Group Theory

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The most useful tool for describing symmetry is the group theory

1) Elements of group theory

A group G is a set of elements $(a, b, c \dots)$ with a multiplication law having the following properties;

(i) Closure. If $a, b \in G$, $c = ab \in G$

(ii) Associative $\quad a(bc) = (ab)c$

(iii) Identity $\quad \exists\, e \in G \ni a = ea = ae \quad \forall a \in G$

(iv) Inverse $\quad$ For every $a \in G$, $\exists\, a^{-1} \ni aa^{-1} = e = a^{-1}a$

Abelian group — elements commute, i.e. $\quad ab = ba \quad \forall a, b \in G$

orthogonal group — $n \times n$ orthogonal matrices

unitary group — $n \times n$ unitary matrices

direct product group — $G = \{g_1, g_2 \cdots\}$, $\quad H = \{h_1, h_2 \cdots\}$ and G commutes with H

direct product group $\quad G \times H = \{g_i h_j\}$ with multiplication law

$$(g_i h_j)(g_m h_n) = (g_i g_m)(h_j h_n)$$

Representation

For a group $G = \{g_1 \cdots g_n \cdots\}$, mapping to a set of matrices $D(g_1), D(g_2) \cdots$
$\ni$ it preserves the group multiplication, i.e.

$$D(g_1)D(g_2) = D(g_1 g_2) \qquad \forall g_1, g_2 \in G$$

If $\exists$ non-singular matrix $M \ni$

$$MD(a)M^{-1} = \begin{bmatrix} D_1(a) & & 0 \\ & D_2(a) & \\ 0 & & \ddots \end{bmatrix} \qquad \text{for all } a \in G.$$

$D(a)$ is called reducible representation

If representation is not reducible, then it is irreducible representation (irrep)

Continuous group : groups parametrized by set of continuous parameters

Example: rotations in 2-dimensions can be parametrized by $0 \le \theta < 2\pi$

2) SU(2) group : set of 2×2 unitary matrices with $\det = 1$

In general, $n \times n$ unitary matrix U can be written as

$$U = e^{iH} \qquad H: n \times n \text{ hermitian matrix}$$

From $\det U = e^{i\,\mathrm{Tr}H}$, we get $\mathrm{Tr}H = 0$ if $\det U = 1$.

Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Complete set of 2×2 hermitian traceless matrices

Define $J_i^{U} = \frac{\sigma_i}{2}$, we can derive

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2 \qquad \text{Lie algebra}$$

This is exactly the same as the commutation relations of angular momentum.

Recall that to get irrep, we define

$$J^2 = J_1^2 + J_2^2 + J_2^3 , \quad \text{with property } [J^2, J_i] = 0, \quad i = 1, 2, 3.$$

Also define $J_\pm \equiv J_1 \pm iJ_2$ then

$$J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2$$

and $[J_+, J_-] = 2J_3$,

For conveience, we choose eigenstates of J^2, J_3,

$$J^2|\lambda, m\rangle = \lambda|\lambda, m\rangle, \qquad J_3|\lambda, m\rangle = m|\lambda, m\rangle$$

from $[J_+, J_3] = -J_+ \implies (J_+J_3 - J_3J_+)|\lambda, m\rangle = -J_+|\lambda, m\rangle$

$$\text{or} \quad J_3(J_+|\lambda, m\rangle) = (m+1)(J_+|\lambda, m\rangle)$$

J_+: raises m to $m+1$, raising operator

Similarly, J_- : lower m to $m-1$,

$$J_3(J_-|\lambda, m\rangle) = (m-1)(J_-|\lambda, m\rangle)$$

Since $J^2 \geq J_3^2$, $\lambda - m^2 \geq 0 \implies m$ is bounded above and below.

