divergent, $\Gamma_a^{(4)}$ cannot be made convergent no matter how many derivatives operate on it, even though the overall superficial degree of divergence is zero. However we have the lower-order counterterm $-\lambda^2\Gamma(0)$ corresponding to the subtraction introduced at the one-loop level. This generates the additional λ^3 contributions of Fig. 2.12(b), (c) with $\Gamma_b^{(4)} \propto -\lambda^3\Gamma(p)\Gamma(0)$ and $\Gamma_c^{(4)} \propto -\lambda^3\Gamma(0)\Gamma(p)$, respectively.

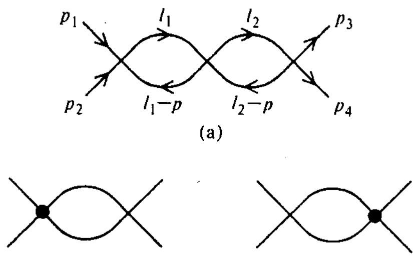

FIG. 2.12. s-channel λ^3 four-point functions. The black spots represent the counterterm $-\lambda^2\Gamma(0)$.

Adding the three graphs, Fig. 2.12(a), (b), (c), we have

$$\begin{aligned}\Gamma^{(4)}(p) &= \Gamma_a^{(4)} + \Gamma_b^{(4)} + \Gamma_c^{(4)} \\ &= -\lambda^3[\Gamma(0)]^2 + \lambda^3[\Gamma(p) - \Gamma(0)]^2 \\ &= \Gamma^{(4)}(0) + \tilde{\Gamma}^{(4)}(p).\end{aligned} \tag{2.68}$$

Only the first term on the right-hand side is divergent and can be removed by a λ^3 counterterm of the form $i\Gamma^{(4)}(0)\phi^4/4!$. We see how, with the inclusion of the lower-order counterterms, divergences take on the form of polynomials in the external momenta. Thus for diagrams with more than one loop it is useful to characterize a divergent contribution as being primitively divergent or not. A *primitively divergent* graph has a non-negative overall superficial degree of divergence but is convergent for all subintegrations. Thus, they are diagrams in which the *only* divergence is caused by *all* of the loop momenta growing large together. In general only primitively divergent graphs such as Fig. 2.13 can have their divergences isolated by direct Taylor-series expansion. For other cases, diagrams with lower-order counterterm insertions must be included in order to aggregate the divergences into the form of polynomials in the external momenta.

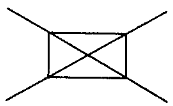

FIG. 2.13. A primitively divergent four-point function.

(3) In the above example of a two-loop, four-point function we have seen how the overall divergence can be isolated when diagrams with lower-order counterterms are included. For such cases where the divergent subintegrations are *disjoint* this can be accomplished in a fairly direct manner. Similarly, it is also relatively easy for cases with *nested divergences*, i.e. for cases where one of each pair of divergent 1PI subgraphs is entirely contained within the other (see the example in Fig. 2.14). After the subgraph divergence

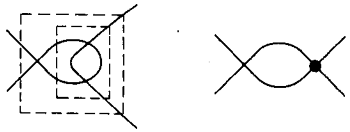

FIG. 2.14. Nested divergences and a diagram with a lower-order counterterm which cancels the subintegration divergence.

is removed by diagrams with lower-order counterterms (Fig. 2.14(b)), the overall divergence is then renormalized by a λ^3 counterterm. Thus for both disjoint and nested divergences the renormalization procedure is rather straightforward. The difficult step in the proof of the convergence (to all orders) involves disentangling the *overlapping divergences*, which are neither disjoint nor nested divergent 1PI diagrams. Fig. 2.2(a) is an example of overlapping divergence. Here it is difficult to see in a simple way how the subintegration divergences can be removed in a systematic fashion because they do not factorize in a simple manner. Nevertheless, this problem has been overcome and we refer the interested reader to the literature (Hepp 1966; Zimmermann 1970; Itzykson and Zuber 1980). The purpose of these comments is to indicate how the proof of renormalizability generally involves complicated graph classifications and combinatorial analysis.

2.3 Regularization schemes

In this section we will give detailed calculations of the various renormalization constants in the renormalized perturbation theory described in the previous sections. To make any meaningful mathematical manipulations on the divergent integrals we must cut off, or regularize, the momentum integration to make the integral finite. The divergent part will then be a function of the cut-off Λ while the finite part will be cut-off-independent in the limit $\Lambda \to \infty$. The cut-off procedure must be chosen in such a way that it maintains the Lorentz invariance and symmetry of the problem. There are two commonly used regularization schemes: the covariant cut-off and dimensional regularization. We shall illustrate them in turn.

