Author: Ling fong Li

1 Weak Interaction

Classification

Because the strong interaction effects are difficult to compute reliably, we distinguish processe involving hadrons from leptons in weak interactions.

(a) Leptonic weak interactions
Here all particles are leptons and there is no complication from strong interactions. Once the interaction is known we can calculate these processes accurately since the interaction is weak and perturbation theory is applicable.
Examples:

$$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e \qquad \nu_\mu + e \to \nu_\mu + e$$
$$\tau^- \to \mu^- + \nu_\tau + \bar{\nu}_\mu \qquad \tau^- \to e^- + \nu_\tau + \bar{\nu}_e$$

(b) Semi-leptonic interactions
These processes involve both leptons and hadrons. Since we can calculate the leptonic parts quite reliabley, these processes can be used to study the properties of hadrons very similar to the case of ep scattering.
Examples :

$$\pi^- \to \mu^- + \nu_\mu \qquad K^+ \to \pi^0 + e^+ + \nu_e$$
$$n \to p + e^- + \bar{\nu}_e \qquad \bar{\nu}_\mu + p \to \mu^+ + n$$

(c) Non-leptonic interactions
Here all particles are hadrons and they are the most difficult reactions to study because of the strong interaction effects. This class of reactions differ from the normal strong interaction in the slower decay rates and smaller crossections.
Example :

$$K^+ \to \pi^+ + \pi^0 \quad K^0 \to \pi^+ + \pi^- + \pi^0$$
$$\Sigma^+ \to P + \pi^0 \qquad \Lambda \to P + \pi^-$$

1.1 Selection Rules in Weak Interaction

(a) Leptonic Interaction

Two neutrino experiments: ν from β-decay and ν from π decay are different
If they were the same then the following chain of reactions should be possible,

$$n \longrightarrow p + e + \nu$$

$$\nu + p \longrightarrow \mu^+ + n$$

However, only e^+ is observed in the final product and no μ^+ has been seen. A simple explanation that is the neutrino from β-decay called ν_e is different from neutrino from π-decay accompanied by μ called ν_μ and there is also muon number and electron number conservation. In these conservation laws we assign the electron number L_e as

$$e^-, \nu_e \qquad L_e = 1$$
$$e^+, \bar{\nu}_e \qquad L_e = -1$$

Similarly, for the muon number L_μ

$$\mu^-, \nu_\mu \qquad L_\mu = 1$$
$$\mu^+, \bar{\nu}_\mu \qquad L_\mu = -1$$

As a consequenc of these conservation laws, the reaction $\mu^{\pm} \longrightarrow e^{\pm} + \gamma$ are forbidden and experimentally this is indeed the case. Lepton number conservations seem to hold up very well for many years until neutrino oscillations have been observed recently.

(b) Semi-leptonic decays

(a) The strangeness changing reactions, $\Delta S \neq 0$ seem to be about a factor of 10 or so smaller than those which conserve strangeness, $\Delta S = 0$.

(b) It has been observed that hadrons in the strangeness changing decays satisfy the selection rule

$$\Delta S = \Delta Q$$

For examples,

$$K^+ \longrightarrow \pi^0 \mu^+ \nu_\mu, \qquad \text{but} \qquad K^+ \nrightarrow \pi^+ \pi^+ e^- \bar{\nu}_e$$

(c) Absence of $\Delta S = 1$ neutral currents
For example,

$$\frac{\Gamma\left(K_L \to \mu^+ \mu^-\right)}{\Gamma\left(K^+ \to \mu^+ \nu\right)} \leq 10^{-9}$$

(d) No $\Delta S = 2$ transistion been obsverbed
For example,

$$\Xi^- \nrightarrow n e^- \bar{\nu}_e$$

When quark model was developed, all these properties can be accormodated by writing the hadronic weak current as

$$J_\mu^{had} = \left[\bar{u}\gamma_\mu \left(1 - \gamma_5\right) d \cos\theta_c + \bar{u}\gamma_\mu \left(1 - \gamma_5\right) s \sin\theta_c\right]$$

where $\theta_c \approx 0.25$ is the Cabbibo angle.

(c) Non-leptonic interactions
Here we havethe $\Delta I = 1/2$ rule,

$$\frac{\Gamma\left(K^+ \longrightarrow \pi^+ \pi^0\right)}{\Gamma\left(K_s \longrightarrow \pi^+ \pi^-\right)} \simeq 1.5 \times 10^{-3}$$

This rule is very difficult to explain because the strong interaction effect is hard to study accurately.

1.2 Milestones of Weak Interaction

(a) **Neutrino and Nuclear β decay**,
The emission of electron e^- from many different nuclei,

$$(A, Z) \to (A, Z+1) + e^-$$

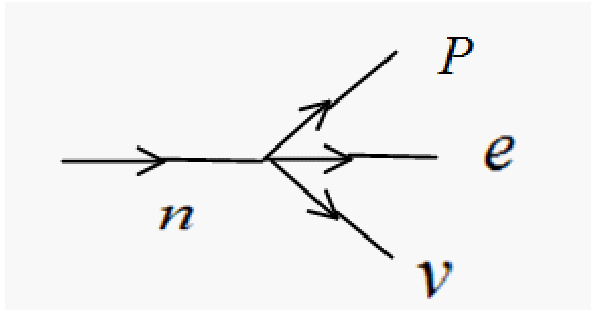
was observed at early study of radioactivity. The unusual feature is that the energy spectrum of e^- is continuous rather than a sharp line as expected in a 2 body decay. If the basic mechanism for e^- emission is

$$n \to p + e^-$$

the energy momentum conservation will require e^- to hav a single energy while experimentally a continuous distribution of energy was observed. Pauli (1930) postulated the presence of **neutrino** which carries away energy and momentum in nuclear β -decay,

$$n \to p + e^- + \bar{\nu}_e$$

so that energy momentum conservation can be saved.