Let j be the largest value of m, then

$$J_+|\lambda, j\rangle = 0$$

$$0 = J_-J_+|\lambda, j\rangle = (J^2 - J_3^2 - J_3)|\lambda, j\rangle = (\lambda - j^2 - j)|\lambda j\rangle$$

$$\lambda = j(j+1)$$

Similarly, let j' be the smallest value of m. then $J_-|\lambda, j'\rangle = 0$

$$\lambda = j'(j'-1)$$

Combining these 2 relations, we get

$$j(j+1) = j'(j'-1) \implies j' = -j$$

and $j - j' = 2j$ = integer

We will use j, m to label the states.

Assume that the states are normalized,

$$\langle jm|jm'\rangle = \delta_{mm'}$$

Write

$$J_\pm|jm\rangle = C_\pm(jm)|j, m\pm 1\rangle$$

Then

$$\langle jm|J_-J_+|jm\rangle = |C_+(j, m)|^2$$

$$= \langle j, m|(J^2 - J_3^2 - J_3)|jm\rangle = j(j+1) - m^2 - m \implies C_+(j, m) = \sqrt{(j-m)(j+m+1)}$$

Similarly $C_-(j, m) = \sqrt{(j+m)(j-m+1)}$

Summary: eigenstates $|jm\rangle$ have the properties

$J_3|j,m\rangle = m|j,m\rangle$. $\quad J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,|j\,m\pm 1\rangle$,

$J^2|j,m\rangle = j(j+1)|j\,m\rangle$

$|j,m\rangle$, $m=-j, -j+1, \dots, j$ are the basis for irreducible representation of SU(2) group

Example : $j=\frac{1}{2}$, $m=\pm\frac{1}{2}$

$$J_3|\tfrac{1}{2},\pm\tfrac{1}{2}\rangle = \pm\tfrac{1}{2}|\tfrac{1}{2},\pm\tfrac{1}{2}\rangle, \qquad J_+|\tfrac{1}{2},\tfrac{1}{2}\rangle = 0, \quad J_+|\tfrac{1}{2},-\tfrac{1}{2}\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$J_-|\tfrac{1}{2},\tfrac{1}{2}\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle \quad J_-|\tfrac{1}{2},-\tfrac{1}{2}\rangle = 0$$

If we write $|\frac{1}{2},\frac{1}{2}\rangle = \alpha = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |\frac{1}{2},-\frac{1}{2}\rangle = \beta = \begin{pmatrix}0\\1\end{pmatrix}$

then we can represent J's by matrices,

$$J_3 = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \qquad J_+ = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \qquad J_- = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$$

$$J_1 = \frac{1}{2}(J_+ + J_-) = \frac{1}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \qquad J_2 = \frac{1}{2i}(J_+ - J_-) = \frac{1}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$$

Within a factor of $\frac{1}{2}$, these are just Pauli matrices

Product representation

Consider 2 spin ½ particles, the total wavefunction is product of wavefunctions for each particle ; $\alpha_1\alpha_2$, $\alpha_1\beta_2$ . . .

Define $\vec{J}^{(1)}$ acts only on particle 1

$\vec{J}^{(2)}$ acts only on particle 2.

$\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$

Use $J_3 = J_3^{(1)} + J_3^{(2)}$, $\quad J_3(\alpha_1\alpha_2) = (J_3^{(1)} + J_3^{(2)})(\alpha_1\alpha_2) = (\alpha_1\alpha_2)$

from $\vec{J}^2 = (\vec{J}^{(1)} + \vec{J}^{(2)})^2 = (\vec{J}^{(1)})^2 + (\vec{J}^{(2)})^2 + 2[\frac{1}{2}(J_+^{(1)}J_-^{(2)} + J_-^{(1)}J_+^{(2)}) + J_3^{(1)}J_3^{(2)}]$

$$\vec{J}^2(\alpha_1\alpha_2) = (\tfrac{3}{4} + \tfrac{3}{4} + \tfrac{2}{4})|\alpha_1\alpha_2\rangle = 2|\alpha_1\alpha_2\rangle \implies j=1 \text{ state. } |1,1\rangle = \alpha_1\alpha_2$$