Covariant regularization

In this procedure (Pauli and Villars 1949) the propagator will be modified as

$$\frac{1}{l^2 - \mu^2 + i\varepsilon} \rightarrow \frac{1}{l^2 - \mu^2 + i\varepsilon} + \sum_i \frac{a_i}{l^2 - \Lambda_i^2 + i\varepsilon} \tag{2.69}$$

where $\Lambda_i^2 \gg \mu^2$ and the a_is are chosen in such a way that in the asymptotic region the modified propagator will have a sufficient number of internal momenta in the denominator so that the integral is convergent.

Let us start with the four-point function. The graph in Fig. 2.5(a) yields a contribution (2.7)

$$\Gamma_a = \Gamma(p^2) = \frac{(-i\lambda)^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{i}{(l-p)^2-\mu^2+i\varepsilon}\frac{i}{l^2-\mu^2+i\varepsilon}. \tag{2.70}$$

Clearly the replacement

$$\frac{1}{l^2-\mu^2+i\varepsilon}\to\frac{1}{l^2-\mu^2+i\varepsilon}-\frac{1}{l^2-\Lambda^2+i\varepsilon}=\frac{\mu^2-\Lambda^2}{(l^2-\mu^2+i\varepsilon)(l^2-\Lambda^2+i\varepsilon)}$$

will be sufficient to render the integral finite. Eqn (2.70) then becomes

$$\Gamma(p^2) = \frac{-\lambda^2\Lambda^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{1}{((l-p)^2-\mu^2+i\varepsilon)(l^2-\mu^2+i\varepsilon)(l^2-\Lambda^2+i\varepsilon)}. \tag{2.71}$$

We choose to make the Taylor expansion around $p^2 = 0$ (or to make subtraction at $p^2 = 0$),

$$\Gamma(p^2) = \Gamma(0) + \tilde{\Gamma}(p^2) \tag{2.72}$$

with

$$\Gamma(0) = \frac{-\lambda^2\Lambda^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\mu^2+i\varepsilon)^2(l^2-\Lambda^2+i\varepsilon)} \tag{2.73}$$

$$\tilde{\Gamma}(p^2) = \frac{-\lambda^2\Lambda^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\mu^2+i\varepsilon)(l^2-\Lambda^2+i\varepsilon)}\times\left[\frac{1}{(l-p)^2-\mu^2+i\varepsilon}-\frac{1}{l^2-\mu^2+i\varepsilon}\right]$$

$$= \frac{\lambda^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{2l\cdot p-p^2}{(l^2-\mu^2+i\varepsilon)^2((l-p)^2-\mu^2+i\varepsilon)} \tag{2.74}$$

where in the last line we have taken the limit $\Lambda\to\infty$ inside the integral because $\tilde{\Gamma}(p^2)$ is convergent. The standard method to evaluate these integrals is to first use the identity to combine the denominator factors

$$\frac{1}{a_1a_2\ldots a_n} = (n-1)!\int_0^1\frac{dz_1\,dz_2\ldots dz_n}{(a_1z_1+a_2z_2+\ldots a_nz_n)^n}\,\delta\left(1-\sum_{i=1}^n z_i\right) \tag{2.75}$$

where the z_is are called the *Feynman parameters*. We can also differentiate with respect to a_1 to get

$$\frac{1}{a_1^2a_2\ldots a_n} = n!\int_0^1\frac{z_1\,dz_1\,dz_2\ldots dz_n}{(a_1z_1+a_2z_2+\ldots a_nz_n)^{n+1}}\,\delta\left(1-\sum_i z_i\right). \tag{2.76}$$

This formula has the advantage that one less Feynman parameter is needed for the case where there are two identical factors in the denominator. Using (2.76), we can combine the denominators in (2.74) to give

$$\frac{1}{(l^2-\mu^2+i\varepsilon)^2}\frac{1}{(l-p)^2-\mu^2+i\varepsilon} = 2\int_0^1\frac{(1-\alpha)\,d\alpha}{A^3} \tag{2.77}$$

where

$$A = (1-\alpha)(l^2-\mu^2)+\alpha[(l-p)^2-\mu^2]+i\varepsilon$$
$$= (l-\alpha p)^2 - a^2 + i\varepsilon$$

with

$$a^2 = \mu^2 - \alpha(1-\alpha)p^2.$$

Thus,

$$\tilde{\Gamma}(p^2) = \lambda^2\int_0^1(1-\alpha)\,d\alpha\int\frac{d^4l}{(2\pi)^4}\frac{2l\cdot p-p^2}{[(l-\alpha p)^2-a^2+i\varepsilon]^3}$$

$$= \lambda^2\int_0^1(1-\alpha)\,d\alpha\int\frac{d^4l}{(2\pi)^4}\frac{(2\alpha-1)p^2}{(l^2-a^2+i\varepsilon)^3} \tag{2.78}$$

where we have changed the variable l to $l+\alpha p$ and have dropped the term linear in l which will vanish upon symmetric integration. It is more convenient to do the integration by the *Wick rotation*, which transforms the Minkowski momentum to the Euclidean momentum. First we note that $d^4l = dl_0\,dl_1\,dl_2\,dl_3$ and

$$l^2 - a^2 + i\varepsilon = l_0^2 - \mathbf{l}^2 - a^2 + i\varepsilon$$
$$= l_0^2 - [(\mathbf{l}^2+a^2)^{1/2} - i\varepsilon]^2.$$

This shows that the integral (2.78) has poles in the complex l_0-plane as shown in Fig. 2.15.