(b) **Fermi Theory**

Fermi (1934) proposed to explain the β decay by making analogy with QED to write down the weak interaction Lagrangian in the form,

$$\mathcal{L}_F = \frac{G_F}{\sqrt{2}}[\bar{p}(x)\gamma_\mu n(x)][\bar{e}(x)\gamma^\mu \nu_e(x)] + h.c. \quad G_F: \text{ Fermi coupling constant}$$

Fitting nuclear β decay reates give

$$G_F \simeq \frac{10^{-5}}{M_p^2}, \qquad M_p \text{ is the proton mass}$$

This theory works very well for $\Delta J = 0$, β decays of many nuclei. Later Gamow-Teller interaction was added

$$\mathcal{L}_{GT} = \frac{-G_F}{\sqrt{2}}[\bar{p}(x)\gamma_\mu\gamma_5 n(x)][\bar{e}(x)\gamma^\mu\gamma_5 \nu_e(x)] + h.c.$$

to account for $\Delta J = 1$ nuclear β decays.

(c) Parity violation and V - A theory

$\theta - \tau$ puzzle

In 1950's, it was observed that there are two decays

$$\theta \to \pi^+ + \pi^-, \qquad \text{(even parity)}$$

$$\tau \to \pi^+ + \pi^- + \pi^0, \qquad \text{(odd parity)}$$

while θ and τ have the same mass, charge and spin. It is very difficult to understand these features if the parity is a good symmetry.

1956 : Lee and Yang proposed that parity is not conserved in weak decays and suggested many experiments to test this hyposethsis.

1957 : C. S. Wu showed that e^- in ^{60}Co decay has the property,

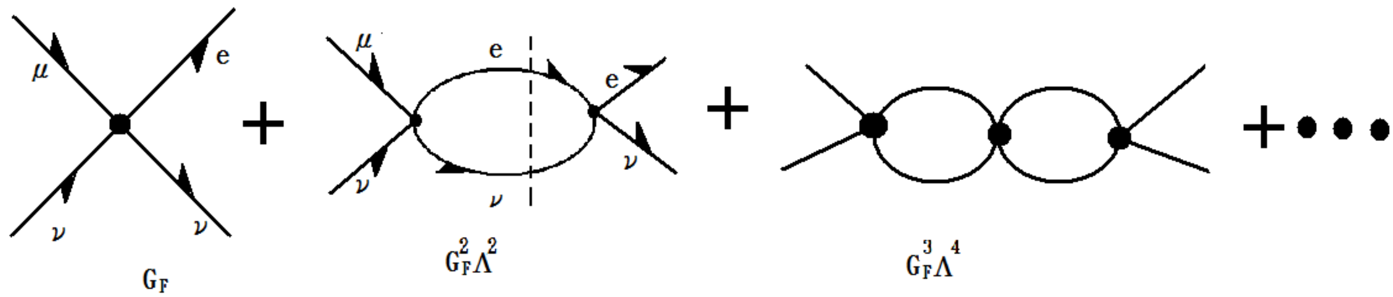
$$\left\langle \vec{\sigma} \cdot \vec{p} \right\rangle \neq 0, \qquad \vec{\sigma}, \ \vec{p} \text{ spin and momentum of } e^-$$

This implies that the parity symmetry is violated in this decay.

(d) V-A theory (1958 Feynman and Gell-Mann, Sudarshan and Marshak, Sakurai)

As a result of parity violation, the effective weak interaction was casted in the form with $V - A$ currents,

$$L_{eff} = \frac{G_F}{\sqrt{2}} J_\mu^\dagger J^\mu + h.c.$$



where

$$J_\lambda(x) = J_{l\,\lambda}(x) + J_{h\,\lambda}(x)$$

$$J_l^\lambda(x) = \bar{\nu}_e\gamma^\lambda(1-\gamma_5)e + \bar{\nu}_\mu\gamma^\lambda(1-\gamma_5)\mu, \qquad \text{leptonic current} \tag{1}$$

and

$$J_h^\lambda(x) = \bar{u}\gamma^\lambda(1-\gamma_5)(\cos\theta_c d + \sin\theta_c s) \qquad \text{hadronic current}$$

$$\theta_c \ : \ \text{Cabibbo angle}$$

Note that in V-A form the fermion fields are all left-handed.
Define

$$\psi_L \equiv \frac{1}{2}(1-\gamma_5)\psi$$

Then we can simplify the form of weak currents,

$$J_l^\lambda(x) = 2\bar{\nu}_{eL}\gamma^\lambda e_L + 2\bar{\nu}_{\mu L}\gamma^\lambda \mu_L + ...$$

Phenomenologically, the V-A theory has been quite successful in most of the weak interactions phenomena.
Difficulties:
(1) **Not renormalizable**

In the Fermi theory, 4 fermions interaction operator has dimension 6 and is not renormalizable. In other words, the higher order graphs are more and more divergent. For example, in μ decay we have
(2) **Violate unitarity**

The tree amplitude for $\nu_\mu + e \to \mu + \nu_e$ has only J = 1 partial wave at high energies and cross section has the form,

$$\sigma(\nu_\mu e) \approx G_F^2 S \ \ S = 2m_e E$$

On the other hand, unitarity for J=1 cross section is

$$\sigma(J=1) < \frac{1}{S}$$

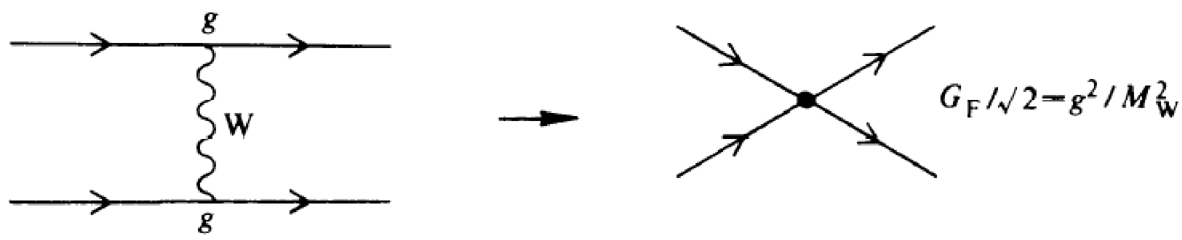
Thus $\sigma(\nu_\mu e)$ violates unitarity for $E \geq 300GeV$. Since unitarity is a consequence of conservation of probablity, this violation is unacceptable.