To get other $j=1$ states, we can use the lowering operator

$$J_-(\alpha_1\alpha_2) = (J_-^{(1)} + J_-^{(2)})(\alpha_1\alpha_2) = (\beta_1\alpha_2 + \alpha_1\beta_2)$$

On the other hand $\quad J_-(\alpha_1\alpha_2) = J_-|1\,1\rangle = \sqrt{(1+1)(1-1+1)}\,|1\,0\rangle = \sqrt{2}\,|1,0\rangle$

$\implies |1,0\rangle = \frac{1}{\sqrt{2}}(\beta_1\alpha_2 + \alpha_1\beta_2)$

Clearly $\quad |1,-1\rangle = \beta_1\beta_2$

The only state left-over is $\quad \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2) \implies |0,0\rangle$ state.

Summary :

1) Among the generators only J_3 is diagonal, — SU(2) is a rank-1 group

2) Irreducible representation is labelled by j and the dimension is $2j+1$;

3) basis states $|j,m\rangle$ $\quad m=j, j-1 \cdots -j$

representation matrices can be obtained from

$$J_3|j,m\rangle = m|j,m\rangle \qquad J_\pm|j\,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\;|j\,m\pm 1\rangle$$

SU(2) & rotation group

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The generators of SU(2) group are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\vec{r} = (x, y, z)$ arbitrary vector in R_3 (3 dimensional coordinate space)

Define a 2x2 matrix h by

$$h = \vec{\sigma} \cdot \vec{r} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

h has the following properties

(1) $h^{+} = h$

(2) $T_r h = 0$

(3) $\det h = -(x^2+y^2+z^2)$

Let U be a 2x2 unitary matrix with $\det U = 1$.

Consider the transformation

$$h \to h' = U h U^{+}$$

Then we have

(1) $h'^{+} = h'$

(2) $T_r h' = 0$

(3) $\det h' = \det h$

Properties (1) & (2) imply that h' can also be expanded in terms of Pauli matrices

$h' = \vec{r}' \cdot \vec{\sigma}$ $\qquad \vec{r}' = (x', y', z')$

$\det h' = \det h \implies \quad x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$

Thus relation between $\vec{r}$ and $\vec{r}'$ is a rotation.

This means that an arbitrary 2x2 unitary matrix U induces a rotation in R_3

Thus this provides a connection between SU(2) and O(3) groups.

Rotation group & QM

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Rotation in R_3 can be represented as linear transformations on $\vec{r}=(x,y,z)=(r_1,r_2,r_3)$

$$r_i \to r_i' = R_{ij} x_j \qquad R R^T = 1 = R^T R$$

$f(r_i)=f(x,y,z)$ function of coordinates

under the rotation $f(r_i) \to f(R_{ij} r_j) = f'(r_i)$

If $f=f'$ we say f is invariant under rotation, eg $f(r_i)=f(r)$, $r=\sqrt{x^2+y^2+z^2}$

In QM, we implement the rotation by

$$\left.\begin{array}{l} |\psi\rangle \to |\psi'\rangle = U|\psi\rangle \\ O \longrightarrow O' = U O U^+ \end{array}\right\} \Rightarrow \langle\psi'|O'|\psi'\rangle = \langle\psi|O|\psi\rangle$$

If $O'=UOU^+=O$, we say the operator O is invariant under rotation

$$\Downarrow \quad UO=OU \qquad \text{or} \qquad [O,U]=0$$

In terms of infinitesmal generators

$$U=e^{i\theta \vec{n}\cdot\vec{J}/\hbar}$$

this implies

$$[J_i, O]=0 \qquad i=1,2,3.$$

For the case where O is the Hamiltonian H, this gives

$$[J_i, H]=0$$

Let $|\psi\rangle$ be an eigenstate of H with eigenvalue E,

$$H|\psi\rangle = E|\psi\rangle$$

then $\qquad (J_i H - H J_i)|\psi\rangle = 0 \Rightarrow H(J_i|\psi\rangle) = E(J_i|\psi\rangle)$

i.e. $\quad |\psi\rangle$ & $J_i|\psi\rangle$ are degenerate

For example, let $|\psi\rangle = |j,m\rangle$ the eigenstates of angular momentum, then

$J_\pm|j,m\rangle$ are also eigenstates if $|\psi\rangle$ is eigenstate of H.

