Quantum Field Theory II
Homework set 1, Due Th March 18

1. Under the rotation about $z-axis$ the coordinate vector $\vec{r}=(x,y,z)$ transforms as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

   Or

$$r'_i = R^{(z)}_{ij}(\theta)\, r_j$$

   Suppose $\theta \ll 1$, and write

$$R^{(z)}_{ij}(\theta) \approx 1 - i\theta L_z$$

   (a) Find the matrix L_z and its eigenvectors and eigenvalues.

   (b) What are the relations between the eigenstates found in part (a) with those eigenstates of J_z given in the Note 1.

   (c) Find the matrices L_x, L_y corresponding to infinitesmal rotations aroud $x-$ and $y-$axises and compute the commutators,

$$[L_x, L_y], \qquad [L_x, L_z], \qquad [L_y, L_z]$$

   (d) For any function $f(x,y,z)$ find the relation between $f(x',y',z')$ and $f(x,y,z)$ for infinitesmal rotation about each of 3 axises.

2. In the usual 3-dimensional space, the coordinate vector $\vec{r}=(x_1,x_2,x_3)$ transforms under rotation as

$$r'_i = R_{ij} r_j, \qquad \text{where} \quad RR^T = R^T R = 1$$

   The matrix given in previous problem is one such example.

   The tensors are those quantanties which transform like product of vectors,

$$T'_{i_1 i_2 \ldots i_n} = R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_n j_n} T_{j_1 j_2 \cdots j_n}, \qquad nth \;\; rank \;\; tensor$$

   (a) Show that the partial derivatives $\dfrac{\partial}{\partial x_i}$ transform in the same way as coordinate vector x_i under the rotation.

   (b) Show that if T_{ij} is a second rank tensor, so are the combinations $(T_{ij}+T_{ji})$ and $(T_{ij}-T_{ji})$.

   (c) Show that the product $x_i \dfrac{\partial}{\partial x_j}$ transform as 2nd rank tensor.

3. Consider a charge particle moving in electormagnetic field non-relativistically.

   (a) Show that the Lagrangian given by

$$L = \frac{1}{2} m \left(\vec{v}\right)^2 + e\vec{A}\cdot\vec{v} - eA_0$$

   gives, through Euler-Lagrange equations, the equation of motion. What happens to this Lagrangian if we perform a gauge transformation ? Discuss the consequence.

   (b) Find the Hamiltonian for this system.

   (c) Show that for the case where particle is moving relativistically, the Lagrangian given by

$$L = -mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}} + e\vec{A}\cdot\vec{v} - eA_0$$

   will give the correctin equation of motion. Write the equation of motion in terms of tensors in Minkowski space. Find the corresponding Hamiltonian.

   (d) Repeat the calculation for part (c) for the case spatial dimension is 2 rather than 3.