Intermidate Boson Theory(IVB)

In analogy with QED, we can introduce vector boson W to couple to the V-A current to mediate the weak interaction

$$\mathcal{L}_W = g(J_\mu W^\mu + h.c.)$$

For example, the μ decay is now mediated by W-exchange.
Since weak interaction is short range, we need W-boson to be massive $M_W \neq 0$. Use the massive W-boson



propagator in the form

$$\frac{-g^{\mu\nu}+\dfrac{k^\mu k^\nu}{M_W^2}}{k^2-M_W^2}\rightarrow\frac{g^{\mu\nu}}{M_W^2}\qquad when\quad |k_\mu|\ll M_W$$

We see that this reproduces 4-fermion interaction with $\dfrac{g^2}{M_W^2}=\dfrac{G_F}{\sqrt{2}}$

In this theory, the scattering $\nu_\mu+e\rightarrow\mu+\nu_e$ no longer violates unitarity. But the violation of unitarity shows up in other processes like

$$\nu+\bar{\nu}\rightarrow W^++W^-$$

and the theory is still non-renormalizable.

1.3 Construction of SU(2)×U(1) model

Choice of group

In IVB theory, the interaction is described by

$$\mathcal{L}_W=g(J_\mu W^\mu+h.c)$$

For simplicity we neglect all other fermions excepts ν, e and write the current as

$$J_\mu=\bar{\nu}\gamma_\mu(1-\gamma_5)e$$

Recall that in the electromagnetic interaction, we have

$$\mathcal{L}_{em}=eJ_\mu^{em}A^\mu,\quad \text{where}\quad J_\mu^{em}=\bar{e}\gamma_\mu e$$

Define the eletromagnetic and weak charges as the intergals of the time-component of the currents

$$T_+=\frac{1}{2}\int d^3xJ_0(x)=\frac{1}{2}\int d^3x\nu^\dagger(1-\gamma_5)e\qquad T_-=(T_+)^\dagger$$

$$Q=\int d^3xJ_0^{em}(x)=-\int d^3xe^\dagger e$$

We can compute the commutator $[T_+,T_-]=2T_3$

$$T_3=\frac{1}{4}\int d^3x[\nu^\dagger(1-\gamma_5)\nu-e^\dagger(1-\gamma_5)e]\neq Q$$

This means that these 3 charges, T_+, T_- and Q don't form a $SU(2)$ algebra. The reason is that in order for the eletric charge operator Q to be a generator of $SU(2)$ it has to be traceless. In our case here it is not the case. Futhre more the weak charges $T_\pm$ have the $V-A$ form while the em charge Q is pure vector.

At this point, there are 2 alternatives:

(a) Introduce another guage boson coupled to T_3. The generatros corresponding to these 4 gauge bosons can then form the group $SU(2)\times U(1)$. This will be the choice we will adapt eventually.

(b) We can add new fermions to modify the currents such that T_+, T_- and Q do form a SU(2) algebra (Georgi and Glashow 1972). Here we introduce new fermions to extend the multiplet into triplets

$$\frac{1}{2}(1-\gamma_5)\begin{pmatrix} E^+ \\ \nu_e\cos\alpha + N\sin\alpha \\ e^- \end{pmatrix}$$

$$\frac{1}{2}(1+\gamma_5)\begin{pmatrix} E^+ \\ N \\ e^- \end{pmatrix}$$

and a singlet

$$\frac{1}{2}(1+\gamma_5)(N\cos\alpha - \nu_e\sin\alpha)$$

The weak charge is then

$$\begin{aligned} T_+ &= \frac{1}{2}\int d^3x\left[E^+(1-\gamma_5)(\nu_e\cos\alpha+N\sin\alpha)\right] \\ &\quad +(\nu_e\cos\alpha+N\sin\alpha)(1-\gamma_5)e+E^+(1+\gamma_5)N+N^\dagger(1+\gamma_5)e \end{aligned}$$

It is then straightforward to verify that

$$[T_+, T_-] = 2Q$$

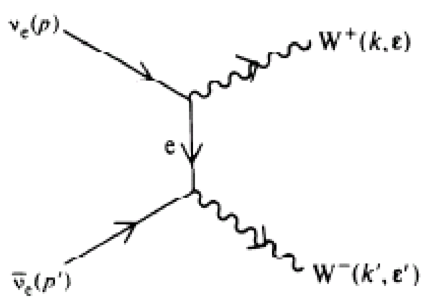
with

$$Q=\int d^3x\left[E^\dagger E - e^\dagger e\right]$$

Clearly, in this model only electromagnetic current is neutral and others all carry charges. When neutral weak current reactions were discovered in 1973, this model is ruled out.