$\Rightarrow$ for a given j, the degeneracy is $(2j+1)$

Abelian gauge theory (QED)

Consider the Lagrangian for a free electron field $\psi(x)$

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)$$

This has global $U(1)$ symmetry,

$$\psi(x) \longrightarrow \psi'(x) = e^{-i\alpha}\psi(x) \qquad \alpha : \text{constant}$$

$$\bar{\psi}(x) \longrightarrow \bar{\psi}'(x) = \bar{\psi}(x)e^{i\alpha}$$

Suppose $\alpha = \alpha(x)$

$$\psi'(x) = e^{-i\alpha(x)}\psi(x), \qquad \bar{\psi}'(x) = \bar{\psi}(x)e^{i\alpha(x)}$$

transformation of derivative

$$\bar{\psi}(x)\partial_\mu\psi(x) \longrightarrow \bar{\psi}'(x)\partial_\mu\psi'(x) = \bar{\psi}(x)\partial_\mu\psi(x) - i(\partial_\mu\alpha)(\bar{\psi}\psi) \qquad \text{not invariant}$$

Introduce gauge field $A_\mu(x)$ to form covariant derivative

$$D_\mu\psi \equiv (\partial_\mu + igA_\mu)\psi(x)$$

So that $D_\mu\psi$ transforms by a phase,

$$(D_\mu\psi)' = e^{-i\alpha(x)}(D_\mu\psi)$$

This requires that

$$(\partial_\mu + igA'_\mu)\psi' = e^{-i\alpha}(\partial_\mu + igA_\mu)\psi$$

$$e^{-i\alpha}[\partial_\mu\psi + i(\partial_\mu\alpha)\psi + igA'_\mu\psi]$$

$$\Rightarrow \qquad A'_\mu = A_\mu - \frac{1}{g}\partial_\mu\alpha$$

Then $\quad \mathcal{L}_0 \to \bar{\psi}i\gamma^\mu(\partial_\mu + igA_\mu)\psi - m\bar{\psi}\psi$

is invariant under local symmetry transformation (local symmetry)

The Lagrangian for gauge field is of the form,

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{invariant}$$

One useful relation is to write $F_{\mu\nu}$ in terms of covariant derivative,

$$D_\mu D_\nu\psi = (\partial_\mu + igA_\mu)(\partial_\nu + igA_\nu)\psi = \partial_\mu\partial_\nu\psi - g^2A_\mu A_\nu\psi + ig(A_\mu\partial_\nu + A_\nu\partial_\mu)\psi + ig(\partial_\mu A_\nu)\psi$$

$$\Rightarrow \quad (D_\mu D_\nu - D_\nu D_\mu)\psi = ig(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi = ig(F_{\mu\nu})\psi$$

From $[(D_\mu D_\nu - D_\nu D_\mu)\psi]' = e^{-i\alpha}(D_\mu D_\nu - D_\nu D_\mu)\psi \implies F'_{\mu\nu} = F_{\mu\nu}$

Thus the Lagrangian of the form

$$\mathcal{L} = \bar{\psi}i\gamma^\mu(\partial_\mu + igA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

is invariant under gauge transformation,

$$\psi(x) \to \psi'(x) = e^{-i\alpha(x)}\psi(x)$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x)$$

Remarks:

1) $A^\mu A_\mu$ term is not gauge invariant $\implies$ A_μ field massless

1) $A^\mu A_\mu$ term is not gauge invariant $\Rightarrow$ A_μ field massless

2) $D_\mu\psi=(\partial_\mu+igA_\mu)\psi$. $\Rightarrow$ minimal coupling determined by $U(1)$ transformation universality.

3) no gauge self coupling. because A_μ does not carry $U(1)$ charge

Non-Abelian symmetry – Yang Mills fields

1954: Yang–Mills generalized $U(1)$ local symmetry to $SU(2)$ local symm.

