

# 1 Group Theory

The most useful tool for studying symmetry is the group theory. We will give a simple discussion of the parts of the group theory which are commonly used in high energy physics.

## 1.1 Elements of group theory

A group  $G$  is a collection of elements ( $a, b, c, \dots$ ) with a multiplication laws having the following properties;

1. Closure.            If  $a, b \in G$  ,  $c = ab \in G$
2. Associative         $a(bc) = (ab)c$
3. Identity             $\exists e \in G \ni a = ea = ae \quad \forall a \in G$
4. Inverse            For every  $a \in G$  ,  $\exists a^{-1} \ni aa^{-1} = e = a^{-1}a$

Examples of groups frequently used in physics are :

1. **Abelian group** — group multiplication commutes, i.e.  $ab = ba \quad \forall a, b \in G$   
e.g. cyclic group of order  $n$ ,  $Z_n$ , consists of  $a, a^2, a^3, \dots, a^n = E$
2. **Orthogonal group** —  $n \times n$  orthogonal matrices,  $RR^T = R^T R = 1$ ,  
 $R : n \times n$  matrix  
e. g. the matrices representing rotations in 2-dimensions,

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

3. **Unitary group** —  $n \times n$  unitary matrices,

We can build larger groups from smaller ones by direct product:

**Direct product group** — Given any two groups ,  $G = \{g_1, g_2, \dots\}$ ,  $H = \{h_1, h_2, \dots\}$  and if  $g$ 's commute with  $h$ 's we can define a direct product group by  $G \times H = \{g_i h_j\}$  with multiplication law

$$(g_i h_j)(g_m h_n) = (g_i g_m)(h_j h_n)$$

## 1.2 Theory of Representation

Consider a group  $G = \{g_1 \cdots g_n \cdots\}$ . If for each group element  $g_i$ , there is an  $n \times n$  matrix  $D(g_i)$  such that it preserves the group multiplication, i.e.

$$D(g_1)D(g_2) = D(g_1g_2) \quad \forall \quad g_1, g_2 \in G$$

then  $D$ 's forms a representation of the group  $G$  ( $n$ -dimensional representation). In other words,  $g_i \longrightarrow D(g_i)$  is a homomorphism. If there exists a non-singular matrix  $M$  such that all matrices in the representation can be transformed into block diagonal form,

$$MD(a)M^{-1} = \begin{pmatrix} D_1(a) & 0 & 0 \\ 0 & D_2(a) & 0 \\ 0 & 0 & \ddots \end{pmatrix} \quad \text{for all } a \in G.$$

$D(a)$  is called reducible representation. If representation is not reducible, then it is irreducible representation (irrep)

**Continuous group:** groups parametrized by set of continuous parameters

Example: Rotations in 2-dimensions can be parametrized by  $0 \leq \theta < 2\pi$

## 1.3 SU(2) group

Set of  $2 \times 2$  unitary matrices with determinant 1 is called  $SU(2)$  group.

In general,  $n \times n$  unitary matrix  $U$  can be written as

$$U = e^{iH} \quad H : n \times n \text{ hermitian matrix}$$

From

$$\det U = e^{iTrH}$$

we get

$$TrH = 0 \quad \text{if} \quad \det U = 1$$

Thus  $n \times n$  unitary matrices  $U$  can be written in terms of  $n \times n$  traceless Hermitian matrices.

Note that Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a complete set of  $2 \times 2$  hermitian traceless matrices. We can use them to describe  $SU(2)$  matrices.

Define  $J_i = \frac{\sigma_i}{2}$ . We can compute the commutators

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2$$

This is the *Lie* algebra of  $SU(2)$  symmetry. This is exactly the same as the commutation relation of angular momentum.

Recall that to construct the irrep of  $SU(2)$  algebra, we define

$$J^2 = J_1^2 + J_2^2 + J_3^2, \quad \text{with property } [J^2, J_i] = 0, \quad i = 1, 2, 3$$

Also define

$$J_{\pm} \equiv J_1 \pm iJ_2 \quad \text{then} \quad J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2 \quad \text{and} \quad [J_+, J_-] = 2J_3$$

For convenience, we choose simultaneous eigenstates of  $J^2, J_3$ ,

$$J^2|\lambda, m\rangle = \lambda|\lambda, m\rangle, \quad J_3|\lambda, m\rangle = m|\lambda, m\rangle$$