Unitarity argument

Equivalently we can argue from unitarity that it is necessary to introduce either new leptons or a new guage boson. Consider in IVB theory the reaction $\nu+\bar{\nu}\longrightarrow W^+ + W^-$ where both $W's$ are longitudinally polarized. The lowest order amplitude corresponding to the following graph is



$$\begin{aligned} T\left(\nu\bar{\nu}\longrightarrow W^+W^-\right) &= -i\bar{v}(p')\left(-ig\not{\varepsilon}'\right)(1-\gamma_5)\frac{i}{\not{p}-\not{k}-m_e}(-ig\not{\varepsilon})(1-\gamma_5)u(p) \qquad (2)\\ &= -2g^2\bar{v}(p')\frac{\not{\varepsilon}'(\not{p}-\not{k})\not{\varepsilon}(1-\gamma_5)}{(p-k)^2-m_e^2}u(p) \end{aligned}$$

The polarization vectors

$$\varepsilon_\mu^{(i)}(k) \quad \text{with} \quad \varepsilon^{(i)}\cdot\varepsilon^{(j)}=-\delta_{ij} \quad \text{and} \quad k\cdot\varepsilon^{(i)}=0$$

maybe chosen in the rest frame of W boson as

$$\varepsilon_0^{(i)} = 0, \qquad \varepsilon_j^{(i)} = \delta_{ij}$$

For a moving W boson with $k_\mu = (E, 0, 0, k)$ with $k = \sqrt{E^2 - M_w^2}$, we can make a Lorentz transformation along the $z-axis$. The transverse polarizations do not change while the longitudinal polarization becomes $\varepsilon_\mu^{(3)} = \frac{1}{M_W}(k, 0, 0, E)$. In the high energy limit with $k = E - \frac{M_W^2}{2E} + \cdots$, we see that

$$\varepsilon_\mu^{(3)} = \frac{k_\mu}{M_W} + O\left(\frac{M_W}{E}\right)$$

Then for longitudinally polarized W bosons, the scattering amplitude in Eq(2) becomes,

$$\begin{aligned} T &\approx -\frac{2g^2}{k^2 - 2p \cdot k} \overline{v}\left(p'\right) \frac{\not{k}'}{M_W} \left(\not{p} - \not{k}\right) \frac{\not{k}}{M_W} \left(1 - \gamma_5\right) u\left(p\right) \\ &\approx \frac{2g^2}{M_W^2} \overline{v}\left(p'\right) \not{k}' \left(1 - \gamma_5\right) u\left(p\right) \end{aligned} \tag{3}$$

To show more explicitly this amplitude is a pure $J = 1$ partial wave, we take

$$p_\mu = (E, 0, 0, E), \qquad p'_\mu = (E, 0, 0, -E)$$

$$k_\mu = \left(E, k\overrightarrow{e}\right), \qquad k'_\mu = \left(E, -k\overrightarrow{e}\right), \qquad \text{with } \overrightarrow{e} = (\sin\theta, 0, \cos\theta)$$

Since ν and $\overline{\nu}$ have opposite helicities, we have

$$u\left(p\right) = \sqrt{E} \begin{pmatrix} 1 \\ \dfrac{\overrightarrow{\sigma} \cdot \overrightarrow{p}}{E} \end{pmatrix} \chi_{-1/2} = \sqrt{E} \begin{pmatrix} 1 \\ \sigma_z \end{pmatrix} \chi_{-1/2}$$

$$\overline{v}\left(p'\right) = \sqrt{E}\chi_{1/2}^\dagger \left(\frac{\overrightarrow{\sigma} \cdot \overrightarrow{p}'}{E}, -1\right) = \sqrt{E}\chi_{1/2}^\dagger \left(-\sigma_z, -1\right)$$

where

$$\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then the combination in Eq(3) becomes,

$$\begin{aligned} \overline{v}\left(p'\right) \not{k}' \left(1 - \gamma_5\right) u\left(p\right) &= E\chi_{1/2}^\dagger \left(-1, -1\right) \begin{pmatrix} E & k\overrightarrow{\sigma} \cdot \overrightarrow{e} \\ -k\overrightarrow{\sigma} \cdot \overrightarrow{e} & -E \end{pmatrix} \\ &\quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \chi_{-1/2} \\ &= -4E\chi_{1/2}^\dagger \left(E - k\overrightarrow{\sigma} \cdot \overrightarrow{e}\right) \chi_{-1/2} = 4Ek\sin\theta \end{aligned}$$

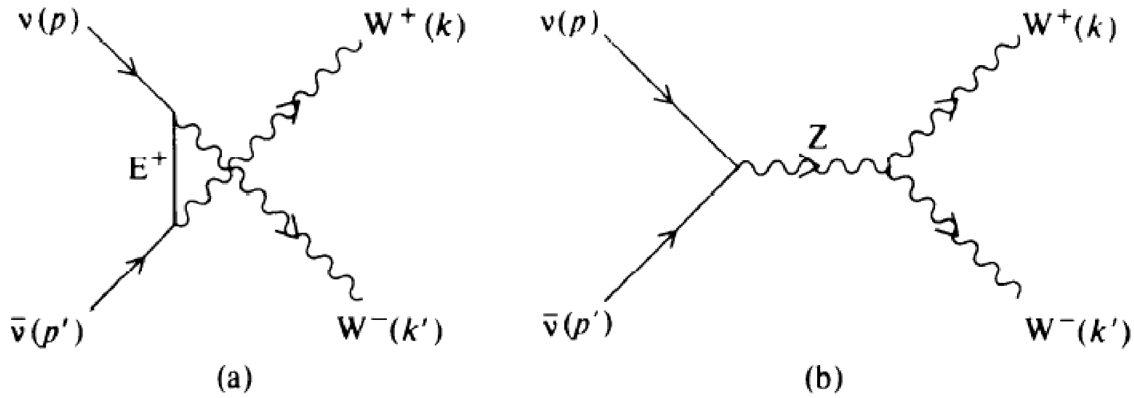
We have then

$$T \approx G_F E^2 \sin\theta \qquad \text{as} \qquad E \to \infty \tag{4}$$

The partial wave expansin for the helicity amplitude is of the form,

$$T_{\lambda_3\lambda_4,\lambda_1\lambda_2}\left(E, \theta\right) = \sum_{J=M}^{\infty} \left(2J + 1\right) T_{\lambda_3\lambda_4,\lambda_1\lambda_2}^J \left(E\right) d_{\mu\lambda}^J \left(\theta\right)$$

where $\lambda_1 = -\lambda_2 = 1/2$ and $\lambda_3 = \lambda_4 = 0$ are the helicities of the initial and final particles with $\lambda = \lambda_1 - \lambda_2 = 1$, $\mu = \lambda_3 - \lambda_4$ and $M = \max\left(\lambda, \mu\right) = 1$. $d_{\mu\lambda}^J\left(\theta\right)$ is the usual rotation matrix with $d_{10}^1\left(\theta\right) = \sin\theta$. It is clear that T in Eq(4) corresponds to a pure $J = 1$ partial wave and violates the unitarity bound of $T^{J=1}\left(E\right) \leq$ constant at high energies. To cancel this bad high energy behavior we need other diagrams for this reaction. There are 2 possibilities: s-channel or u-channel exchange diagrams.