Consider an isospin doublet

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

Under $SU(2)$ transformation

$$\psi(x)\longrightarrow\psi'(x)=\exp\left\{-\frac{i\vec{\tau}\cdot\vec{\theta}}{2}\right\}\psi(x) \qquad \vec{\tau}=(\tau_1,\tau_2,\tau_3)\ \text{Pauli matrices}$$

$$\left[\frac{\tau_i}{2},\frac{\tau_j}{2}\right]=i\epsilon_{ijk}\left(\frac{\tau_k}{2}\right)$$

Start with free Lagrangian

$$\mathcal{L}_0=\bar{\psi}(x)(i\gamma^\mu\partial_\mu-m)\psi$$

Under local symmetry transformation,

$$\psi(x)\longrightarrow\psi'(x)=U(\theta)\psi(x) \qquad U(\theta)=\exp\left\{-\frac{i\vec{\tau}\cdot\vec{\theta}(x)}{2}\right\}$$

Derivative term

$$\partial_\mu\psi(x)\rightarrow\partial_\mu\psi'(x)=U\partial_\mu\psi+(\partial_\mu U)\psi$$

Introduce gauge fields $\vec{A}_\mu$ to form covariant derivative,

$$D_\mu\psi(x)\equiv\left(\partial_\mu-ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\right)\psi$$

Require that

$$[D_\mu\psi]'=U[D_\mu\psi]$$

$$\Rightarrow\left(\partial_\mu-ig\frac{\vec{\tau}\cdot\vec{A}_\mu'}{2}\right)(U\psi)=U\left(\partial_\mu-ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\right)\psi$$

or $$-ig\left(\frac{\vec{\tau}\cdot\vec{A}_\mu'}{2}\right)U+\partial_\mu U=U\left(-ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\right)$$

$$\boxed{\frac{\vec{\tau}\cdot\vec{A}_\mu'}{2}=U\left(\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\right)U^{-1}-\frac{i}{g}(\partial_\mu U)U^{-1}}$$

We can use covariant derivatives to construct field tensor

$$D_\mu D_\nu\psi=\left(\partial_\mu-ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\right)\left(\partial_\nu-ig\frac{\vec{\tau}\cdot\vec{A}_\nu}{2}\right)\psi=\partial_\mu\partial_\nu\psi-ig\left(\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\partial_\nu\psi+\frac{\vec{\tau}\cdot\vec{A}_\nu}{2}\partial_\mu\psi\right)$$
$$-ig\,\partial_\mu\left(\frac{\vec{\tau}\cdot\vec{A}_\nu}{2}\right)\psi+(-ig)^2\left(\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\right)\left(\frac{\vec{\tau}\cdot\vec{A}_\nu}{2}\right)\psi$$

Antisymmetrization

$$(D_\mu D_\nu-D_\nu D_\mu)\psi\equiv ig\left(\frac{\vec{\tau}\cdot\vec{F}_{\mu\nu}}{2}\right)\psi$$

$$\frac{\vec{\tau}\cdot\vec{F}^{\mu\nu}}{2}=\frac{\vec{\tau}}{2}\cdot(\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu)-ig\left[\frac{\vec{\tau}\cdot\vec{A}_\mu}{2},\frac{\vec{\tau}\cdot\vec{A}_\nu}{2}\right]$$

or $$F^i_{\mu\nu}=\partial_\mu A^i_\nu-\partial_\nu A^i_\mu+g\epsilon^{ijk}A^j_\mu A^k_\nu$$

or $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu \underbrace{+ g\epsilon^{ijk} A^j_\mu A^k_\nu}_{\text{new term}}$

Under gauge transformation.

$$\vec{\tau}\cdot\vec{F}'_{\mu\nu} = U(\vec{\tau}\cdot\vec{F}_{\mu\nu})U^{-1}$$

<u>Infinitesmal transformation</u> $\theta(x) \ll 1$.