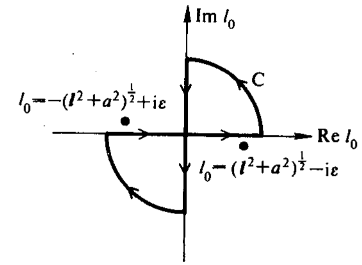

FIG. 2.15.

Using Cauchy's theorem we then have

$$\oint_C dl_0\, f(l_0) = 0 \tag{2.79}$$

where

$$f(l_0) = \frac{1}{[l_0^2 - ((\mathbf{l}^2 + a^2)^{1/2} - i\varepsilon)^2]^3}.$$

Since $f(l_0) \to l_0^{-6}$ as $l_0 \to \infty$, the contribution from the circular part of contour C vanishes. Eqn. (2.79) implies that

$$\int_{-\infty}^{\infty} dl_0 f(l_0) = \int_{-i\infty}^{+i\infty} dl_0 f(l_0).$$

Thus, the integration along the real axis has been rotated to that along the imaginary axis. Change the variable $l_0 = il_4$ so that l_4 is real and

$$\int_{-i\infty}^{+i\infty} dl_0 f(l_0) = i\int_{-\infty}^{\infty} dl_4 f(il_4)$$

$$= -i\int_{-\infty}^{\infty} \frac{dl_4}{(l_1^2 + l_2^2 + l_3^2 + l_4^2 + a^2 - i\varepsilon)^3}. \tag{2.80}$$

If we define Euclidean momentum $k_i = (l_1, l_2, l_3, l_4)$ with $k^2 = l_1^2 + l_2^2 + l_3^2 + l_4^2$, then the results in eqns (2.79) and (2.80) may be written

$$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - a^2 + i\varepsilon)^3} = -i\int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + a^2 - i\varepsilon)^3} \tag{2.81}$$

where $d^4k = dl_1\, dl_2\, dl_3\, dl_4$. Using polar coordinates in four-dimensional Euclidean space, we have

$$\int d^4 k = \int_0^\infty k^3\, dk \int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta \int_0^\pi \sin^2\chi\, d\chi \tag{2.82}$$

and

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + a^2 - i\varepsilon)^3} = 2\pi^2 \int_0^\infty \frac{k^3\, dk}{(2\pi)^4} \frac{1}{(k^2 + a^2 - i\varepsilon)^3}$$

$$= \frac{1}{16\pi^2} \int_0^\infty \frac{k^2\, dk^2}{(k^2 + a^2 - i\varepsilon)^3}. \tag{2.83}$$

Using the formula for beta functions

$$\int_0^\infty \frac{t^{m-1}\, dt}{(t + a^2)^n} = \frac{1}{(a^2)^{n-m}} \frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)}, \tag{2.84}$$

we obtain

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + a^2 - i\varepsilon)^3} = \frac{1}{32\pi^2(a^2 - i\varepsilon)} \tag{2.85}$$

or the vertex function in eqn (2.78) becomes

$$\tilde{\Gamma}(p^2) = \frac{-i\lambda^2}{32\pi^2} \int_0^1 \frac{d\alpha(1-\alpha)(2\alpha - 1)p^2}{[\mu^2 - \alpha(1-\alpha)p^2 - i\varepsilon]}. \tag{2.86}$$

Since $0 < \alpha < 1$ we get $\mu^2 - \alpha(1-\alpha)p^2 > 0$ for $p^2 < 4\mu^2$ and we can drop $i\varepsilon$ in the denominator. It is straightforward to evaluate the integral to give

$$\tilde{\Gamma}(p^2) = \tilde{\Gamma}(s) = \frac{i\lambda^2}{32\pi^2}\left\{2 + \left(\frac{4\mu^2 - s}{|s|}\right)^{\frac{1}{2}} \ln[\{(4\mu^2 - s)^{\frac{1}{2}} - (|s|)^{\frac{1}{2}}\}/\{(4\mu^2 - s)^{\frac{1}{2}} + (|s|)^{\frac{1}{2}}\}]\right\} \quad \text{for} \quad s < 0$$

$$= \frac{i\lambda^2}{32\pi^2}\left\{2 + 2\left(\frac{4\mu^2 - s}{s}\right)^{\frac{1}{2}} \tan^{-1}\left(\frac{s}{4\mu^2 - s}\right)^{\frac{1}{2}}\right\} \quad \text{for} \quad 0 < s < 4\mu^2$$