(a) The heavy lepton alternative; the u-channel exchange in the following diagram(a) yields the amplitude,

$$\begin{aligned} T_u\left(\nu\bar{\nu} \longrightarrow W^+W^-\right) &= -2g'^2\bar{v}\left(p'\right)\frac{\not{\varepsilon}'\left(\not{p}-\not{k}'\right)\not{\varepsilon}\left(1-\gamma_5\right)}{\left(p-k'\right)^2-m_E^2}u\left(p\right) \\ &= \frac{-2g'^2}{M_W^2}\bar{v}\left(p'\right)\not{k}'\left(1-\gamma_5\right)u\left(p\right) \end{aligned}$$

If $g^2 = g'^2$, this will cancel the bad behavior given in Eq(3)

(b) The neutral vector boson alternative; the s-channel exchange in the diagram (b) given above gives the amplitude

$$\begin{aligned} T_s\left(\nu\bar{\nu} \longrightarrow W^+W^-\right) &= -i\bar{v}\left(p'\right)\left(-if\gamma_\beta\right)\left(1-\gamma_5\right)u\left(p\right)L_{\alpha\mu\nu}\varepsilon'^\mu\left(k'\right)\varepsilon^\nu\left(k\right) \\ &\times i\left[-g^{\alpha\beta}+\frac{\left(k+k\right)^\alpha\left(k+k\right)^\beta}{M_Z^2}\right]\left[\frac{1}{\left(k+k\right)^2-M_Z^2}\right] \end{aligned}$$

Choose the ZWW coupling to have Yang-Mills structure

$$L_{\alpha\mu\nu} = -if'\left[\left(k'-k\right)_\alpha g_{\mu\nu} - \left(2k'+k\right)_\nu g_{\mu\alpha} + \left(k'+2k\right)_\mu g_{\alpha\nu}\right]$$

we get

$$\begin{aligned} L_{\alpha\mu\nu}\varepsilon'^\mu\left(k'\right)\varepsilon^\nu\left(k\right) &= -if'\left[\left(k'-k\right)_\alpha\varepsilon\cdot\varepsilon' - \left(2k'\cdot\varepsilon\right)\varepsilon'_\alpha + \left(2k\cdot\varepsilon'\right)\varepsilon_\alpha\right] \\ &\approx \frac{if'}{M_W^2}\left[\left(k'-k\right)_\alpha\left(k\cdot k'\right)\right] \end{aligned}$$

and

$$T_s \simeq -\frac{ff'}{M_W^2}\bar{v}\left(p'\right)\not{k}'\left(1-\gamma_5\right)u\left(p\right)$$

Thus if we choose $ff' = 2g^2$, this will also canel the amplitude in Eq(3). This corresponds to the case of adding an additional $U(1)$ symmetry discussed before.
In fact if one demands that all the amplitudes which violate unitarity be cancelled out, one ends up with a renormalizable Lagrangian which is the same as the one derived from the algebraic approach.

We now choose the gauge group to be $SU(2) \times U(1)$. The Lagrangian for the gauge fields is then

$$L = -\frac{1}{4}F^{i\mu\nu}F^i_{\mu\nu} - \frac{1}{4}G^{\mu\nu}G_{\mu\nu}$$

where

$$F_{\mu\nu}^{i} = \partial_\mu A_\nu^i - \partial_\nu A_\nu^i + g\epsilon^{ijk} A_\mu^j A_\nu^k \qquad SU(2) \quad \text{gauge fields}$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\mu B_\mu \qquad U(1) \quad \text{gauge field}$$

Fermions

Clearly, from the stuctrue of the weak charge current given in Eq(1) ν, e form a doublet under SU(2),

$$l_L = \frac{1}{2}(1-\gamma_5)\begin{pmatrix} \nu \\ e \end{pmatrix}$$

For convenience, we introduced left-handed and right-handed fields

$$\psi_L \equiv \frac{1}{2}(1-\gamma_5)\psi, \quad \psi_R \equiv \frac{1}{2}(1+\gamma_5)\psi, \qquad \psi = \psi_L + \psi_R$$

Then

$$T_+ = \int (\nu_L^+ e_L) d^3x, \quad T_- = \int (e_L^+ \nu_L) d^3x, \quad Q = \int (e_L^+ e_L + e_R^+ e_R)$$

Note that

$$Q - T_3 = \int [-\frac{1}{2}(\nu_L^+ \nu_L + e_L^+ e_L) - e_R^+ e_R] d^3x$$

It is straightforward to show that

$$[Q - T_3, T_i] = 0\,, \quad i = 1, 2, 3$$

Thus we can take $Q - T_3$ to be $U(1)$ charge $Y \equiv 2(Q - T_3)$, sometime called **weak hypercharge**. The Y charges for fermions are

$$l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \qquad Y = -1, \qquad e_R \;\; Y = -2$$

With these quantum numbers, the Lagrangian for gauge coupling is

$$\mathcal{L}_2 = \bar{l}_L i\gamma^\nu D_\nu l_l + \bar{l}_R i\gamma^\nu D_\nu l_R \tag{5}$$

where

$$D_\nu \psi = (\partial_\nu - ig\frac{\vec{\tau}\cdot\vec{A}_\nu}{2} - ig^{'}\frac{Y}{2}B_\nu)\psi$$