$$A'^{i\mu} = A^\mu + \epsilon^{ijk}\theta^j A^k_\mu - \frac{1}{g}\partial_\mu\theta^i$$

$$F'^i_{\mu\nu} = F^i_{\mu\nu} + \epsilon^{ijk}\theta^j F^k_{\mu\nu}$$

Remarks

1) Again $A^a_\mu A^{a\mu}$ is not gauge invariant $\Rightarrow$ gauge boson massless $\Downarrow$ long range force

2) A^a_μ carries the symmetry charge (e.g. color ---)

3) $F^{a\mu\nu} \sim \partial A - \partial A + \underbrace{g\,AA}_{\text{term responsible for Asymptotic freedom.}}$

Maxwell Equation

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$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0},\quad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \qquad \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}=\epsilon_0\frac{\partial\vec{E}}{\partial t}+\vec{J}$$

Source free Equations can be solved

$$\vec{B}=\vec{\nabla}\times\vec{A}, \quad \Rightarrow \quad \vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0$$

$$\vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t} \qquad \partial^\mu A^\nu-\partial^\nu A^\mu=F^{\mu\nu}$$

$$F^{ij}\sim\epsilon^{ijk}B_k \qquad F^{0i}\sim E^i$$

Gauge invariance

$$\left.\begin{array}{l}\phi\to\phi-\frac{\partial\alpha}{\partial t}\\ \vec{A}\to\vec{A}+\vec{\nabla}\alpha\end{array}\right\} \quad \begin{array}{l}A^\mu=\left(\frac{\phi}{c},\vec{A}\right)\\ A^\mu\to A^\mu-\partial^\mu\alpha\end{array}$$

Schrodinger Eq for a charged particle

$$\left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}-e\vec{A}\right)^2-e\phi\right]\psi=i\hbar\frac{\partial\psi}{\partial t}$$

To get gauge invariance, need to transform ψ

$$\psi\to e^{ie\alpha/\hbar}\psi$$

Spontaneous symmetry breaking — ground state does not have the symmetry of the Hamiltonian

$\Rightarrow$ If the symmetry is continuous one, there will be massless scalar fields

Example: ferromagnetism

$T > T_c$ (Curie temp) all dipoles are randomly oriented — rotational invariant

$T < T_c$ all dipoles are oriented in some direction.

Ginzburgh–Landau theory

Free energy as function of magnetization $\vec{M}$ (averaged)

$$u(\vec{M}) = (\partial_i \vec{M})^2 + \alpha_1(T)\, \vec{M}\cdot\vec{M} + \alpha_2 (\vec{M}\cdot\vec{M})^2$$

$$\alpha_2 > 0, \quad \alpha_1(T) = \alpha(T - T_c) \qquad \alpha > 0$$

ground state $\qquad \vec{M}(\alpha_1 + 2\alpha_2 \vec{M}\cdot\vec{M}) = 0$

$T > T_c$ only solution is $\vec{M} = 0$.

$T < T_c$ non-trivial sol $|\vec{M}| = +\sqrt{\frac{-\alpha_1}{2\alpha_2}} \neq 0$

$\Rightarrow$ ground state with $\vec{M}$ in some direction is no longer rotational invariant.

Nambu–Goldstone theorem:

Noether's theorem: continuous symmetry $\Rightarrow$ conserved charge Q

Suppose there are 2 local operators A, B with property

$$[Q, B] = A \qquad Q = \int d^3x\, j_0(x) \quad \text{indep of time}$$

Suppose $\langle 0|A|0\rangle = v \neq 0$ (Symmetry breaking condition)

$$\Rightarrow 0 \neq \langle 0|[Q,B]|0\rangle = \int d^3x \langle 0|[j_0(x), B]|0\rangle$$

$$= \sum_n (2\pi)^3 \delta^3(\vec{P}_n) \{\langle 0|J_0(0)|n\rangle\langle n|B|0\rangle e^{-iE_n t} - \langle 0|B|n\rangle\langle n|J_0|0\rangle e^{iE_n t}\} = v$$

Since $v \neq 0$ and time-independent, we need to a state such that

$$E_n \to 0 \quad \text{for } \vec{P}_n = 0$$

massless excitation. For the case of relativistic particle with energy momentum relation $E = \sqrt{\vec{p}^2 + m^2}$, this implies massless particle – Goldstone boson.