$$= \frac{i\lambda^2}{32\pi^2}\left\{2 + \left(\frac{s - 4\mu^2}{s}\right)^{\frac{1}{2}} \ln\left[\frac{s^{\frac{1}{2}} - (s - 4\mu^2)^{\frac{1}{2}}}{s^{\frac{1}{2}} + (s - 4\mu^2)^{\frac{1}{2}}}\right] + i\pi\right\}$$

$$\text{for} \quad s > 4\mu^2. \tag{2.87}$$

With the same procedure, the divergent term $\Gamma(0)$ given in eqn (2.73) can be calculated

$$\Gamma(0) = \frac{i\lambda^2\Lambda^2}{32\pi^2} \int_0^1 \frac{\alpha\, d\alpha}{\alpha(\mu^2 - \Lambda^2) + \Lambda^2}. \tag{2.88}$$

For large Λ^2, this gives

$$\Gamma(0) \simeq \frac{i\lambda^2}{32\pi^2} \ln\frac{\Lambda^2}{\mu^2}. \tag{2.89}$$

Thus the one-loop contribution to the four-point function is

$$\Gamma^{(4)}_{\text{1-loop}}(s, t, u) = 3\Gamma(0) + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \tag{2.90}$$

where the cut-off-dependent $\Gamma(0)$ is given by eqn (2.89) and the finite $\tilde{\Gamma}(s)$ is given by eqn (2.87). We have to add the counterterm $(3i\Gamma(0)/4!)\phi^4$ to cancel these divergences. By (2.36) this corresponds to the renormalization constant

$$Z_\lambda^{-1} = 1 + \frac{3i\Gamma(0)}{\lambda} = 1 - \frac{3\lambda}{32\pi^2} \ln\frac{\Lambda^2}{\mu^2}. \tag{2.91}$$

Having cancelled the divergences, the total four-point function up to this order is then given by (2.42)

$$\Gamma^{(4)}_{R}(s, t, u) = -i\lambda + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u). \tag{2.92}$$

For the two-point function of eqn (2.6), corresponding to the graph in Fig. 2.4, we have

$$-i\Sigma(p^2) = \frac{-i\lambda}{2} \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2 - \mu^2 + i\varepsilon}. \tag{2.93}$$

This is a quadratically divergent integral and it can be regularized by choosing a_1 and a_2 in eqn (2.69) such that

$$\frac{1}{l^2-\mu^2+i\varepsilon}+\frac{a_1}{l^2-\Lambda_1^2+i\varepsilon}+\frac{a_2}{l^2-\Lambda_2^2+i\varepsilon}\to\frac{1}{l^6}\quad\text{as}\quad l^2\to\infty.$$

It is not difficult to see that we need

$$a_1=\frac{\mu^2-\Lambda_2^2}{\Lambda_2^2-\Lambda_1^2}\quad\text{and}\quad a_2=\frac{\Lambda_1^2-\mu^2}{\Lambda_2^2-\Lambda_1^2}.$$

Then the modified propagator becomes

$$\frac{1}{l^2-\mu^2+i\varepsilon}+\frac{a_1}{l^2-\Lambda_1^2+i\varepsilon}+\frac{a_2}{l^2-\Lambda_2^2+i\varepsilon}$$
$$=\frac{(\Lambda_1^2-\mu^2)(\Lambda_2^2-\mu^2)}{(l^2-\mu^2)(l^2-\Lambda_1^2)(l^2-\Lambda_2^2)}\to\frac{\Lambda^4}{(l^2-\mu^2)(l^2-\Lambda^2)^2}$$

for Λ_1 and Λ_2 both approach a large Λ. The regularized self-energy is

$$-i\Sigma(p^2)=\frac{\lambda}{2}\int\frac{d^4l}{(2\pi)^4}\frac{\Lambda^4}{(l^2-\mu^2+i\varepsilon)(l^2-\Lambda^2+i\varepsilon)^2}$$
$$=\frac{-i\lambda\Lambda^4}{32\pi^2}\int_0^1\frac{\alpha\,d\alpha}{\alpha\Lambda^2+(1-\alpha)\mu^2}$$
$$=\frac{-i\lambda}{32\pi^2}\left[\Lambda^2-\mu^2\ln\frac{\Lambda^2}{\mu^2}\right]. \tag{2.94}$$

Since it is independent of the external momentum p, the Taylor expansion is trivial,

$$\Sigma(p^2)=\Sigma(0)\simeq\frac{\lambda}{32\pi^2}\Lambda^2. \tag{2.95}$$

As we have mentioned before, this p-independence is a special property of the one-loop approximation in $\lambda\phi^4$ theory. For a more general self-energy graph, $\Sigma(p)$ will have a nontrivial dependence on p and the Taylor series around $p^2=0$ will be

$$\Sigma(p^2)=\Sigma(0)+p^2\Sigma'(0)+\tilde{\Sigma}(p^2) \tag{2.96}$$

where $\Sigma(0)$ and $\Sigma'(0)$ are cut-off-dependent and $\tilde{\Sigma}(p^2)$ is finite. And we have to add $\frac{1}{2}\Sigma(0)\phi^2$ and $\frac{1}{2}\Sigma'(0)(\partial_\mu\phi)^2$ counterterms to cancel these divergences.