For example,

$$D_\nu l_L = (\partial_\nu - ig\frac{\vec{\tau}\cdot\vec{A}_\nu}{2} - ig^{'}\frac{Y}{2}B_\nu) l_L$$

Spontaneous Symmetry Breaking

The symmetry braking pattern we want is $SU(2)\times U(1) \rightarrow U(1)_{em}$. Choose scalar fields in SU(2) doublet with hypercharge $Y = 1$,

$$\phi = \begin{pmatrix} \phi^\dagger \\ \phi^0 \end{pmatrix}, \; Y = 1$$

The Lagrangian contaning ϕ is of the form,

$$\mathcal{L}_3 = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi)$$

where

$$D_\mu \phi = (\partial_\mu - \frac{ig}{2}\vec{\tau}\cdot\vec{A}_\mu - \frac{ig^{'}}{2}B_\mu)\phi$$

and

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2$$

In addition there is a coupling between leptons and scalar field ϕ,

$$\mathcal{L}_4 = f\bar{L}_L\phi e_R + h.c.$$

As we have seen before, the spontaneous symmetry breaking is generated by the vaccum expectation value

$$<\phi>_0 = \langle 0|\,\phi\,|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \qquad \text{with} \quad v = \sqrt{\frac{\mu^2}{\lambda}}$$

Write the scalar field in the form

$$\phi(x) = U^{-1}(\vec{\xi})\begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix} \quad \text{where} \;\; U(\vec{\xi}) = \exp[\frac{i\vec{\xi}(x)\cdot\vec{\tau}}{v}] \tag{6}$$

Gauge Transformation

The scalar field in Eq(6) is in the form of gauge transformation. We can then simplify the form by a gauge transformation

$$\phi' = U(\vec{\xi})\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+\eta(x) \end{pmatrix}$$

$$\frac{\vec{\tau}\cdot\vec{A}''_\mu}{2} = U(\vec{\xi})\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}U^{-1}(\vec{\xi}) - \frac{i}{g}(\partial_\mu U)U^{-1}$$

Now the field $\vec{\xi}(x)$ disappears from the Lagrangian as a consequence of gauge invariance. From $\mathcal{L}_4$ (Yukawa coupling), VEV of the scalar field gives

$$\mathcal{L}_4 = f\frac{1}{\sqrt{2}}(\bar{l}_L <\phi> e_R + h.c.) + f\frac{\eta(x)}{\sqrt{2}}(\bar{e}_L e_R + h.c.)$$

as consequence the electron is now massive with electron mass

$$m_e = \frac{f}{\sqrt{2}}v$$

Mass spectrum

We now list the mass spectrum of the theory after the spontaneous symmetry breaking:

(a) Fermion mass

$$m_e = \frac{fv}{\sqrt{2}}$$

(b) Scalar mass(Higgs)

$$V(\phi') = \mu^2\eta^2 + \lambda v\eta^3 + \frac{\lambda}{4}\eta^4 \qquad \rightarrow \; m_\eta = \sqrt{2}\mu$$

(c) Gauge boson masses
From the covariant derivative in $\mathcal{L}_3$

$$\mathcal{L}_3 = \frac{v^2}{2}\chi^\dagger(g\frac{\vec{\tau}\cdot\vec{A}''_\mu}{2} + \frac{g'B'_\mu}{2})(g\frac{\vec{\tau}\cdot\vec{A}''^\mu}{2} + \frac{g'B'^\mu}{2})\chi + \cdots, \qquad \chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we get the mass terms for the gauge bosons,

$$\begin{aligned}\mathcal{L}_3 &= \frac{v^2}{8}\{g^2[(A^{\mathbf{1}}_\mu)^2 + (A^{\mathbf{2}}_\mu)^2] + (gA^{\mathbf{3}}_\mu - g'B_\mu)^2\} + \cdots \\ &= M_W^2 W^{+\mu}W^-_\mu + \frac{1}{2}M_Z^2 Z^\mu Z_\mu + \cdots\end{aligned}$$

where

$$W_\mu^+ = \frac{1}{\sqrt{2}}(A_\mu^{\mathbf{1}} - iA_\mu^{\mathbf{2}}), \qquad M_W^2 = \frac{g^2 v^2}{4}$$

$$Z_\mu = \frac{1}{\sqrt{g^2 + g^{'2}}}(g^{'} A^{\mathbf{3}} - gB_\mu), \quad M_Z^2 = \frac{g^2 + g^{'2}}{4} v^2$$

The field$A_\mu = \frac{1}{\sqrt{g^2+g^{'2}}}(g^{'} A_\mu^{\mathbf{3}} + gB_\mu)$ massless photon
does not appear in $\mathcal{L}_3$ and is massless. This clearly corresponds to photon field. For convience we define

$$\tan\theta_W = \frac{g^{'}}{g} \quad \theta_W : \text{Weinberg angle or weak mixing angle}$$

Then we can write

$$Z_\mu = \cos\theta_W A_\mu^{\mathbf{3}} - \sin\theta_W B_\mu \quad M_Z^2 = \frac{g^2 v^2}{4}\sec^2\theta_W$$

$$A_\mu = \sin\theta_W A_\mu^{\mathbf{3}} - \cos\theta_W B_\mu$$

Note that there is a relation between M_W, M_Z and θ_W of the form,

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1$$

which is a consequence of the doublet nature of the scalar fields.

The weak interactions mediated by W and Z bosons can be read out from Eq(5)

(a) Charged current

$$\mathcal{L}_{cc} = \frac{g}{\sqrt{2}}(J_\mu^\dagger W^{\dagger\mu} + h.c.) \quad J_\mu^\dagger = J_\mu^1 + iJ_\mu^2 = \frac{1}{2}\bar{\nu}\gamma_\mu(1-\gamma_5)e$$

Again to get 4-fermion interaction as low energy limit, we require

$$\frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}$$

which implies that

$$v = \sqrt{\frac{\sqrt{2}}{G_F}} \approx 246Gev$$

This is usually referred to as the **weak scale**.