From

$$[J_+, J_3] = -J_+$$

we get

$$(J_+J_3 - J_3J_+)|\lambda, m\rangle = -J_+|\lambda, m\rangle$$

Or

$$J_3(J_+|\lambda, m\rangle) = (m+1)(J_+|\lambda, m\rangle)$$

Thus  $J_+$  raises the eigenvalue from  $m$  to  $m+1$  and is called *raising operator*. Similarly,  $J_-$  lowers  $m$  to  $m-1$ ,

$$J_3(J_-|\lambda, m\rangle) = (m-1)(J_-|\lambda, m\rangle)$$

Since

$$J^2 \geq J_3^2, \quad \lambda - m^2 \geq 0$$

we see that  $m$  is bounded above and below. Let  $j$  be the largest value of  $m$ , then

$$J_+|\lambda, j\rangle = 0$$

Then

$$0 = J_-J_+|\lambda, j\rangle = (J_3^2 - J_3^2 - J_3)|\lambda, j\rangle = (\lambda - j^2 - j)|\lambda, j\rangle$$

and

$$\lambda = j(j+1)$$

Similarly, let  $j'$  be the smallest value of  $m$ , then

$$J_-|\lambda, j'\rangle = 0 \quad \lambda = j'(j' - 1)$$

Combining these 2 relations, we get

$$j(j+1) = j'(j'-1) \Rightarrow j' = -j \text{ and } j - j' = 2j = \text{integer}$$

We will use  $j, m$  to label the states. Assume that the states are normalized,

$$\langle jm | jm' \rangle = \delta_{mm'}$$

Write

$$J_{\pm}|jm\rangle = C_{\pm}(jm)|j, m \pm 1\rangle$$

then

$$\begin{aligned} \langle jm | J_- J_+ | jm \rangle &= |C_+(j, m)|^2 \\ LHS &= \langle j, m | (J^2 - J_3^2 - J_3) | jm \rangle = j(j+1) - m^2 - m \end{aligned}$$

This gives

$$C_+(j, m) = \sqrt{(j-m)(j+m+1)}$$

Similarly

$$C_-(j, m) = \sqrt{(j+m)(j-m+1)}$$

Summary: eigenstates  $|jm\rangle$  have the properties

$$J_3|j, m\rangle = m|j, m\rangle \quad J_{\pm}|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|jm \pm 1\rangle, \quad J^2|j, m\rangle = j(j+1)|jm\rangle$$

$|j, m\rangle$ ,  $m = -j, -j+1, \dots, j$  are the basis for irreducible representation of  $SU(2)$  group. From these relations we can construct the representation matrices. We will illustrate these by following examples.

Example:  $j = \frac{1}{2}$ ,  $m = \pm\frac{1}{2}$

$$J_3 = \left| \frac{1}{2}, \pm\frac{1}{2} \right\rangle \left\langle \pm\frac{1}{2} \left| \frac{1}{2}, \pm\frac{1}{2} \right. \right\rangle$$

$$J_+ \left| \frac{1}{2}, \frac{1}{2} \right\rangle = 0, \quad J_+ \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle, \quad J_- \left| \frac{1}{2}, \frac{1}{2} \right\rangle = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle, \quad J_- \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = 0$$

If we write

$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle = \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then we can represent  $J'$ s by matrices,

$$J_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J_1 = \frac{1}{2}(J_+ + J_-) = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{2i}(J_+ - J_-) = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Within a factor of  $\frac{1}{2}$ , these are just Pauli matrices

### Product representation

Let  $\alpha$  be the spin-up and  $\beta$  the spin-down states. Then for 2 spin  $\frac{1}{2}$  particles, the total wavefunction is product of wavefunctions of the form,  $\alpha_1\alpha_2, \alpha_1\beta_2 \dots$

Define  $\vec{J}^{(1)}$  acts only on particle 1 and  $\vec{J}^{(2)}$  acts only on particle 2.

$$\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$$

Use

$$J_3 = J_3^{(1)} + J_3^{(2)} \quad , \quad J_3(\alpha_1\alpha_2) = (J_3^{(1)} + J_3^{(2)})(\alpha_1\alpha_2) = (\alpha_1\alpha_2)$$

From

$$\vec{J}^2 = (\vec{J}^{(1)} + \vec{J}^{(2)})^2 = (\vec{J}^{(1)})^2 + (\vec{J}^{(2)})^2 + 2[\frac{1}{2}(J_+^{(1)}J_-^{(2)} + J_-^{(1)}J_+^{(2)} + J_3^{(1)}J_3^{(2)})]$$

$$\vec{J}^2(\alpha_1\alpha_2) = (\frac{3}{4} + \frac{3}{4} + \frac{2}{4})|\alpha_1\alpha_2\rangle = 2|\alpha_1\alpha_2\rangle$$