To summarize, the total Lagrangian up to one loop has the form

$$\mathscr{L}^{(1)}=\mathscr{L}^{(0)}+\Delta\mathscr{L}^{(1)} \tag{2.97}$$

where

$$\mathscr{L}^{(0)}=\frac{1}{2}[(\partial_\mu\phi)^2-\mu^2\phi^2]-\frac{\lambda}{4!}\phi^4$$
$$\Delta\mathscr{L}^{(1)}=\frac{3i\Gamma(0)}{4!}\phi^4+\frac{1}{2}\Sigma(0)\phi^2+\frac{1}{2}\Sigma'(0)(\partial_\mu\phi)^2.$$

Combining terms of the same structure, we can write (2.97) as

$$\mathscr{L}^{(1)}=\frac{Z_\phi}{2}(\partial_\mu\phi)^2-\frac{(\mu^2-\delta\mu^2)\phi^2}{2}-\frac{\lambda Z_\lambda}{4!}\phi^4 \tag{2.98}$$

with

$$Z_\phi=1+\Sigma'(0),$$
$$\lambda Z_\lambda^{-1}=\lambda+3i\Gamma(0),$$
$$\delta\mu^2=\Sigma(0). \tag{2.99}$$

The values of these renormalization constants in the one-loop approximation are

$$Z_\phi=1\quad\text{since}\quad\Sigma'(0)=0,$$
$$Z_\lambda=1+\frac{3\lambda}{32\pi^2}\ln\frac{\Lambda^2}{\mu^2},$$
$$\delta\mu^2=\frac{\lambda}{32\pi^2}\Lambda^2. \tag{2.100}$$

If we express everything in terms of the bare quantities through eqns (2.49), (2.50), and (2.51), we find

$$\mathscr{L}^{(1)}=\frac{1}{2}[(\partial_\mu\phi_0)^2-\mu_0^2\phi_0^2]-\frac{\lambda_0}{4!}\phi_0^4 \tag{2.101}$$

which is exactly the same as the unrenormalized Lagrangian (2.1) as it should be.

Finally we comment on the convention used in making the Taylor series expansions (2.72) and (2.96) around $p_i=0$ to fix the finite part of the Green's function. An equivalent way to state the same convention is to specify the normalization conditions of Green's function. From (2.96), the finite part of the self-energy has the properties

$$\tilde{\Sigma}(p^2)|_{p^2=0}=0 \tag{2.102}$$

and

$$\left.\frac{\partial\tilde{\Sigma}(p^2)}{\partial p^2}\right|_{p^2=0}=0. \tag{2.103}$$

These properties imply that the full propagator

$$i\Delta_R(p^2)=\frac{i}{p^2-\mu^2-\tilde{\Sigma}(p^2)+i\varepsilon} \tag{2.104}$$

will satisfy the normalization conditions

$$\Delta_R^{-1}(p^2)|_{p^2=0}=-\mu^2 \tag{2.105}$$

and

$$\left.\frac{\partial\Delta_R^{-1}}{\partial p^2}\right|_{p^2=0}=1. \tag{2.106}$$

Similarly from (2.72) and thus from $\tilde{\Gamma}(0)=0$, we have from (2.92) the normalization condition for the vertex function

$$\Gamma_R^{(4)}(0, 0, 0) = -i\lambda. \tag{2.107}$$

(*Remark:* Although (2.104) was originally derived with a Taylor expansion of $\Sigma(p^2)$ around $p^2 = \mu^2$ it also holds for the present $p^2 = 0$ expansion as a derivation entirely similar to eqns (2.14)–(2.22) will show.)

In short, one can use conditions (2.105), (2.106), and (2.107) to replace the prescription 'Taylor expansion around $p_i = 0$' to fix the finite part of Green's function.

In this connection we observe that the renormalized coupling constant defined by (2.107) differs from that defined by eqn (2.41) where a Taylor expansion has been made around the symmetric point $s_0 = t_0 = u_0 = 4\mu^2/3$. It implies condition (2.33)

$$\Gamma_R^{(4)}(s_0, t_0, u_0) = -i\lambda \tag{2.108}$$

to be contrasted with (2.107). Thus, different Taylor expansions or subtraction points yield different definitions of the coupling constant. This leads to the concept of a running coupling constant (see Chapter 3). Clearly the physics should not depend on the choice of subtraction point which is purely a convention. In practice how is this apparent difference taken care of? Consider the two-body scattering cross-sections calculated using two different definitions of the coupling constant. The calculated cross-sections may appear to be different by an overall constant (the angular distributions are identical). But this is immaterial because we need to define the coupling constant operationally as the value of the cross-section at some kinematical point. Thus the difference is only apparent and the two seemingly different calculations really yield the same result.