(b) Neutral Current

The Largrangian for the neutral currents is

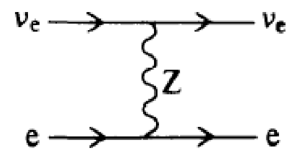
$$\mathcal{L}_{NC} = gJ_\mu^3 A^{\mathbf{3}\mu} + \frac{g^{'}}{2}J_\mu^Y B^\mu = eJ_\mu^{em}A^\mu + \frac{g}{\cos\theta_W}J_\mu^Z Z^\mu$$

where

$$e = g\sin\theta_W,$$

and

$$J_\mu^Z = J_\mu^{\mathbf{3}} - \sin^2\theta_W J_\mu^{em}$$



is the weak neutral current. We can define the weak neutral charge as

$$Q^Z = \int J_0^Z d^3x = (T_3 - \sin^2\theta_W Q)$$

This means that the coupling strength of fermions to Z-boson is proportional to the quantum number $T_3 - \sin^2\theta_W Q$.
In particluar, Z boson can contribute to the scattering

$$\nu_e + e \rightarrow \nu_e + e$$

The measurement of this corss section in the 1970's give $\sin^2\theta_W \approx 0.22$. This yields $M_W \approx 80$ GeV and $M_Z \approx 90$ GeV.

1.3.1 Generalization to more than one family.

From 4-fermion and IVB theory, the form of weak currents of leptons and hadrons gives the following multiplets structure,

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \quad e_R, \mu_R \quad \begin{pmatrix} u \\ d_\theta \end{pmatrix}_L \quad u_R, d_R, s_R$$

where

$$d_\theta = \cos\theta_C d + \sin\theta_C s$$

The neutral current in the down quark sector is of the form

$$\begin{aligned} \mathcal{L}_{NC} &= [\bar{d}_\theta \gamma_\mu (-\frac{1}{2} + \sin^2\theta_C \frac{1}{3}) d_{\theta L} - \sin^2\theta_W \frac{1}{3} (\bar{d}_R \gamma_\mu d_R + \bar{s}_R \gamma_\mu s_R)] Z^\mu \\ &= (-\frac{1}{2} + \sin^2\theta_W \frac{1}{3})[(\bar{d}_L \gamma_\mu d_L + \bar{s}_L \gamma_\mu s_L) + \sin\theta_W \cos\theta_W (\bar{d}_L \gamma_\mu s_L + \bar{s}_L \gamma_\mu d_L) + ... \end{aligned}$$

The term $\left(\bar{d}_L \gamma_\mu s_L + \bar{s}_L \gamma_\mu d_L\right)$ gives rise to $\Delta S = 1$ neutral current processes, e.g. $K_L \rightarrow \mu^+ + \mu^-$ with same order of magnitude as charged current interaction. But experimentally,

$$R = \frac{\Gamma(K_L \rightarrow \mu^+ + \mu^-)}{\Gamma(K^+ \rightarrow \mu + \nu)} \leq 10^{-8}$$

Thus we can not have $\Delta S = 1$ neutral curent process at the same order of magnitude as the charged current process.

GIM mechanism

Glashow, Iliopoulos and Maiani (1970) suggested that there is a 4-th quark, the charm quark c, which couples to the orthogonal combination $s_\theta = -\sin\theta_c d + \cos\theta_c s$ so that the multiplets look like

$$\begin{pmatrix} u \\ d_\theta \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s_\theta \end{pmatrix}_L$$

As a result , the $\Delta S = 1$, neutral current is canceled out. The new current is of the form

$$\bar{d}_\theta(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W)\gamma_\mu d_\theta + \bar{s}_\theta(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W)\gamma_\mu s_\theta = (-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W)(\bar{d}\gamma_\mu d + \bar{s}\gamma_\mu s)$$

which conserves the strangeness.

Quark mixing

Before the spontaneous symmetry braking, fermions are all massless becauseψ_L and ψ_R have different quantum numbers under $SU(2)\times U(1)$. So the mass term $\left(\bar{\psi}_L\psi_R + h.c.\right)$ is not invariant under $SU(2)\times U(1)$ groups and can not appear in the Lagrangian. When we have more than one doublets, ψ_{iR}, ψ_{iL} all have the same quantum numbers with respect to $SU(2)\times U(1)$ group we call them "weak eigenstates". Here the index i labels the different families of weak eigenstates. When spontaneous symmetry breaking takes place, fermions obtain their masses through Yukawa coupling.

$$\mathcal{L}_Y = (f_{ij}\bar{g}_{iL}u_{Rj} + f'_{ij}\bar{g}_{iL}d_{Rj})\phi + h.c.$$

Note that from the requirement of renormalizability we need to write down all possible terms consistent with $SU(2)\times U(1)$ symmetry. Since Yukawa coupling constants f_{ij},, f'_{ij} are arbitrary, the fermion mass matrices are in general not diagonal. When mass matrices are diagonalized the mass matrix we obtain the mass eigenstates which are not the same as the weak eigenstates. The mass matrices in the up and down sectors are given by

$$m^{(u)}_{ij} = f_{ij}\frac{v}{\sqrt{2}} \quad m^{(d)}_{ij} = f'_{ij}\frac{v}{\sqrt{2}}$$

These matrices which are sandwiched between left and right handed fields can be diagonalized by bi-unitary transformations, i.e. given a mass matrix m_{ij},there exits unitary matrices S and T such that

$$S^\dagger m T = m_d$$

is diagonal. Basically, S is the unitary matrix which diagnoalizes the hermitian combination mm^+, i. e.

$$S^+(mm^+)S = m_d^2$$

<u>Biunitary transformation</u>

Write

$$m_d^2 = \begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix}$$

Define

$$m_d = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix}$$

and

$$H = S m_d S^\dagger \qquad \text{hermitian}$$

Define a matrix V by

$$V \equiv H^{-1}m$$

Then

$$VV^\dagger = H^{-1}mm^\dagger H^{-1} = H^{-1}Sm_d^2S^\dagger H^{-1} = H^{-1}H^2H^{-1} = 1$$