This means  $|1, 1\rangle = \alpha_1\alpha_2$  is a  $j = 1$  state. To get other  $j = 1$  states, we can use the lowering operator

$$J_-(\alpha_1\alpha_2) = (J_-^{(1)} + J_-^{(2)})(\alpha_1\alpha_2) = (\beta_1\alpha_2 + \alpha_1\beta_2)$$

On the other hand

$$\begin{aligned} J_-(\alpha_1\alpha_2) &= J_-|11\rangle = \sqrt{(1+1)(1-1+1)}|1, 0\rangle = \sqrt{2}|1, 0\rangle \\ \Rightarrow |1, 0\rangle &= \frac{1}{\sqrt{2}}(\beta_1\alpha_2 + \alpha_1\beta_2) \end{aligned}$$

The only state left-over is

$$\frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2)$$

This is a  $|0, 0\rangle$  state.

Summary:

1. Among the generator only  $J_3$  is diagonal, —  $SU(2)$  is a rank-1 group
2. Irreducible representation is labeled by  $j$  and the dimension is  $2j + 1$
3. Basis states  $|j, m\rangle$   $m = j, j - 1 \dots (-j)$  representation matrices can be obtained from

$$J_3|j, m\rangle = m|j, m\rangle \quad J_{\pm}|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

## 1.4 SU(2) and rotation group

The generators of  $SU(2)$  group are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let  $\vec{r} = (x, y, z)$  be arbitrary vector in  $R_3$  (3 dimensional coordinate space). Define a  $2 \times 2$  matrix  $h$  by

$$h = \vec{\sigma} \cdot \vec{r} = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

$h$  has the following properties

1.  $h^\dagger = h$
2.  $Tr h = 0$
3.  $\det h = -(x^2 + y^2 + z^2)$

Let  $U$  be a  $2 \times 2$  unitary matrix with  $\det U = 1$ . Consider the transformation

$$h \rightarrow h' = U h U^\dagger$$

Then we have

1.  $h'^\dagger = h'$
  2.  $Tr h' = 0$
  3.  $\det h' = \det h$
- , (3)

Properties (1)&(2) imply that  $h'$  can also be expanded in terms of Pauli matrices

$$h' = \vec{r}' \cdot \vec{\sigma} = (x', y', z')$$

$$\det h' = \det h \Rightarrow x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$$

Thus relation between  $\vec{r}$  and  $\vec{r}'$  is a rotation. This means that an arbitrary  $2 \times 2$  unitary matrix  $U$  induces a rotation in  $R_3$ . This provides a connection between  $SU(2)$  and  $O(3)$  groups.

## 1.5 Rotation group & QM

Rotation in  $R_3$  can be represented as linear transformations on

$$\vec{r}(x, y, z) = (r_1, r_2, r_3) \quad , \quad r_i \rightarrow r'_i = R_{ij}r_j \quad RR^T = 1 = R^T R$$

Consider an arbitrary function of coordinates,  $f(\vec{r}) = f(x, y, z)$ . Under the rotation, the change in  $f$

$$f(r_i) \rightarrow f(R_{ij}r_j) = f'(r_i)$$

If  $f = f'$  we say  $f$  is invariant under rotation, eg  $f(r_i) = f(r)$ ,  $r = \sqrt{x^2 + y^2 + z^2}$

In QM, we implement the rotation by

$$|\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle, \quad O \rightarrow O' = UOU^\dagger$$

so that

$$\Rightarrow \langle\psi'|O'|\psi'\rangle = \langle\psi|O|\psi\rangle$$

If  $O' = O$ , we say the operator  $O$  is invariant under rotation

$$\rightarrow UO = OU \quad [O, U] = 0$$

In terms of infinitesimal generators, we have

$$U = e^{-i\theta\vec{n}\cdot\vec{J}/\hbar}$$

This implies

$$[J_i, O] = 0, \quad i = 1, 2, 3$$

For the case where  $O$  is the Hamiltonian  $H$ , this gives  $[J_i, H] = 0$ . Let  $|\psi\rangle$  be an eigenstate of  $H$  with eigenvalue  $E$ ,

$$H|\psi\rangle = E|\psi\rangle$$

then

$$(J_i H - H J_i)|\psi\rangle = 0 \quad \Rightarrow \quad H(J_i|\psi\rangle) = E(J_i|\psi\rangle)$$

i.e.  $|\psi\rangle$  &  $J_i|\psi\rangle$  are degenerate. For example, let  $|\psi\rangle = |j, m\rangle$  the eigenstates of angular momentum, then  $J_\pm|j, m\rangle$  are also eigenstates if  $|\psi\rangle$  is eigenstate of  $H$ . This means for a given  $j$ , the degeneracy is  $(2j + 1)$ .