Dimensional regularization

The basic idea of this scheme ('t Hooft and Veltman 1972; Bollini and Giambiagi 1972; Ashmore 1972; Cicuta and Montaldi 1972) is that, since the ultraviolet divergences in Feynman diagrams come from the integration of internal momenta in four-dimensional space, the integrals can be made finite by lowering the dimensionalities of the space–time. Then the Feynman integrals can be defined as analytic functions of the space–time dimension n. The ultraviolet divergences will manifest themselves as singularities as $n \to 4$. As before, the finite part can be obtained by subtracting out the first few terms in the Taylor expansion. This regularization scheme has the important advantage that it will not destroy any algebraic relations among Green's functions that do not depend on space–time dimensions. In particular, the Ward identities, which are relations among Green's functions resulting from the symmetries of the theory, can be maintained in this dimensional regularization scheme. For a review see Leibrandt (1975).

We will illustrate this method with an example. Consider the one-loop four-point Green's function in eqn (2.7) corresponding to the diagram in Fig. 2.5(a). It is proportional to the integral

$$I = \int d^4 l \frac{1}{(l-p)^2 - \mu^2 + i\varepsilon} \frac{1}{l^2 - \mu^2 + i\varepsilon} \tag{2.109}$$

which is logarithmically divergent. To define the integral in n-dimensional space, we take the internal momentum to have n components: $l_\mu = (l_0, l_1, \ldots, l_{n-1})$, while the external momentum has four nonvanishing components: $p_\mu = (p_0, p_1, p_2, p_3, 0 \ldots 0)$. The integral in n-dimensional space is then defined as

$$I(n) = \int d^n l \frac{1}{(l-p)^2 - \mu^2 + i\varepsilon} \frac{1}{l^2 - \mu^2 + i\varepsilon} \tag{2.110}$$

which is convergent for $n < 4$. To define this integral for non-integer values of n, we first combine the denominators using Feynman parameters and make the Wick rotation (eqn (2.75)),

$$I(n) = \int_0^1 d\alpha \int \frac{d^n l}{[(l - \alpha p)^2 - a^2 + i\varepsilon]^2}$$

$$= i \int_0^1 d\alpha \int \frac{d^n l}{[l^2 + a^2 - i\varepsilon]^2} \tag{2.111}$$

with $a^2 = \mu^2 - \alpha(1-\alpha)p^2$.

The integrand is now independent of the angles of the integration momentum, which can then be integrated out

$$\int d^n l = \int_0^\infty l^{n-1}\, dl \int_0^{2\pi} d\theta_1 \int_0^\pi \sin\theta_2\, d\theta_2 \int_0^\pi \sin^2\theta_3\, d\theta_3 \ldots$$

$$\times \int_0^\pi \sin^{n-2}\theta_{n-1}\, d\theta_{n-1}$$

$$= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty l^{n-1}\, dl \tag{2.112}$$

where we have used the formula

$$\int_0^\pi \sin^m\theta\, d\theta = \frac{\pi^{1/2}\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}. \tag{2.113}$$

Thus eqn (2.111) may be written

$$I(n) = \frac{2i\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 d\alpha \int_0^\infty \frac{l^{n-1}\, dl}{[l^2 + a^2 - i\varepsilon]^2}. \tag{2.114}$$

The dependence on n is now explicit. For complex n, the integral is well-defined as long as $0 < \mathrm{Re}(n) < 4$; the lower bound results from the apparent divergence of the integral at the $l = 0$ limit. This infrared divergence is actually an artefact of our procedure as it is cancelled by the singularity in $\Gamma(\frac{1}{2}n)$ as $n \to 0$. We can extend this domain of analyticity by integration by parts

$$\frac{1}{\Gamma\left(\frac{n}{2}\right)}\int_0^\infty \frac{l^{n-1}\,\mathrm{d}l}{[l^2+a^2-\mathrm{i}\varepsilon]^2} = \frac{-2}{\Gamma\left(\frac{n}{2}+1\right)}\int_0^\infty \mathrm{d}l\,\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{1}{[l^2+a^2-\mathrm{i}\varepsilon]^2}\right) \tag{2.115}$$

where we have used

$$z\Gamma(z) = \Gamma(z+1). \tag{2.116}$$

The integral is now well defined for $-2 < \mathrm{Re}(n) < 4$. If we repeat this procedure v times, the analyticity domain is extended to $-2v < \mathrm{Re}(n) < 4$ and eventually to $\mathrm{Re}(n) \to -\infty$. Thus the integral given in eqn (2.114) can be taken as an analytic function for $\mathrm{Re}(n) < 4$. To see what happens as $n \to 4$, we use (2.84) to evaluate the integral for $n < 4$,

$$I(n) = \mathrm{i}\pi^{n/2}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 \frac{\mathrm{d}\alpha}{[a^2-\mathrm{i}\varepsilon]^{2-n/2}}. \tag{2.117}$$

