Quantum Field Theory II

May 20, 2011

Homework set 4, Due Thu June 9

1. In the parton model , if we assume that the proton quark sea has the same number of up and down quark pairs, i.e. in terms of the antiquark densiy, $\bar{u}(x) = \bar{d}(x)$, show that

$$\int_0^1 \frac{dx}{x}\left[F_2^p(x) - F_2^n(x)\right] = \frac{1}{3}$$

2. Consider a free scalar field theory. Show that the matrix element,

$$\langle p\left|\phi(x+\varepsilon)\,\phi(x-\varepsilon)\right|p\rangle$$

is singular as $\varepsilon \to 0$ while

$$\langle p\left|:\phi(x+\varepsilon)\,\phi(x-\varepsilon):\right|p\rangle$$

is finite in the same limit.

3. The propagator for a massless scalar field can be written in the form,

$$\Delta_F(x) = \int \frac{d^4x}{(2\pi)^4}\frac{e^{ik\cdot x}}{k^2 + i\varepsilon}$$

Carrying out the integration to show that

$$\Delta_F(x) = \frac{i}{4\pi}\frac{1}{x^2 - i\varepsilon}$$

4. Consider the process $e^+e^- \longrightarrow hadrons$ through one-photon annilation.

   (a) Show that the total hadronic cross section (summing over hadronic final states) can be written as

$$\sigma_{total}\left(e^+e^- \longrightarrow hadrons\right) = \frac{8\pi^2\alpha^2}{3\left(q^2\right)^2}\int d^4x e^{iq\cdot x}\langle 0\left|\left[J_\mu(x), J^\mu(0)\right]\right|0\rangle$$

   where $J_\mu(x)$ is the electromagnetic current and q^μ the 4-momentum of the intermediate photon.

   (b) Suppose that $J_\mu(x)$ is made out of free quarks:

$$J_\mu(x) =: \bar{q}(x)\,\gamma_\mu Q q(x) := \sum_i :\bar{q}_i(x)\,\gamma_\mu e_i q_i(x):$$

   where Q is the charge matrix and i is the flavor index, calculate the commutator

$$\left[J_\mu(x), J^\mu(0)\right]$$

   and show that

$$\sigma_{total}\left(e^+e^- \longrightarrow hadrons\right) = \frac{4\pi\alpha^2}{3\left(q^2\right)^2}Tr\left(Q^2\right)$$

   (c) Suppose that the current $J_\mu(x)$ is made out of free elementary scalar fields,

$$J_\mu(x) =: \sum_{i,j}\left[\phi_i^\dagger Q_{ij}\partial_\mu\phi_j - \partial_\mu\phi_i^\dagger Q_{ij}\phi_j\right]$$

   calculate the commutator

$$\left[J_\mu(x), J^\mu(0)\right]$$

   and

$$\sigma_{total}\left(e^+e^- \longrightarrow hadrons\right)$$

5. Suppose hermitian and traceless matrices $\frac{\lambda^a}{2}$, $a = 1, 2, \cdots, n^2 - 1$, are the generators of $SU(n)$ Lie algebra,

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = iC^{abc}\frac{\lambda^c}{2}$$

where C^{abc} are the structure constants of the algebra.

(a) Show that

$$\sum_d (\lambda^d)_{ij}(\lambda^d)_{kl} = 2\left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right)$$

(b) Show that the matrices defined by

$$(T^a)_{bc} = iC^{abc}$$

satisfy the same algebra

$$\left[T^a, T^b\right] = iC^{abc}T^c$$

(Hint: use Jacobi identity)

(c) Show that

$$\sum_a \left(T^a T^b\right) = C_2 \delta^{ab}$$

and

$$C_2 = n$$