## 1.6 Gauge Theory

### 1.6.1 Abelian gauge theory(QED)

Maxwell Equation

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad , \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad , \quad \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J}$$

Source free equations can be solved by introducing  $\vec{A}$  and  $\phi$

$$\vec{B} = \nabla \times \vec{A} \quad , \quad \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

$$\partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu} \quad , \quad \text{with} \quad F^{ij} \sim \epsilon^{ijk} B_k \quad F^{0i} \sim E^i$$

$\vec{E}$  and  $\vec{B}$  are unchanged under the transformation

$$\phi \rightarrow \phi - \frac{\partial \alpha}{\partial t} \quad , \quad \vec{A} \rightarrow \vec{A} + \vec{\nabla} \alpha$$

Or

$$A^\mu \rightarrow A^\mu + \partial^\mu \alpha, \quad \text{with} \quad A^\mu = \left( \frac{\phi}{c}, \vec{A} \right)$$

This is called the **gauge invariance** of electrodynamics. Classically, it not clear as to the physical significance of this invariance. It takes quantum theory to realize its meaning.

Schrodinger Equation for a charged particle is of the form,

$$\left[ \frac{1}{2m} \left( \frac{\hbar}{i} \vec{\nabla} - e\vec{A} \right)^2 - e\phi \right] \psi = i\hbar \frac{\partial \psi}{\partial t}$$

To get the same physics, we need to transform  $\psi$  by

$$\psi \rightarrow e^{ie\alpha/\hbar} \psi, \quad U(1) \quad \text{phase transformation}$$

This provides a **connection between gauge transformation with symmetry transformation**.

Consider the Lagrangian for a free electron field  $\psi(x)$

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$$



This has global U(1) symmetry,

$$\psi(x) \rightarrow \psi'^{-i\alpha}\psi(x) \quad \alpha : \text{constant}$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = \bar{\psi}(x)e^{i\alpha}$$

Suppose now the phase is space-time dependent,  $\alpha = \alpha(x)$

$$\psi = e^{-i\alpha(x)}\psi(x) \quad , \quad \bar{\psi}'(x) = \bar{\psi}(x)e^{i\alpha(x)}$$

The transformation of derivative is

$$\bar{\psi}(x)\partial_\mu\psi(x) \rightarrow \bar{\psi}'(x)\partial_\mu\psi'(x) = \bar{\psi}(x)\partial_\mu\psi(x) - i(\partial_\mu\alpha)(\bar{\psi}\psi)$$

which is not invariant. Introduce gauge field  $A_\mu(x)$  to form **covariant derivative**

$$D_\mu\psi \equiv (\partial_\mu + igA_\mu)\psi(x)$$

So that  $D_\mu\psi$  transforms the same way as  $\psi$ ,

$$(D_\mu\psi)' = e^{-i\alpha(x)}(D_\mu\psi)$$

This requires that

$$(\partial_\mu + igA'_\mu)\psi' = e^{-i\alpha}(\partial_\mu + igA_\mu)\psi$$

which implies

$$A'_\mu = A_\mu - \frac{1}{g}\partial_\mu\alpha$$

Then

$$\mathcal{L} = \bar{\psi}i\gamma^\mu(\partial_\mu + igA_\mu)\psi - m\bar{\psi}\psi$$

is invariant under local symmetry transformation (**local symmetry**)

The Lagrangian for gauge field is of the form,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

One useful relation is to write  $F_{\mu\nu}$  in terms of covariant derivative,

$$\begin{aligned} D_\mu D_\nu\psi &= (\partial_\mu + igA_\mu)(\partial_\nu + igA_\nu)\psi \\ &= \partial_\mu\partial_\nu\psi - g^2 A_\mu A_\nu\psi + ig(A_\mu\partial_\nu + A_\nu\partial_\mu)\psi + ig(\partial_\mu A_\nu)\psi \end{aligned}$$

Antisymmetrization gives

$$(D_\mu D_\nu - D_\nu D_\mu)\psi = ig(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi = ig(F_{\mu\nu})\psi$$