Using formula (2.116)

$$\Gamma\left(2-\frac{n}{2}\right) = \frac{\Gamma\left(3-\frac{n}{2}\right)}{2-\frac{n}{2}} \to \frac{2}{4-n} \quad \text{as} \quad n \to 4,$$

we see that the singularity at $n = 4$ is a simple pole. If we now expand everything around $n = 4$

$$\Gamma\left(2-\frac{n}{2}\right) = \frac{2}{4-n} + A + (n-4)B + \ldots \tag{2.118}$$

$$a^{n-4} = 1 + (n-4)\ln a + \ldots, \tag{2.119}$$

where A and B are some constants, we obtain the limit

$$I(n) \underset{n\to 4}{\to} \frac{2\mathrm{i}\pi^2}{4-n} - \mathrm{i}\pi^2\int_0^1 \mathrm{d}\alpha\,\ln[\mu^2-\alpha(1-\alpha)p^2] + \mathrm{i}\pi^2 A. \tag{2.120}$$

With the one-loop contribution of (2.7) ($\Gamma = \lambda^2 I/32\pi^4$), we have

$$\Gamma(p^2) = \frac{\lambda^2}{32\pi^2}\left\{\frac{2\mathrm{i}}{4-n} - \mathrm{i}\int_0^1 \mathrm{d}\alpha\,\ln[\mu^2-\alpha(1-\alpha)p^2] + \mathrm{i}A\right\}. \tag{2.121}$$

The Taylor expansion around $p^2 = 0$ gives

$$\Gamma(p^2) = \Gamma(0) + \tilde{\Gamma}(p^2) \tag{2.122}$$

where

$$\Gamma(0) = \frac{\lambda^2}{32\pi^2}\left(\frac{2\mathrm{i}}{4-n} - \mathrm{i}\ln\mu^2 + \mathrm{i}A\right) \simeq \frac{\mathrm{i}\lambda^2}{16\pi^2(4-n)} \tag{2.123}$$

and

$$\tilde{\Gamma}(p^2) = \frac{-\mathrm{i}\lambda^2}{32\pi^2}\int_0^1 \mathrm{d}\alpha\,\ln\left[\frac{\mu^2-\alpha(1-\alpha)p^2}{\mu^2}\right] = \frac{-\mathrm{i}\lambda^2}{32\pi^2}\int_0^1 \frac{\mathrm{d}\alpha(1-\alpha)(2\alpha-1)p^2}{[\mu^2-\alpha(1-\alpha)p^2]} \tag{2.124}$$

where we have performed an integration by parts. Clearly the finite part is exactly the same as that given by the method of covariant regularization in eqn (2.86). Thus the finite part of Green's function is independent of the regularization schemes as it should be and only depends on the subtraction point. The $\Gamma(0)$ term diverges as a simple pole at $n = 4$ corresponding to the $\ln \Lambda$ term (2.89) in the covariant regularization calculation.

The one-loop self-energy (Fig. 2.4) is given by eqn (2.6) which in the dimensional-regularization scheme becomes

$$-\mathrm{i}\Sigma(p^2) = \frac{\lambda}{2}\int \frac{\mathrm{d}^n l}{(2\pi)^4}\frac{1}{l^2-\mu^2+\mathrm{i}\varepsilon} = \frac{-\mathrm{i}\lambda\pi^{n/2}\Gamma\left(1-\frac{n}{2}\right)}{32\pi^4(\mu^2)^{1-n/2}}. \tag{2.125}$$

Since, from eqn (2.116),

$$\Gamma\left(1-\frac{n}{2}\right) = \frac{\Gamma\left(3-\frac{n}{2}\right)}{\left(1-\frac{n}{2}\right)\left(2-\frac{n}{2}\right)}, \tag{2.126}$$

the quadratic divergent term (2.95) has poles at $n = 4$ and also at $n = 2$. For $n \to 4$ we have

$$-\mathrm{i}\Sigma(0) = \frac{\mathrm{i}\lambda\mu^2}{16\pi^2}\left(\frac{1}{4-n}\right). \tag{2.127}$$

To compare the two regularization methods we list the results for the divergences in Table 2.1. Thus divergent Feynman integrals when evaluated in n-dimensional space appear as poles of the resulting Γ function at

$n = 4, \ldots$ etc., keeping in mind that the quadratic divergence also has a pole at $n = 2$, see eqn (2.126).