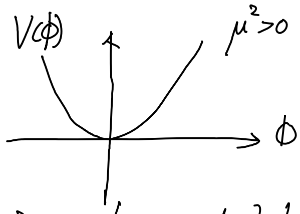
Discrete symmetry case

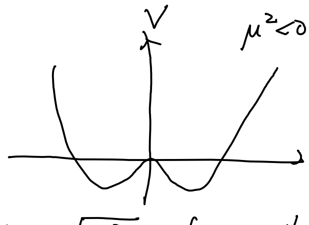
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \qquad \phi \to -\phi \text{ symmetry}$$

The Hamiltonian density

$$H = \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

Effective energy $\qquad u(\phi) = \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi), \quad V(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$





$\Rightarrow \mu^2 < 0$ the ground state has $\phi = \pm\sqrt{-\mu^2}$ classically

For $\mu<0$, the ground state has $\phi=\pm\sqrt{\frac{\mu^2}{\lambda}}$ classically.

This means the quantum ground state $|0\rangle$ will have the property

$$\langle 0|\phi|0\rangle = v \neq 0 \qquad \text{symmetry breaking condition}$$

Define quantum field ϕ' by

$$\phi' = \phi - v$$

then

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi')^2 - (-\mu^2)\phi'^2 - \lambda v \phi'^3 - \frac{\lambda}{4}\phi'^4$$

No Goldstone boson — discrete symmetry

Abelian symmetry case.

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2\right] - V(\sigma^2+\pi^2)$$

with $\quad V(\sigma^2+\pi^2) = -\frac{\mu^2}{2}(\sigma^2+\pi^2) + \frac{\lambda}{4}(\sigma^2+\pi^2)^2$

$O(2)$ symmetry $\qquad \begin{pmatrix}\sigma\\ \pi\end{pmatrix} \to \begin{pmatrix}\sigma'\\ \pi'\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\sigma\\ \pi\end{pmatrix}$

minimum $\qquad \sigma^2+\pi^2 = \frac{\mu^2}{\lambda} \equiv v^2 \qquad$ circle in σ-π plane

For conveience choose $\quad \langle 0|\sigma|0\rangle = v \qquad \langle 0|\pi|0\rangle = 0$

New quantum field $\quad \sigma' = \sigma - v, \qquad \pi' = \pi$

New Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \sigma')^2 + (\partial_\mu \pi)^2\right] - \mu^2\sigma'^2 - \lambda v \sigma'(\sigma'^2+\pi'^2) - \frac{\lambda}{4}(\sigma'^2+\pi'^2)^2 \qquad O(2)$$

no π'^2 term, $\Rightarrow$ π' massless Goldstone boson

Non-Abelian case — σ model

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2\right] + \bar{N} i\gamma^\mu \partial_\mu N + g\bar{N}(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)N - V(\sigma^2+\vec{\pi}^2) + (f_\pi m_\pi^2 \sigma)$$

$$V(\sigma^2+\vec{\pi}^2) = -\frac{\mu^2}{2}(\sigma^2+\vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2+\vec{\pi}^2)^2$$

minimum $\qquad \sigma^2+\vec{\pi}^2 = v^2 = \frac{\mu^2}{\lambda}$

choose $\quad \langle\sigma\rangle = v, \qquad \langle\vec{\pi}\rangle = 0$

Then $\quad \vec{\pi}$ are Goldstone bosons

When we combine spontaneous symmetry breaking with local symmetry, a very interesting phenomena occurs. This was discovered in the 60's by Higgs, Englert & Brout, Guralnik, Hagen & Kibble independently.