From

$$[(D_\mu D_\nu - D_\nu D_\mu)\psi]' = e^{-i\alpha}(D_\mu D_\nu - D_\nu D_\mu)\psi$$

we get

$$F'_{\mu\nu} = F_{\mu\nu}$$

The advantage of this relation is that the gauge transformation of  $F_{\mu\nu}$  is automatically determined by the covariant derivative. Thus the Lagrangian of the form

$$\mathcal{L} = \bar{\psi}i\gamma^\mu(\partial_\mu + igA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

is invariant under gauge transformation

$$\psi(x) \rightarrow \psi' = e^{-i\alpha(x)}\psi(x)$$

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x)$$

Remarks:

1.  $A_\mu A^\mu$  term is not gauge invariant  $\Rightarrow$  gauge field massless.
2.  $D_\mu\psi = (\partial_\mu + igA_\mu)\psi \Rightarrow$  minimal coupling determined by U(1) transformation is universal
3. no gauge self coupling because  $A_\mu$  does not carry U(1) charge.

### 1.6.2 Non-Abelian symmetry-Yang Mills fields

1954: Yang-Mills generalized U(1) local symmetry to SU(2) local symmetry.

Consider an isospin doublet  $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

Under SU(2) transformation

$$\psi(x) \rightarrow \psi'(x) = \exp\left\{-\frac{i\vec{\tau} \cdot \vec{\theta}}{2}\right\}\psi(x)$$

where  $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$  are Pauli matrices, with

$$\left[\frac{\tau_i}{2}, \frac{\tau_j}{2}\right] = i\epsilon_{ijk}\left(\frac{\tau_k}{2}\right)$$

Start with free Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi$$

which is invariant under global  $SU(2)$  transformation.

Under local symmetry transformation, we have

$$\psi(x) \rightarrow \psi'(x) = U(\theta)\psi(x) \quad U(\theta) = \exp\left\{-\frac{i\vec{\tau} \cdot \theta(x)}{2}\right\}$$

Derivative term

$$\partial_\mu \psi(x) \rightarrow \partial_\mu \psi'(x) = U \partial_\mu \psi + (\partial_\mu U) \psi$$

is not invariant. Introduce gauge fields  $\vec{A}_\mu$  to form the covariant derivative,

$$D_\mu \psi(x) \equiv (\partial_\mu - ig \frac{\vec{\tau} \cdot \vec{A}_\mu}{2}) \psi$$

Require that

$$[D_\mu \psi]' = U[D_\mu \psi]$$

Or

$$(\partial_\mu - ig \frac{\vec{\tau} \cdot \vec{A}_\mu'}{2})(U\psi) = U(\partial_\mu - ig \frac{\vec{\tau} \cdot \vec{A}_\mu}{2})\psi$$

This gives the transformation of gauge field,

$$\boxed{\frac{\vec{\tau} \cdot \vec{A}_\mu'}{2} = U\left(\frac{\vec{\tau} \cdot \vec{A}_\mu}{2}\right)U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}}$$

We can use covariant derivatives to construct field tensor

$$\begin{aligned} D_\mu D_\nu \psi &= (\partial_\mu - ig \frac{\vec{\tau} \cdot \vec{A}_\mu}{2})(\partial_\nu - ig \frac{\vec{\tau} \cdot \vec{A}_\nu}{2})\psi = \partial_\mu \partial_\nu \psi - ig\left(\frac{\vec{\tau} \cdot \vec{A}_\mu}{2}\partial_\nu \psi + \frac{\vec{\tau} \cdot \vec{A}_\nu}{2}\partial_\mu \psi\right) \\ &\quad - ig\partial_\mu\left(\frac{\vec{\tau} \cdot \vec{A}_\nu}{2}\right)\psi + (-ig)^2\left(\frac{\vec{\tau} \cdot \vec{A}_\mu}{2}\right)\left(\frac{\vec{\tau} \cdot \vec{A}_\nu}{2}\right)\psi \end{aligned}$$

Antisymmetrize this to get the field tensor,

$$(D_\mu D_\nu - D_\nu D_\mu)\psi \equiv ig\left(\frac{\vec{\tau} \cdot \vec{F}_{\mu\nu}}{2}\right)\psi$$

then

$$\frac{\vec{\tau} \cdot \vec{F}_{\mu\nu}}{2} = \frac{\vec{\tau}}{2} \cdot (\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu) - ig\left[\frac{\vec{\tau} \cdot \vec{A}_\mu}{2}, \frac{\vec{\tau} \cdot \vec{A}_\nu}{2}\right]$$

Or in terms of components,

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\epsilon^{ijk} A_\mu^j A_\nu^k$$

