

# Quantization of Gauge Theory

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Canonical quantization of gauge theory is difficult because the gauge invariance implies that not all components of gauge fields are real physical degrees of freedom. To eliminate those components which are dependent, it is easier to use path integral quantization.

Consider for simplicity,  $SU(2)$  Yang-Mills fields,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \quad a=1,2,3$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \epsilon^{abc} A_\mu^b A_\nu^c$$

We can write the generating functional as

$$W[\vec{J}] = \int [d\vec{A}_\mu] \exp \left\{ i \int d^4x [\mathcal{L} + \vec{J}_\mu \cdot \vec{A}^\mu] \right\}$$

The free-field part is then

$$W_0[\vec{J}] = \int [d\vec{A}_\mu] \exp \left\{ i \int d^4x [\mathcal{L}_0 + \vec{J}_\mu \cdot \vec{A}^\mu] \right\}$$

$$\begin{aligned} \text{where } \int d^4x \mathcal{L}_0(x) &= -\frac{1}{4} \int d^4x (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \\ &= \frac{1}{2} \int d^4x A_\mu^a(x) (g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu) A_\nu^a(x) \end{aligned}$$

The general formula for the Gaussian integral is of the form,

$$\int [d\phi] \exp \left[ -\frac{1}{2} \langle \phi | K | \phi \rangle + \langle J | \phi \rangle \right] \sim \frac{1}{\sqrt{\det K}} \exp \langle J | K^{-1} | J \rangle$$

However, in our case the operator  $K$

$$K_{\mu\nu}(x-y) = (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) \delta^4(x-y)$$

has the property of a projection operator, i.e.

$$\int d^4y K_{\mu\nu}(x-y) K^{\nu\lambda}(y-z) \propto K_{\mu\lambda}(x-z)$$

This means that  $K_{\mu\nu}(x-y)$  has no inverse and the Gaussian integral is singular.

The reason that  $W_0[\vec{J}]$  is singular is due to the gauge invariance. In the path integral for  $W_0[\vec{J}]$  we have summed over all field configurations, including "orbits" that are related by gauge transformation. This overcounting is the root of divergent integral.

## Isolating path integral volume factor

Simple illustrative example

Consider a 2-dimensional integral

$$W = \int dx dy e^{iS(x,y)} = \int d^2\vec{r} e^{iS(\vec{r})} \quad \vec{r} = (r, \theta)$$

Suppose  $S(\vec{r})$  is invariant under rotation

$$S(\vec{r}) = S(\vec{r}_\theta) \quad \text{when } \vec{r}_\theta = (r, \theta + \phi)$$

Thus  $S(\vec{r})$  is constant over (circular) orbit and  $W$  is proportional to the length of the orbit. Thus if we only wish to sum over contribution from inequivalent  $S(\vec{r})$ 's we can simply divide out the volume factor corresponding to polar angle integration  $\int d\theta = 2\pi$ . We will use a more complicated procedure which can be generalized to more general cases.

Insert the identity,

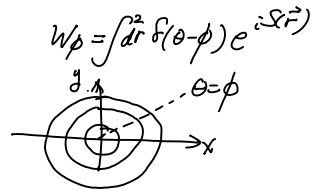
$$1 = \int d\phi \delta(\theta - \phi)$$

into  $W$

$$W = \int d\phi \int d^2r e^{iS(\vec{r})} \delta(\theta - \phi) = \int d\phi W_\phi$$

Using invariant property of  $S(\vec{r}) = S(\vec{r}_\phi)$ , we have

$$W_\phi = W_{\phi'} \Rightarrow W_\phi \text{ is indep of } \phi$$



$W = \int d\phi W_\phi = W_\phi \int d\phi = 2\pi W_\phi$   
We can impose a more complicate constraint,

which intersects each of the orbits once,

We need to compute

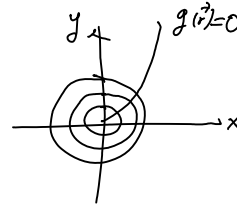
$$[\Delta_g(\vec{r})]^{-1} = \int d\phi \delta[g(\vec{r}_\phi)]$$

We can show that  $\Delta_g(\vec{r})$  is rotational invariant

$$[\Delta_g(\vec{r}_\phi)]^{-1} = \int d\phi' \delta[g(\vec{r}_{\phi+\phi'})] = \int d\phi'' \delta[g(\vec{r}_{\phi''})] = [\Delta_g(\vec{r})]^{-1}$$

