Composite Operator

2010年5月18日
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In some cases, we need to consider Green's function of composite operator.

Consider the operator $\Omega(x) = \frac{1}{2}\phi^2(x)$ in $\lambda\phi^4$ theory. Green's function with one insertion of O is

$$G^{(n)}_{\Omega}(x; x_1, x_2 \cdots x_n) = \langle 0| T(\tfrac{1}{2}\phi^2(x)\, \phi(x_1)\phi(x_2)\cdots\phi(x_n))|0\rangle$$

In momentum space

$$(2\pi)^4\delta^4(p+p_1+\cdots+p_n)\, G^{(n)}_{\phi^2}(p; p_1 \cdots p_n) = \int d^4x\, e^{-ip\cdot x}\int \prod_{i=1}^{n} d^4x_i\, e^{-ip_i\cdot x_i}\, G^{(n)}_{O}(x; x_1 \cdots x_n)$$

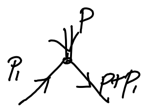
In perturbation theory, we can use Wick's theorem to work out these Green's functions in terms of Feynman diagram

Example, to lowest order in λ

$$G^{(2)}_{\phi^2}(x; x_1 x_2) = \frac{1}{2}\langle 0|T(\phi^2(x)\phi(x_1)\phi(x_2))|0\rangle$$
$$= i\Delta(x-x_1)\,\Delta(x-x_2)$$

or in momentum space

$$G^{(2)}_{\phi^2}(p; p_1, p_2) = i\Delta(p_1)\, i\Delta(p+p_1)$$



If we truncate the external propagators, we have

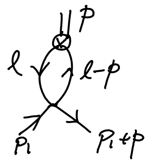
$$\Gamma^{(2)}_{\phi^2}(p, p_1, -p_1-p) = 1$$

To first order in λ, we have

$$G^{(2)}_{\phi^2}(x, x_1, x_2) = \int\langle 0|T(\tfrac{1}{2}\phi^2(x)\,\phi(x_1)\phi(x_2)\, \tfrac{(-i\lambda)}{4!}\phi^4(y))|0\rangle\, d^4y$$
$$= \int d^4y \left(\frac{-i\lambda}{2}\right)[i\Delta(x-y)]^2\, i\Delta(x_1-y)\;\; i\Delta(x_2-y)$$

The amputated 1PI momentum space Green's is

$$\Gamma^{(2)}_{\phi^2}(p; p_1, -p-p_1) = \left(\frac{-i\lambda}{2}\right)\int\frac{d^4 l}{(2\pi)^4}\left(\frac{i}{l^2-\mu^2+i\varepsilon}\right)\frac{i}{(l-p)^2-\mu^2+i\varepsilon}$$



To calculate this type of Green's functions systematically, we can add a term $\chi(x)\Omega(x)$ to $\mathcal{L}$

$$\mathcal{L}[\chi] = \mathcal{L}[0] + \chi(x)\Omega(x)$$

where $\chi(x)$ is a c-number source function. We can construct the generating functional $W[\chi]$ in the presence of this external source. We obtain the connected Green's function by differentiating $\ln W[\chi]$ with respect to χ and then setting χ to zero.

Renormalization of Composite operators

Superficial degrees of divergence

$$D_{\Omega} = D + \delta_{\Omega} = D + (d_{\Omega} - 4)$$

where d_{Ω} is the canonical dimension of Ω

For the case of $\Omega(x) = \frac{1}{2}\phi^2(x)$, $d_{\phi^2} = 2$. and $D_{\phi^2} = 2-n \Rightarrow$ only $\Gamma^{(2)}_{\phi^2}$ is divergent.

Taylor expansion

$$\Gamma^{(2)}_{\phi^2}(p; p_1) = \Gamma^{(2)}_{\phi^2}(0,0) + \Gamma^{(2)}_{\phi^2 R}(p, p_1)$$

We can combine the counter term

$$-\frac{i}{2}\,\Gamma^{(2)}_{\phi^2}(0,0)\,\chi(x)\,\phi^2(x)$$

with the original term to write

$$-\frac{i}{2}\chi\phi^2 - \frac{i}{2}\Gamma^{(3)}_{\phi^2}(0,0)\chi\phi^2 = \frac{-i}{2}Z_{\phi^2}\chi\phi^2 \qquad Z_{\phi^2} = 1+\Gamma_{\phi^2}(0,0)$$

In general, we need to insert counterterm $\Delta\Omega$ into the original addition

$$\mathcal{L} \to \mathcal{L} + \chi(\Omega + \Delta\Omega)$$

If $\Delta\Omega = C\Omega$, as in the case of $\Omega = \frac{i}{2}\phi^2$, we have

$$\mathcal{L}[\chi] = \mathcal{L}[0] + \chi Z_\Omega \Omega = \mathcal{L}[0] + \chi\Omega_0$$

with $\Omega_0 = Z_\Omega \Omega = (1+C)\Omega$

Such composite operators are said to be multiplicative renormalizable.
and Green's functions of unrenormalized operator Ω_0 is related to that of renormalized operator Ω by

$$G^{(n)}_{\Omega_0}(x; x_1 \cdots x_n) = \langle 0 | T(\Omega_0(x)\phi_0(x_1)\cdots\phi_0(x_n)) | 0 \rangle$$
$$= Z_\Omega Z_\phi^{n/2} G^{(n)}_{\Omega R}(x; x_1 \cdots x_n)$$

For more general cases, $\Delta\Omega \neq C\Omega$. and the renormalization of a composite operator may require counterterm proportional to other composite operators

Example, 2 composite operators A and B

$$\mathcal{L}[\chi] = \mathcal{L}[0] + \chi_A(A + \Delta A) + \chi_B(B + \Delta B)$$

The counter terms ΔA and ΔB are linear combinations of A and B

$$\Delta A = C_{AA}A + C_{AB}B$$
$$\Delta B = C_{BA}A + C_{BB}B$$

We can write

$$\mathcal{L}[\chi] = \mathcal{L}[0] + (\chi_A, \chi_B)\,C\begin{pmatrix} A \\ B \end{pmatrix} \qquad C = \begin{pmatrix} 1+C_{AA} & C_{AB} \\ C_{BA} & 1+C_{BB} \end{pmatrix}$$

Diagonalize C by bi-unitary transformation

$$UCV^{+} = \begin{pmatrix} Z_{A'} & 0 \\ 0 & Z_{B'} \end{pmatrix}$$

Then

$$\mathcal{L}[\chi] = \mathcal{L}[0] + Z_{A'}\chi_{A'}A' + Z_{B'}\chi_{B'}B'$$

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = V\begin{pmatrix} A \\ B \end{pmatrix} \qquad (\chi_{A'}, \chi_{B'}) = (\chi_A, \chi_B)U$$

A', B' are multiplicatively renormalizable.