The the term quadratic in  $A$  is new in Non-Abelian symmetry. Under the gauge transformation we have

$$\vec{\tau} \cdot \vec{F}_\mu \nu' = U(\vec{\tau} \cdot \vec{F}_\mu \nu) U^{-1}$$

Infinitesimal transformation  $\theta(x) \ll 1$

$$A^{i/\mu} = A^\mu + \epsilon^{ijk} \theta^j A_\mu^k - \frac{1}{g} \partial_\mu \theta^i$$

$$F_{\mu\nu}^{/i} = F_{\mu\nu}^i + \epsilon^{ijk} \theta^j F_{\mu\nu}^k$$

Remarks

1. Again  $A_\mu^a A^{a\mu}$  is not gauge invariant  $\Rightarrow$  gauge boson massless  $\Rightarrow$  long range force
2.  $A_\mu^a$  carries the symmetry charge (e.g. color —)
3. The quadratic term in  $F^{a\mu\nu} \sim \partial A - \partial A + g A A$  is for asymptotic freedom.

### Recipe for the construction of theory with local symmetry

1. Write down a Lagrangian with local symmetry
2. Replace the usual derivative  $\partial_\mu \phi$  by the covariant derivative  $D_\mu \phi \sim (\partial_\mu - ig A_\mu^a t^a) \phi$  where gauge fields  $A_\mu^a$  have been introduced.
3. Use the antisymmetric combination  $(D_\mu D_\nu - D_\nu D_\mu) \phi \sim F_{\mu\nu}^a \phi$  to construct the field tensor  $F_{\mu\nu}^a$  and add  $-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}$  to the Lagrangian density

## 2 Spontaneous symmetry breaking

Spontaneous symmetry breaking—ground state does not have the symmetry of the Hamiltonian

$\Rightarrow$  If the symmetry is continuous one, there will be massless scalar fields—Goldstone boson

Example: ferromagnetism

$T > T_c$  (Curie temp) all dipoles are randomly oriented—rotational invariant

$T < T_c$  all dipoles are oriented in some direction

Ginzburgh-Landau theory

Free energy as function of magnetization  $\vec{m}$  (averaged)

$$\mu(\vec{M}) = (\partial_t \vec{M})^2 + \alpha_1(T) \vec{M} \cdot \vec{M} + \alpha_2(\vec{M} \cdot \vec{M})^2$$

We take  $\alpha_2 > 0$  so that the free energy is positive for large  $M$  and  $\alpha_1(T) = \alpha(T - T_c)$   $\alpha > 0$  so that there is a transition going through Curie temperature  $T_c$ . It is easy to see that the ground state is governed by

$$\vec{M}(\alpha_1 + 2\alpha_2 \vec{M} \cdot \vec{M}) = 0$$

For  $T > T_c$  only solution is  $\vec{M} = 0$  and  $T < T_c$  non-trivial sol  $|\vec{M}| = \sqrt{\frac{\alpha_1}{2\alpha_2}} \neq 0$

$\Rightarrow$  ground state with  $\vec{M}$  in some direction is no longer rotational invariant.

## 2.1 Nambu-Goldstone theorem

Recall that Noether's theorem says that a continuous symmetry will give conserved charge  $Q$ . Suppose there are 2 local operators  $A, B$  with property

$$[Q, B] = A \quad Q = \int d^3x j_0(x) \quad \text{indep of time}$$

Suppose  $\langle 0|A|0\rangle = v \neq 0$  (symmetry breaking condition)

$$\begin{aligned} \Rightarrow 0 \neq \langle 0|[Q, B]|0\rangle &= \int d^3x \times \langle 0|[j_0(x), B]|0\rangle \\ &= \sum_n (2\pi)^3 \delta^3(\vec{P}_n) \{ \langle 0|j_0(0)|n\rangle \langle n|B|0\rangle e^{-iE_n t} - \langle 0|B|n\rangle \langle n|j_0(0)|0\rangle e^{-iE_n t} \} = v \end{aligned}$$

Since  $v \neq 0$  and time-independent, we need a state such that

$$E_n \rightarrow 0 \quad \text{for } \vec{P}_n = 0$$

massless excitation. For the case of relativistic particle with energy momentum rotation  $E = \sqrt{\vec{P}^2 + m^2}$  this implies massless particle- Goldstone boson.

### Discrete symmetry case

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4, \quad \phi \rightarrow -\phi \quad \text{symmetry}$$

The Hamiltonian density

$$H = \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\vec{\nabla} \phi)^2 + \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

Effective energy

$$\mu(\phi) = \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi) \quad , \quad V(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

For  $\mu^2 < 0$  the ground state has  $\phi = \pm\sqrt{\frac{-\mu^2}{\lambda}}$  classically.

