)}(\sigma p_i, \lambda, \mu) = 0$$

or $$\left[\frac{\partial}{\partial t} - \beta(\lambda)\frac{\partial}{\partial\lambda}\right] F^{(n)}(e^t p_i, \lambda, \mu) = 0 \qquad \text{where } t = \ln\sigma$$

Introduce the effective, or running constant $\bar{\lambda}$ as solution to the equation

$$\frac{d\bar{\lambda}(t,\lambda)}{dt} = \beta(\bar{\lambda}) \qquad \text{with initial condition } \bar{\lambda}(0,\lambda) = \lambda.$$

This equation has the solution

$$t = \int_\lambda^{\bar{\lambda}(t,\lambda)} \frac{dx}{\beta(x)}$$

It is straightforward to show that

$$\frac{1}{\beta(\bar\lambda)}\frac{d\bar\lambda}{d\lambda} = \beta(\lambda) \qquad \text{and} \qquad \left[\frac{\partial}{\partial t} - \beta(\lambda)\frac{\partial}{\partial\lambda}\right]\bar\lambda(t,\lambda) = 0$$

In other words, $F^{(n)}$ depends on t and λ only through the combination $\bar\lambda(t,\lambda)$

$$F^{(n)} = F^{(n)}(p_i, \bar\lambda(t,\lambda), \mu)$$

Also

$$\exp\left[n\int_0^\lambda \frac{\gamma(\lambda)}{\beta(\lambda)}d\lambda\right] \sim \exp\left[n\int_0^{\bar\lambda}\frac{\gamma(x)}{\beta(x)}dx + n\int_{\bar\lambda}^{\lambda}\frac{\gamma(x)}{\beta(x)}dx\right] = H(\bar\lambda)\exp\left[-n\int_\lambda^{\bar\lambda}\frac{\gamma(x)}{\beta(x)}dx\right]$$

where $\quad H(\bar\lambda) = \exp\left[n\int_0^{\bar\lambda}\frac{\gamma(x)}{\beta(x)}dx\right]$

The solution is then

$$\Gamma_{as}^{(n)}(\sigma p_i,\lambda,\mu) = \sigma^{4-n}\exp\left[-n\int_0^t \gamma(\bar\lambda(x',\lambda))dx'\right] H(\bar\lambda) F^{(n)}(p_i,\bar\lambda(t,\lambda),\mu)$$

If we set $t=0$ (or $\sigma=1$), we see that

$$\Gamma_{as}^{(n)}(p_i,\lambda,\mu) = H(\lambda)F^{(n)}(p_i,\lambda,\mu)$$

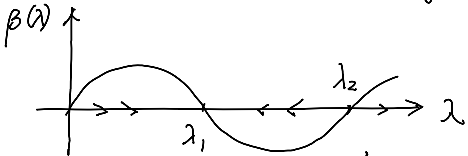
Thus the solution has the simple form

$$\Gamma_{as}^{(n)}(\sigma p_i,\lambda,\mu) = \sigma^{4-n}\exp\left[-n\int_0^t \gamma(\bar\lambda(x',\lambda))dx'\right]\Gamma_{as}^{(n)}(p_i,\bar\lambda(t,\lambda),\mu)$$

Effective coupling constant $\bar\lambda$

$$\frac{d\bar\lambda(t,\lambda)}{dt} = \beta(\bar\lambda) \qquad \text{initial condition } \bar\lambda(0,\lambda)=\lambda$$

Suppose $\beta(\lambda)$ has the following simple behavior



Suppose $0<\lambda<\lambda_1$, then at $t=0$, $\left.\frac{d\bar\lambda}{dt}\right|_{t=0} > 0 \Rightarrow \bar\lambda$ increases as t increases

This increase will continue until $\bar\lambda$ reaches λ_1 where $\frac{d\bar\lambda}{dt}=0$.

On the other hand, if initially $\lambda_1<\lambda<\lambda_2$, then $\left.\frac{d\bar\lambda}{dt}\right|_{t=0}<0$, $\bar\lambda$ will decrease until it reaches λ_1. Thus as $t\to\infty$, we get

$$\lim_{t\to\infty}\bar\lambda(t,\lambda) = \lambda_1 \qquad \lambda_1: \text{ultraviolet stable fixed point}$$

and $\quad \Gamma_{as}^{(n)}(p_i,\bar\lambda(t,\lambda),\mu) \xrightarrow[t\to\infty]{} \Gamma_{as}^{(n)}(p_i,\lambda_1,\mu)$

Example: Suppose $\beta(x)$ has a simple zero at $\lambda=\lambda_1$,

$$\beta(\lambda) \simeq a(\lambda_1-\lambda) \qquad a>0$$

then $\quad \frac{d\bar\lambda}{dt} = a(\lambda_1-\bar\lambda) \implies \bar\lambda = \lambda_1 + (\lambda-\lambda_1)e^{-at}$

... approach to fixed point is exponential in t, or power in $\sigma = \ln t$

dt

i.e. the approach to fixed point is exponential in t, or power in $\sigma = \ln t$

Also the prefactor can be simplified,

$$\int_0^t \gamma(\bar\lambda(x,\lambda))\,dx = \int_\lambda^{\bar\lambda} \frac{\gamma(y)dy}{\beta(y)} \approx \frac{-\gamma(\lambda_1)}{a}\int_\lambda^{\bar\lambda}\frac{d\lambda'}{\lambda'-\lambda_1} = \frac{-\gamma(\lambda_1)}{a}\ln\left(\frac{\bar\lambda-\lambda_1}{\lambda-\lambda_1}\right)$$

$$= \gamma(\lambda_1)t = \gamma(\lambda_1)\ln\sigma$$

$$\lim_{\sigma\to\infty}\Gamma_{as}^{(n)}(\sigma p_i,\lambda,\mu) = \sigma^{4-n[1+\gamma(\lambda_1)]}\,\Gamma_{as}^{(n)}(p_i,\lambda_1,\mu)$$

Thus the asymptotic behavior in field theory is controlled by the fixed point λ_1 and $\gamma(\lambda_1)$: anormalous dimension