Abelian case

Consider the Lagrangian given by

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where $D^\mu\phi = (\partial^\mu - igA^\mu)\phi$, $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

the Lagrangian is invariant under the local gauge transformation

$$\phi(x) \longrightarrow \phi'(x) = e^{-i\alpha(x)}\phi(x)$$

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x)$$

The spontaneous symm. breaking is generated by the potential

$$V(\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$$

which has a minimum at $\phi^\dagger\phi = \frac{v^2}{2} = \frac{1}{2}\left(\frac{\mu^2}{\lambda}\right)$

For the quantum theory, we can choose

$$|\langle 0|\phi|0\rangle| = \frac{v}{\sqrt{2}}$$

Or if we write $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

this corresponds to

$$\langle\phi_1\rangle = v, \qquad \langle\phi_2\rangle = 0 \qquad\qquad \phi_2 : \text{Goldstone boson}$$

Define the quantum fields by

$$\phi_1' = \phi_1 - v, \qquad \phi_2' = \phi_2$$

Covariant derivative terms gives

$$(D_\mu\phi)^\dagger(D^\mu\phi) = [(\partial_\mu + igA_\mu)\phi^\dagger][(\partial^\mu - igA^\mu)\phi]$$

$$= \frac{1}{2}(\partial_\mu\phi_1' + gA_\mu\phi_2')^2 + \frac{1}{2}(\partial_\mu\phi_2' - gA_\mu\phi_1')^2 + \underbrace{\frac{g^2v^2}{2}A^\mu A_\mu}_{\text{mass term for } A^\mu} + \cdots$$

Write the scalar field as

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x))\, e^{i\xi(x)/v}$$

"Gauge" transformation:

$$\phi'(x) = e^{-i\xi/v}\phi(x), \qquad B_\mu = A_\mu(x) - \frac{1}{gv}\partial_\mu\xi$$

$\xi(x)$ disappears from the Lagrangian

Roughly speaking, massless gauge field A_μ combine with Goldstone boson $\xi(x)$ to become massive gauge boson. As a consequence, two long range forces (from Goldstone boson $\xi(x)$ and $A_\mu(x)$) disappear

Non-Abelian case

SU(2) group : $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ doublet

$$\mathcal{L} = (D_\mu\phi)^+(D^\mu\phi) - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$D_\mu\phi = \left(\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\right)\phi\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$V(\phi) = -\mu^2(\phi^+\phi) + \lambda(\phi^+\phi)^2$$

Spontaneous symmetry breaking :

$$\langle\phi\rangle_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \qquad v = \sqrt{\frac{\mu^2}{\lambda}}$$

Define $\phi' = \phi - \langle\phi\rangle_0$

From covariant derivative

$$(D_\mu\phi)^+(D^\mu\phi) = \left[\left(\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\right)(\phi' + \langle\phi\rangle_0)\right]^+\left[\left(\partial^\mu - ig\frac{\vec{\tau}\cdot\vec{A}^\mu}{2}\right)(\phi' + \langle\phi\rangle_0)\right]$$

$$\longrightarrow \frac{1}{4}g^2\langle\phi\rangle_0(\vec{\tau}\cdot\vec{A}_\mu)(\vec{\tau}\cdot\vec{A}^\mu)\langle\phi\rangle_0 = \frac{1}{2}\left(\frac{gv}{2}\right)^2\vec{A}_\mu\cdot\vec{A}^\mu$$

All gauge bosons get masses

$$M_A = \frac{1}{2}gv$$

The symmetry is completely broken.

Write

$$\phi(x) = \exp\left\{\frac{i\vec{\tau}\cdot\vec{\xi}(x)}{v}\right\}\begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix}$$

"gauge" transformation

$$\phi'(x) = U(x)\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+\eta(x) \end{pmatrix}$$

$$\frac{\vec{\tau}\cdot\vec{B}_\mu}{2} = U(x)\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}U^{-1} - \frac{i}{g}[\partial_\mu U]U^{-1}(x)$$

where $U(x) = \exp\left\{\frac{i\vec{\tau}\cdot\vec{\xi}}{v}\right\}$