This means the quantum ground state  $|0\rangle$  will have the property

$$\langle 0|\phi|0\rangle = \nu \neq 0 \quad \text{symmetry breaking condition}$$

Define quantum field  $\phi'$  by  $\phi' = \phi - \nu$

$$\text{then} \quad \mathcal{L} = \frac{1}{2}(\partial_\mu\phi'^2 - (-\mu^2)\phi'^2 - \lambda\nu\phi'^3 - \frac{\lambda}{4}\phi'^4$$

No Goldstone boson—discrete symmetry

#### Abelian symmetry case

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2] - V(\sigma^2 + \pi^2)$$

with

$$V(\sigma^2 + \pi^2) = -\frac{\mu^2}{2}(\sigma^2 + \pi^2) + \frac{\lambda}{4}(\sigma^2 + \pi^2)^2$$

This system has O(2) symmetry,

$$\begin{pmatrix} \sigma \\ \pi \end{pmatrix} \rightarrow \begin{pmatrix} \sigma' \\ \pi' \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix}$$

The minimum is located at

$$\sigma^2 + \pi^2 = \frac{\mu^2}{\lambda} = \nu^2$$

This is a circle *in  $\sigma-\pi$  plane*. For convenience choose  $\langle 0|\sigma|0\rangle=\nu$   $\langle 0|\pi|0\rangle=0$ .

New quantum field

$$\sigma' = \sigma - \nu \quad , \quad \pi' = \pi$$

The new Lagrangian is

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\sigma'^2 + (\partial_\mu\pi')^2] - \mu^2\sigma'^2 - \lambda\nu\sigma'(\sigma'^2 + \pi'^2) - \frac{\lambda}{4}(\sigma'^2 + \pi'^2)^2$$

Note that there is no  $\pi'^2$  term,  $\Rightarrow \pi'$  massless Goldstone boson

#### Non-Abelian case

$\sigma$ -model

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \sigma^2 + (\partial_\mu \vec{\pi})^2] + \bar{N}i\gamma^\mu \partial_\mu N + g\bar{N}(\sigma + i\vec{\tau} \cdot \vec{\pi}\gamma_5)N - V(\sigma^2 + \vec{\pi}^2)$$

with

$$V(\sigma^2 + \vec{\pi}^2) = -\frac{\mu^2}{2}(\sigma^2 + \vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2$$

It has the following symmetries, in the infinitesimal forms,

$$\left\{ \begin{array}{l} \sigma \longrightarrow \sigma' = \sigma \\ \vec{\pi} \longrightarrow \vec{\pi}' = \vec{\pi} + i\vec{\alpha} \times \vec{\pi} \\ N \longrightarrow N' = N - i\frac{\vec{\alpha} \cdot \vec{\tau}}{2}N \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \sigma \longrightarrow \sigma' = \sigma + \vec{\beta} \cdot \vec{\pi} \\ \vec{\pi} \longrightarrow \vec{\pi}' = \vec{\pi} - \vec{\beta}\sigma \\ N \longrightarrow N' = N - i\frac{\vec{\beta} \cdot \vec{\tau}}{2}\gamma_5 N \end{array} \right.$$

where  $\vec{\alpha}$  and  $\vec{\beta}$  are arbitrary parameters. The symmetry is of the form,  $SU(2) \times SU(2)$ . This Lagrangian has been used in 1960's to describe the interaction between pions and nucleons. Note that the nucleon is massless due to the symmetry under the axial transformation. As for the spontaneous symmetry breaking, it is easy to see that the minimum is located at

$$\sigma^2 + \vec{\pi}^2 = \nu^2 = \frac{\mu^2}{\lambda}$$

If we choose

$$\langle \sigma \rangle = \nu \quad , \quad \langle \vec{\pi} \rangle = 0$$

The  $\vec{\pi}$  are Goldstone bosons. The symmetry is broken from  $SU(2) \times SU(2)$  to  $SU(2)$ . Note that the spontaneous symmetry also give mass to the nucleon,

$$M_N = gv.$$

## 2.2 Higgs Phenomena

When we combine spontaneous symmetry breaking with local symmetry, a very interesting phenomena occurs. This was discovered in the 60's by Higgs, Englert & Brout, Guralnik, Hagen & Kibble independently