Integrating over  $\delta[g(\vec{r}_\phi)]$ , we get

$$\Delta_g(\vec{r}) = \left. \frac{\partial g(\vec{r})}{\partial \theta} \right|_{g=0}$$



The integral is then

$$W = \int d\phi W_\phi \quad \text{where} \quad W_\phi = \int d^2r e^{iS(\vec{r})} \Delta_g(\vec{r}) \delta[g(\vec{r}_\phi)]$$

Again  $W_\phi$  is rotational invariant

$$\begin{aligned} W_{\phi'} &= \int d^2r e^{iS(\vec{r})} \Delta_g(\vec{r}) \delta[g(\vec{r}_{\phi'})] \\ &= \int d^2r' e^{iS(\vec{r}')} \Delta_g(\vec{r}') \delta[g(\vec{r}'_{\phi'})] = W_\phi \end{aligned} \quad \vec{r}' = (r, \phi')$$

### Volume factor in gauge theories

1

Gauge transf  $\vec{A}_\mu \rightarrow \vec{A}_\mu^\theta$

$$\text{where } \vec{A}_\mu^\theta = U(\theta) \left[ \vec{A}_\mu + \frac{1}{ig} U^{-1}(\theta) \partial_\mu U(\theta) \right] U^{-1}(\theta) \quad \text{where } U(\theta) = \exp\left[-\frac{i\vec{\theta} \cdot \vec{\tau}}{2}\right]$$

We restrict the path integration to hypersurface which intersects each orbit once. If we choose the hypersurface as

$$f_a(\vec{A}_\mu) = 0 \quad a=1,2,3$$

so that the equation

$$f_a(\vec{A}_\mu^\theta) = 0$$

must have a unique solution  $\vec{\theta}$  for a given  $\vec{A}_\mu$ .

In the neighbourhood of identity, we can write

$$U(\theta) = 1 + i \frac{\vec{\tau} \cdot \vec{\theta}}{2} + O(\theta^2)$$

the integration over group space can be chosen as

$$[d\theta] = \prod_{a=1}^3 d\theta_a$$

Define

then

$$\Delta_f^{-1}[\vec{A}_\mu] = \int [d\theta(x)] \delta[f_a(\vec{A}_\mu^0)]$$

then

$$\Delta_f[\vec{A}_\mu] = \det M_f \quad \text{where } (M_f)_{ab} = \frac{\delta f_a}{\delta \theta_b}$$

Thus  $M_f$  is the response of  $f_a[\vec{A}_\mu]$  to the infinitesimal gauge transformation. Recall that the infinitesimal gauge transformation is of the form,

$$A_\mu^{oa} = A_\mu^a + \epsilon^{ab} \theta^b A_\mu^c - \frac{1}{g} \partial_\mu \theta^a$$

and the response of  $f_a[\vec{A}_\mu]$  is

$$f_a[\vec{A}_\mu^0(x)] = f_a[\vec{A}_\mu(x)] + \int d^4y [M_f(x,y)]_{ab} \theta_b(y) + O(\theta^2)$$

Again  $\Delta_f[\vec{A}_\mu]$  is gauge invariant,

write

$$\Delta_f^{-1}[\vec{A}_\mu] = \int [d\theta(x)] \delta[f_a(\vec{A}_\mu^0)]$$

$$\begin{aligned} \text{then } \Delta_f^{-1}[\vec{A}_\mu] &= \int [d\theta'] \delta[f_a(\vec{A}_\mu^{0\theta'})] = \int [d(\theta\theta')](x) \delta[f_a(\vec{A}_\mu^{0\theta'})] \\ &= \int [d\theta'(x)] \delta[f_a(\vec{A}_\mu^{0\theta'})] = \Delta_f^{-1}[\vec{A}_\mu] \end{aligned}$$

The path integral is then

$$\begin{aligned} \int [d\vec{A}_\mu] \exp \left\{ i \int d^4x \mathcal{L}(x) \right\} &= \int [d\theta(x)] [d\vec{A}_\mu] \Delta_f(\vec{A}_\mu) \delta[f_a(\vec{A}_\mu^0)] \exp \left\{ i \int d^4x \mathcal{L}(x) \right\} \\ &= \int [d\theta(x)] [d\vec{A}_\mu(x)] \Delta_f(\vec{A}_\mu) \delta[f_a(\vec{A}_\mu)] \exp \left\{ i \int d^4x \mathcal{L}(x) \right\} \end{aligned}$$