TABLE 2.1

	Covariant regularization	Dimensional regularization
$\Gamma(0)$	$\frac{i\lambda^2}{32\pi^2}\ln\frac{\Lambda^2}{\mu^2}$	$\frac{i\lambda^2}{32\pi^2}\left(\frac{2}{4-n}\right)$
$\Sigma(0)$	$\frac{\lambda}{32\pi^2}\Lambda^2$	$\frac{\lambda}{32\pi^2}\left(\frac{-2\mu^2}{4-n}\right)$

2.4 Power counting and renormalizability

In the previous sections the renormalization procedure in $\lambda\phi^4$ theory has been illustrated in some detail. Here we will discuss the problem of renormalization for the more general types of interaction. The BPH renormalization procedure will be followed in this discussion. In a later part of this section, renormalization of composite operators will also be examined.

Theories with fermion and scalar particles

For simplicity we shall first concentrate on theories with spin-1/2 and spin-0 particles only. For the Lagrangian density, $\mathscr{L} = \mathscr{L}_0 + \Sigma_i \mathscr{L}_i$, where $\mathscr{L}_0$ is the free Lagrangian quadratic in the fields and the $\mathscr{L}_i$s are the interaction terms (for example, $\mathscr{L}_i = g_1\bar{\psi}\gamma_\mu\psi\,\partial^\mu\phi,\ g_2(\bar{\psi}\psi)^2,\ g_3\bar{\psi}\psi\phi,\ g_4\phi^3,\ g_5\phi^4, \ldots$), for a given graph we can define the quantities

n_i = number of ith type vertices;
b_i = number of scalar lines in the ith type vertex;
f_i = number of fermion lines in the ith type vertex;
d_i = number of derivatives in the ith type vertex;
B = number of external scalar lines;
F = number of external fermion lines;
IB = number of internal scalar lines;
IF = number of internal fermion lines.

Thus for $\mathscr{L}_1 = g_1\bar{\psi}\gamma_\mu\psi\,\partial^\mu\phi$ we have $b_1 = 1$, $f_1 = 2$, $d_1 = 1$. From the structure of the graph we have relations like that of (2.58)

$$B + 2(IB) = \sum_i n_i b_i \tag{2.129a}$$

$$F + 2(IF) = \sum_i n_i f_i. \tag{2.129b}$$

Just as in (2.59), the number of loop integrations L can be calculated

$$L = (IB) + (IF) - n + 1 \tag{2.130}$$

where

$$n = \sum_i n_i. \tag{2.131}$$

The superficial degree of divergence D is then given by

$$\begin{aligned} D &= 4L - 2(IB) - (IF) + \sum_i n_i d_i \\ &= 4 + 2(IB) + 3(IF) + \sum_i n_i(d_i - 4). \end{aligned} \tag{2.132}$$

Using (2.129) we can eliminate IB and IF,

$$D = 4 - B - \tfrac{3}{2}F + \sum_i n_i\,\delta_i \tag{2.133}$$

where

$$\delta_i = b_i + \tfrac{3}{2}f_i + d_i - 4 \tag{2.134}$$

is called the *index of divergence* of the interaction $\mathscr{L}_i$. For $\lambda\phi^4$ theory, $\delta = 0$ and (2.133) reduces to (2.61). In general δ_i can be related to the dimension of the coupling constant in units of mass. Knowing that the Lagrangian density has dimension four and that the scalar field, the fermion field, and the derivative have dimensions 1, 3/2, and 1, respectively, the dimension of the coupling constant is given by

$$\dim(g_i) = 4 - b_i - \tfrac{3}{2}f_i - d_i = -\delta_i. \tag{2.135}$$

From (2.133) we see that, for a fixed number of external lines, the superficial degree of divergence will have different behaviour for the following three cases.

(1) *g_i has positive dimension (or $\delta_i < 0$).* Then D decreases with the number of ith type vertices. In this case $\mathscr{L}_i$ is called a *super-renormalizable* interaction and the divergences are restricted to a finite number of graphs. For example, consider the graphs for the two-point Green's functions in the super-renormalizable $\lambda\phi^3$ theory. The one-loop diagram in Fig. 2.16(a) is divergent while the two-loop one in Fig. 2.16(b) is not.

FIG. 2.16.

(2) *g_i is dimensionless (or $\delta_i = 0$).* Here D is independent of the number of ith type vertices. The divergences are present in all higher-order diagrams of a finite number of Green's functions. $\mathscr{L}_i = g_1\phi^4,\ g_2\bar{\psi}\psi\phi$ are such examples, and they are called *renormalizable* interactions.

(3) *g_i has negative dimension (or $\delta_i > 0$).* In this case, D increases with the number of ith type vertices and all Green's functions are divergent for sufficiently large n_i. These types of interactions are *non-renormalizable*, and are exemplified by $\mathscr{L}_i = g_1\bar{\psi}\gamma_\mu\psi\,\partial^\mu\phi,\ g_2(\bar{\psi}\psi)^2,\ g_3\phi^5, \ldots$ etc.