Abelian case

Consider the Lagrangian given by

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi \phi^\dagger - \lambda (\phi^\dagger \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where

$$D^\mu \phi = (\partial^\mu - igA^\mu)\phi \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The Lagrangian is invariant under the local gauge transformation

$$\phi(x) \rightarrow \phi' = e^{-i\alpha(x)} \phi(x)$$

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{g} \partial_\mu \alpha(x)$$

The spontaneous symm. breaking is generated by the potential

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

which has a minimum at

$$\phi^\dagger \phi = \frac{\nu^2}{2} = \frac{1}{2} \left( \frac{\mu^2}{\lambda} \right)$$

For the quantum theory, we can choose

$$|\langle 0 | \phi | 0 \rangle| = \frac{\nu}{\sqrt{2}}$$

Or if we write

$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2)$$

With the choice

$$\langle \phi_1 \rangle = \nu \quad , \quad \langle \phi_2 \rangle = 0$$

$\phi_2$  corresponds to Goldstone boson as before. Define the quantum fields by

$$\phi'_1 = \phi_1 - \nu \quad , \quad \phi'_2 = \phi_2$$

Covariant derivative terms gives

$$(D_\mu \phi)^\dagger (D^\mu \phi) = [(\partial_\mu + igA_\mu)\phi^\dagger][(\partial^\mu - igA^\mu)\phi]$$



$$\frac{-1}{2}(\partial_\mu \phi'_1 + g A_\mu \phi'_2)^2 + \frac{1}{2}(\partial_\mu \phi'_2 - g A_\mu \phi'_1)^2 + \frac{g^2 \nu^2}{2} A^\mu A_\mu + \dots \text{ mass terms for } A^\mu$$

Write the scalar field as

$$\phi(x) = \frac{1}{\sqrt{2}}(\nu + \eta(x))e^{i\xi(x)/\nu}$$

"Gauge" transformation:

$$\phi \longrightarrow \phi' = e^{-i\xi(x)/\nu} \phi(x) \quad , \quad B_\mu = A_\mu(x) - \frac{1}{g\nu} \partial_\mu \xi$$

$\xi(x)$  disappears from the Lagrangian

Roughly speaking, massless gauge field  $A_\mu$  combine with Goldstone boson  $\xi(x)$  to become massive gauge boson. As a consequence, two long range forces (from Goldstone boson  $\xi(x)$  and  $A_\mu(x)$ ) disappear.

Non-Abelian case

SU(2) group:  $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$  doublet

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$V(\phi) = -\mu^2 (\phi^\dagger \phi) + \lambda (\phi^\dagger \phi)^2$$

Spontaneous symmetry breaking:

$$\langle \phi \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \nu \end{pmatrix} \quad \nu = \sqrt{\frac{\mu^2}{\lambda}}$$

Define  $\phi' = \phi - \langle \phi \rangle_0$

From covariant derivative

$$\begin{aligned} (D_\mu \phi)^\dagger (D^\mu \phi) &= [\partial_\mu - ig \frac{\vec{\tau} \cdot \vec{A}_\mu}{2} (\phi' + \langle \phi \rangle_0)]^\dagger [\partial^\mu - ig \frac{\vec{\tau} \cdot \vec{A}^\mu}{2} (\phi' + \langle \phi \rangle_0)] \\ &\rightarrow \frac{1}{4} g^2 \langle \phi \rangle_0 (\vec{\tau} \cdot \vec{A}_\mu) (\vec{\tau} \cdot \vec{A}^\mu) \langle \phi \rangle_0 = \frac{1}{2} \left( \frac{g\nu}{2} \right)^2 \vec{A}_\mu \cdot \vec{A}^\mu \end{aligned}$$

All gauge bosons get masses

$$M_A = \frac{1}{2} g \nu$$

The symmetry is completely broken. Write

$$\phi(x) = \exp\left\{ \frac{i \vec{\tau} \cdot \vec{\xi}(x)}{\nu} \right\} \begin{pmatrix} 0 \\ \frac{\nu + \eta(x)}{\sqrt{2}} \end{pmatrix}$$

Use "gauge" transformation

$$\phi'(x) = U(x)\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \nu + \eta(x) \end{pmatrix}$$

$$\frac{\vec{\tau} \cdot \vec{B}_\mu}{2} = U \frac{\vec{\tau} \cdot \vec{A}_\mu}{2} U^{-1} - \frac{i}{g} [\partial_\mu U] U^{-1}$$

$$\text{where } U(x) = \exp\left\{\frac{\vec{\tau} \cdot \vec{\xi}}{\nu}\right\}$$

to transform away  $\vec{\xi}(x)$ .