1. In Yang-Mills theory with isospin \(SU(2)\) symmetry, consider a \(SU(2)\) doublet of the form
\[
\psi = \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right)
\]
which transform as
\[
\psi \rightarrow \psi' = U(\theta) \psi, \quad \text{with} \quad U(\theta) = \exp \left( -i \frac{\vec{\tau} \cdot \vec{A}_\mu}{2} \right)
\]
The covariant derivative is of the form
\[
D_\mu \psi = \left( \partial_\mu - ig \frac{\vec{\tau} \cdot \vec{A}_\mu}{2} \right) \psi
\]
Under the gauge transformation we have
\[
\frac{\vec{\tau} \cdot \vec{A}'_\mu}{2} = U \left( \frac{\vec{\tau} \cdot \vec{A}_\mu}{2} \right) U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1}
\]
Show that \(\vec{A}_\mu\) transform as \(I = 1\) triplet if \(U\) is a global transformation.

2. Suppose the set of scalar fields \(\vec{\phi} = (\phi_1, \phi_2, \phi_3)\) transform as a vector in \(O(3)\). The self interaction of \(\vec{\phi}\) is of the form,
\[
L = \frac{1}{2} \left( \partial_\mu \vec{\phi} \right)^2 - V \left( \vec{\phi} \right)
\]
where
\[
V \left( \vec{\phi} \right) = -\frac{\mu^2}{2} \left( \vec{\phi} \cdot \vec{\phi} \right) + \frac{\lambda}{4} \left( \vec{\phi} \cdot \vec{\phi} \right)^2
\]
(a) Find the minimum of \(V \left( \vec{\phi} \right)\).
(b) Find the combinations of \(\phi'_i\)s, which are Goldstone bosons.

3. The \(SU(n)\) group consist of \(n \times n\) unitary unimodular matrices, \(UU^\dagger = U^\dagger U = 1\). For infinitesimal transformation, we can write
\[
U_{jk} = \delta_{jk} + \varepsilon_{jk}
\]
where \(\varepsilon\) is a Hermitian matrix,
\[
\varepsilon_{jk} = \varepsilon_{kj}^*
\]
It is more convenient to use upper or lower indices so that
\[
\varepsilon_{jk} \equiv \varepsilon^k_j
\]
and complex conjugation interchanges upper and lower indices,
\[
\varepsilon^k_j = (\varepsilon^j_k)^*
\]
Then the hermiticity condition becomes
\[
\varepsilon^k_j = \varepsilon^k_j
\]
The \(n\)-dimensional vector \(\phi_i\) and its complex conjugate \(\phi^j\) have the following transformation,
\[
\phi_i \rightarrow \phi'_i = \phi_i + i \varepsilon^k_i \phi_k, \quad \phi^j \rightarrow \phi'^j = \phi^j - i \varepsilon^j_i \phi_i
\]
For the fields in a joint representation \(\phi_i^j\), we have
\[
\phi_i^j \rightarrow \phi'_i^j = \phi_i^j + i \varepsilon^k_i \phi_k^j - i \varepsilon^j_i \phi_i^k
\]
(a) Construct the covariant derivative for \( \phi_i \) and \( \phi_i' \), respectively. Show that the transformation for the gauge field boson is

\[
W_{\mu i}^j \rightarrow W_{\mu i}^j = W_{\mu i}^j + i\varepsilon_i^k W_{\mu}^k - i\varepsilon_i^j W_{\mu}^k - \frac{1}{g} \partial_{\mu} \varepsilon_i^j
\]

(b) Construct the field strength tensor \( F_{\mu\nu}^i \) for the gauge field \( W_{\mu i}^j \).

(c) Construct the covariant derivative for scalar fields in the adjoint representation.

4. Consider the Lagrangian for scalar QED with Higgs phenomena,

\[
L = (D_{\mu} \phi)^\dagger (D^{\mu} \phi) + \frac{\mu^2}{2} \phi \phi - \frac{\lambda}{4} \left( \phi^\dagger \phi \right)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
\]

with

\[
(D^{\mu} \phi) = (\partial^{\mu} - ieA^{\mu}) \phi, \quad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}
\]

Consider the static case where \( \partial_0 \phi = \partial_0 A = 0 \) and \( A_0 = 0 \).

(a) Show that the equation of motion for \( \vec{A} \) is of the form,

\[
\vec{\nabla} \times \vec{B} = \vec{J}, \quad \text{with} \quad \vec{J} = ie \left[ \phi^\dagger (\vec{\nabla} - ie\vec{A}) \phi - ((\vec{\nabla} + ie\vec{A}) \phi^\dagger ) \phi \right]
\]

(b) Show that with spontaneous symmetry breaking, in the classical approximation \( \phi = v = \sqrt{\frac{\mu^2}{\lambda}} \), the current is of the form

\[
\vec{J} = e^2 v^2 \vec{A}, \quad \text{London equation}
\]

and thus

\[
\nabla^2 \vec{B} = e^2 v^2 \vec{B}, \quad \text{Meissner effect}
\]

(c) The resistivity \( \rho \) for the system is defined as

\[
\vec{E} = \rho \vec{J}
\]

Show that, in the case of spontaneous symmetry breaking, \( \rho = 0 \), and we have superconductivity.

5. The \( \sigma - \) model Lagrangian is of the form,

\[
L = \frac{1}{2} [ (\partial_\mu \sigma^2 + (\partial_\mu \pi)^2) + \bar{N} i \gamma^\mu \partial_\mu N + g \bar{N} (\sigma + i \bar{\tau} \cdot \pi \gamma_5) N - V(\sigma^2 + \pi^2) ]
\]

with

\[
V(\sigma^2 + \pi^2) = -\frac{\mu^2}{2} (\sigma^2 + \pi^2) + \frac{\lambda}{4} (\sigma^2 + \pi^2)^2
\]

(a) Show that it has the following symmetries, in the infinitesimal forms,

\[
\begin{align*}
\sigma &\rightarrow \sigma' = \sigma \\
\bar{\pi} &\rightarrow \bar{\pi}' = \bar{\pi} + i \vec{\alpha} \times \bar{\pi} \\
N &\rightarrow N' = N - i \frac{\vec{\alpha} \cdot \vec{\tau}}{2} N
\end{align*}
\]

and

\[
\begin{align*}
\sigma &\rightarrow \sigma' = \sigma + \vec{\beta} \cdot \bar{\pi} \\
\bar{\pi} &\rightarrow \bar{\pi}' = \bar{\pi} - \vec{\beta} \sigma \\
N &\rightarrow N' = N - i \frac{\vec{\beta} \cdot \vec{\tau}}{2} \gamma_5 N
\end{align*}
\]

where \( \vec{\alpha} \) and \( \vec{\beta} \) are arbitrary parameters.

(b) Compute the charges corresponding to these symmetries and their commutators.