Quantum Field Theory II
Homework set 2, Due Tu Apr 12

1. In Yang-Mills theory with isospin $SU(2)$ symmetry, consider a $SU(2)$ doublet of the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

which transform as

$$\psi \to \psi' = U(\theta)\psi, \qquad \text{with} \quad U(\theta) = \exp\left(-i\frac{\vec{\tau}\cdot\vec{\theta}(x)}{2}\right)$$

The covariant derivative is of the form

$$D_\mu\psi = \left(\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\right)\psi$$

Under the gauge transformation we have

$$\frac{\vec{\tau}\cdot\vec{A'}_\mu}{2} = U\left(\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\right)U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}$$

Show that $\vec{A}_\mu$transform as $I = 1$ triplet if U is a global transformation.

2. Suppose the set of scalar fields $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ transform as a vector in $O(3)$. The self interaction of $\vec{\phi}$ is of the form,

$$L = \frac{1}{2}\left(\partial_\mu\vec{\phi}\right)^2 - V\left(\vec{\phi}\right)$$

where

$$V\left(\vec{\phi}\right) = -\frac{\mu^2}{2}\left(\vec{\phi}\cdot\vec{\phi}\right) + \frac{\lambda}{4}\left(\vec{\phi}\cdot\vec{\phi}\right)^2$$

   (a) Find the minimum of $V\left(\vec{\phi}\right)$.

   (b) Find the combinations of $\phi_i's$, which are Goldstone bosons.

3. The $SU(n)$ group consist of $n\times n$ unitary unimodular matrices, $UU^\dagger = U^\dagger U = 1$. For infinitesmal transformation, we can write

$$U_{jk} = \delta_{jk} + \varepsilon_{jk}$$

where ε is a Hermitian matrix,

$$\varepsilon_{jk} = \varepsilon^*_{kj}$$

It is more convenient to use upper or lower indices so that

$$\varepsilon_{jk} \equiv \varepsilon_j^{\ k}$$

and complex conjugation interchanges upper and lower indices,

$$\varepsilon_j^{\ k} = \left(\varepsilon^j_{\ k}\right)^*$$

Then the hermiticity condition becomes

$$\varepsilon_j^{\ k} = \varepsilon^k_{\ j}$$

The n-dimensionalvector ϕ_i and its complex conjugate ϕ^i have the following transformation,

$$\phi_i \longrightarrow \phi_i' = \phi_i + i\varepsilon_i^{\ k}\phi_k, \qquad \phi^i \longrightarrow \phi'^i = \phi^i - i\varepsilon^i_{\ j}\phi^k$$

For the fields in ajoint representation ϕ_i^j, we have

$$\phi_i^j \longrightarrow \phi_i'^{\ j} = \phi_i^j + i\varepsilon_i^{\ k}\phi_k^j - i\varepsilon^j_{\ k}\phi_i^k$$

(a) Construct the covariant derivative for ϕ_i and ϕ^i, respectively. Show that the transformation for the gauge field boson is

$$W_{\mu i}^{\ j} \longrightarrow W_{\mu i}'^{\ j} = W_{\mu i}^{\ j} + i\varepsilon_i^{\ k} W_{\mu k}^{\ j} - i\varepsilon^j_{\ k} W_{\mu i}^{\ k} - \frac{1}{g}\partial_\mu \varepsilon_i^{\ j}$$

(b) Construct the field strength tensor $F^i_{\mu\nu j}$ for the gauge field $W_{\mu i}^{\ j}$.

(c) Construct the covariant derivative for scalar fields in the adjoint representation.

4. Consider the Lagrangian for scalar QED with Higgs phenomena,

$$\mathcal{L} = (D_\mu\phi)^\dagger (D^\mu\phi) + \frac{\mu^2}{2}\phi^\dagger\phi - \frac{\lambda}{4}\left(\phi^\dagger\phi\right)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

with

$$(D^\mu\phi) = (\partial^\mu - ieA^\mu)\phi, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Consider the static case where $\partial_0\phi = \partial_0\vec{A} = 0$ and $A_0 = 0$.

(a) Show that the equation of motion for $\vec{A}$ is of the form,

$$\vec{\nabla} \times \vec{B} = \vec{J}, \qquad \text{with } \vec{J} = ie\left[\phi^\dagger(\vec{\nabla} - ie\vec{A})\phi - ((\vec{\nabla} + ie\vec{A})\phi^\dagger)\phi\right]$$

(b) Show that with spontaneous symmetry breaking, in the classical approximation $\phi = v = \sqrt{\frac{\mu^2}{\lambda}}$, the current is of the form

$$\vec{J} = e^2v^2\vec{A}, \qquad \text{London equation}$$

and thus

$$\nabla^2\vec{B} = e^2v^2\vec{B}, \qquad \text{Meissner effect}$$

(c) The resistivity ρ for the system is defined as

$$\vec{E} = \rho\vec{J}$$

Show that, in the case of spontaneous symmetry breaking, $\rho = 0$, and we have superconductivity.

5. The $\sigma-$model Lagrangian is of the form,

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\sigma^2 + (\partial_\mu\vec{\pi})^2] + \bar{N}i\gamma^\mu\partial_\mu N + g\bar{N}(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)N - V(\sigma^2 + \vec{\pi}^2)$$

with

$$V(\sigma^2 + \vec{\pi}^2) = -\frac{\mu^2}{2}(\sigma^2 + \vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2$$

(a) Show that it has the following symmetries, in the infinitesmal forms,

$$\begin{cases} \sigma \longrightarrow \sigma' = \sigma \\ \vec{\pi} \longrightarrow \vec{\pi}' = \vec{\pi} + i\vec{\alpha} \times \vec{\pi} \\ N \longrightarrow N' = N - i\dfrac{\vec{\alpha}\cdot\vec{\tau}}{2}N \end{cases}$$

and

$$\begin{cases} \sigma \longrightarrow \sigma' = \sigma + \vec{\beta}\cdot\vec{\pi} \\ \vec{\pi} \longrightarrow \vec{\pi}' = \vec{\pi} - \vec{\beta}\sigma \\ N \longrightarrow N' = N - i\dfrac{\vec{\beta}\cdot\vec{\tau}}{2}\gamma_5 N \end{cases}$$

where $\vec{\alpha}$ and $\vec{\beta}$ are arbitrary parameters.

(b) Compute the charges corresponding to these symmetries and their commutators.