We can now drop the "volume factor"  $\int [d\theta(x)]$  to write the generating functional as

$$W_f[\vec{J}] = \int [d\vec{A}_\mu] (\det M_f) \delta[f_a(\vec{A}_\mu)] \exp \left\{ i \int d^4x [\mathcal{L}(x) + \vec{J}_\mu \cdot \vec{A}^\mu] \right\}$$

This is called Faddeev - Popov ansatz

The factor  $(\det M_f)$  can be written as

$$(\det M_f) \sim \int [d\vec{c}] [d\vec{c}^\dagger] \exp \left\{ i \int d^4x d^4y \sum_{a,b} c_a^\dagger(x) [M_f(x,y)]_{ab} c_b(y) \right\}$$

where  $\vec{c}$ : Grassmannian fields

We want now to convert  $\delta[f_a(\vec{A}_\mu)]$  into exponential form.

Suppose we choose the gauge fixing term to be

$$f_a[\vec{A}_\mu] = B_a(x)$$

$$\text{Then } \int [d\theta(x)] \Delta_f[\vec{A}_\mu] \delta[f_a(\vec{A}_\mu^0) - B_a(x)] = 1$$

will give the same  $\Delta_f[\vec{A}_\mu]$  as before

$$\text{Note that } \int [d\theta(x)] \exp \left\{ -\frac{i}{2\xi} \vec{B}^2(x) \right\} \sim \text{constant}$$

We can then write

$$\begin{aligned} W[\vec{J}] &= \int [d\vec{A}_\mu] [d\vec{B}(x)] (\det M_f) \delta[f_a(\vec{A}_\mu) - B_a] \exp \left\{ i \int d^4x [\mathcal{L}(x) - \vec{J}^\mu \cdot \vec{A}_\mu - \frac{1}{2\xi} \vec{B}^2(x)] \right\} \\ &= \int [d\vec{A}_\mu] (\det M_f) \exp \left\{ i \int d^4x [\mathcal{L}(x) - \vec{J}^\mu \cdot \vec{A}_\mu - \frac{1}{2\xi} [f_a(\vec{A}_\mu)]^2] \right\} \quad \xi \text{ arbitrary} \end{aligned}$$

Put all these together we can write

$$W[\vec{J}] = \int [d\vec{A}_\mu] [d\vec{c}(x)] [d\vec{c}^\dagger(x)] \exp \left\{ i S_{eff}[\vec{J}] \right\}$$

where

$$S_{eff}[\vec{J}] = S[\vec{J}] + S_{gf} + S_{FPG}$$

$$S_{gf} = -\frac{1}{2\xi} \int d^4x \{f_a[\vec{A}_\mu(x)]\}^2$$

$$S_{FP\theta} = \int d^4x d^4y \sum_{a,b} C_a^\dagger(x) [M_f(x,y)]_{ab} C_b(y)$$

Covariant Gauge

$$f_a(\vec{A}_\mu) = \partial^\mu A_\mu^a = 0$$

under infinitesimal gauge transf

$$U(\theta(x)) = 1 + i\vec{\theta} \cdot \vec{c}/2 + O(\theta^2)$$

$$A_\mu^{ab} = A_\mu^a + \epsilon^{abc} \theta^b(x) A_\mu^c - \frac{1}{g} \partial_\mu \theta^a(x)$$

$$\begin{aligned} \text{then } f^a(A_\mu^\theta) &= f^a(A_\mu) + \partial^\mu [\epsilon^{abc} \theta^b(x) A_\mu^c - \frac{1}{g} \partial_\mu \theta^a(x)] \\ &= f^a(A_\mu) + \int d^4y [M_f(x,y)]_{ab} \theta^b(y) \end{aligned}$$

$$\text{with } [M_f(x,y)]_{ab} = -\frac{1}{g} \partial^\mu [\delta^{ab} \partial_\mu - g \epsilon^{abc} A_\mu^c] \delta^4(x-y)$$

$$\text{Then } S_{gf} = -\frac{1}{2\xi} \int d^4x (\partial^\mu \vec{A}_\mu)^2$$

$$S_{FP\theta} = \frac{1}{g} \int d^4x \sum_{a,b} C_a^\dagger(x) \partial^\mu [\delta_{ab} \partial_\mu - g \epsilon_{abc} A_\mu^c] C_b(x)$$