So V is unitary and we have

$$S^\dagger H S = m_d, \qquad \Longrightarrow \qquad S^\dagger m V^\dagger S = m_d$$

Or

$$S^\dagger m T = m_d, \qquad \text{with} \quad T = V^\dagger S$$

If we write the left-handed doublets, (weak eigenstates) as

$$q_{1L} = \begin{pmatrix} u' \\ d' \end{pmatrix}_L \quad q_{2L} = \begin{pmatrix} c' \\ s' \end{pmatrix}_L$$

These weak eigenstates are related to mass eigenstates by unitary transformations,

$$\begin{pmatrix} u' \\ c' \end{pmatrix} = S_u \begin{pmatrix} u \\ c \end{pmatrix}, \qquad \begin{pmatrix} d' \\ s' \end{pmatrix} = S_d \begin{pmatrix} d \\ s \end{pmatrix}$$

Note that in the coupling to charged gauge boson $W^{\pm}$, we have

$$\mathcal{L}_W = \frac{g}{\sqrt{2}} W_\mu [\bar{q}_{1L}\gamma^\mu \tau^\dagger q_{1L} + \bar{q}_{2L}\gamma^\mu \tau^\dagger q_{2L}] + h.c.$$

and is invariant under unitary transformation in q_{1L}, q_{2L} space, i.e.

$$\begin{pmatrix} q'_{1L} \\ q'_{2L} \end{pmatrix} = V \begin{pmatrix} q_{1L} \\ q_{2L} \end{pmatrix} \quad VV^\dagger = 1 = V^\dagger V$$

We can use this feature to put all mixing in the down quark sector,

$$q'_{iL} = \begin{pmatrix} u \\ d'' \end{pmatrix}_L, \begin{pmatrix} c \\ s'' \end{pmatrix}_L, \qquad \text{where} \quad \begin{pmatrix} d'' \\ s'' \end{pmatrix} = U \begin{pmatrix} d \\ s \end{pmatrix}$$

Here U is a 2×2 unitary matrix. Clearly, we can extend this to 3 generations with result

$$q_{iL} : \begin{pmatrix} u \\ d'' \end{pmatrix}, \begin{pmatrix} c \\ s'' \end{pmatrix}, \begin{pmatrix} t \\ b'' \end{pmatrix} \quad \begin{pmatrix} d'' \\ s'' \\ b'' \end{pmatrix} = U \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

Now U is a 3×3 unitary matrix, usually called the Cabibbo-Kobayahsi-Maskawa (CKM) matrix.

CP violation Phase

CP violetion can come form complex coupling to gauge bosons. The gauge coupling of $W^{\pm}$ to quarks is governed by the 3×3 unitary matrix U discussed above. This unitary matrix U can have many complex entries. However, in diagonalzing the mass matrices, $S^\dagger(mm^\dagger)S = m_d^2$ There is ambiguity in the matrix S, in the form of diagonal phases. In other words, if S diagonalizes the mass matrix, so does S'

$$S' = S \begin{pmatrix} e^{i\alpha_1} & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & e^{\alpha_n} \end{pmatrix}$$

We can use this property to redefine the quark fields to reduce the phase in U. It turns out that for n×n unitary matrix, number of independent physical phases left over is

$$\frac{(n-1)(n-2)}{2}$$

Thus to get CP violetion we need to go to 3 generations or more (Kobayashi Mskawa). Here we give a constructive proof of this statement. Let us start with a first doublet written in the form,

$$q_{1L} = \begin{pmatrix} u \\ U_{11}d + U_{12}s + U_{13}b \end{pmatrix}$$

If U_{11} has phase δ,

$$U_{11} = R_{11}e^{i\delta}, \qquad R_{11} \qquad \text{real}$$

then this phase δ can be absorbed in the redefiniton of the $u-$quark field

$$u \longrightarrow u' = ue^{-i\delta}$$

and we can write

$$q_{1L} = e^{i\delta} \begin{pmatrix} u' \\ R_{11}d + U'_{12}s + U'_{13}b \end{pmatrix}$$

Similarly, we can factor out the complex phases of U_{21} and U_{31} by redefinition of c and t quark fields. These overall phases are immaterial because there are no gauge couplings between doublets with different family indices. Finally we can absorb two more phases of U_{12} and U_{13} by a redefinition of the s and b fields. The doublets now take the form

$$\begin{pmatrix} u' \\ R_{11}d + R_{12}s + R_{13}b \end{pmatrix}_L, \qquad \begin{pmatrix} c' \\ R_{21}d + R_{22}e^{i\delta_1}s + R_{23}e^{i\delta_2}b \end{pmatrix}_L,$$

$$\begin{pmatrix} t' \\ R_{31}d + R_{32}e^{i\delta_3}s + R_{33}e^{i\delta_4}b \end{pmatrix}_L,$$

Now we have reduced the number of parameters to 13. The normalization conditons of each down-like state gives 3 real conditions aen orthogonality conditions among different states give 6 real conditions on the parameters, Now we are down to 4 parameters. Since we need 3 parameters for the real orthogonal matrix, we end up with one independent phase.

Flavor consevation in neutral current interaction

It turns out that the coupling of neutral Z boson to the fermions conserve flavors. This can be illustrated as follows. We first write the neutral currents in terms of quark fields which are weak eigenste,

$$J_\mu^Z = \sum_i \bar{\psi}_i \gamma_\mu \left[T_3(\psi_i) - \sin^2\theta_W Q(\psi_i)\right]\psi_i$$

Separate into left- and right-handed fields and distingush the up and down components,

$$\begin{aligned} J_\mu^Z &= \sum_i (\bar{u}'_{Li}\gamma_\mu \left[\frac{1}{2} - \sin^2\theta_W\left(\frac{2}{3}\right)\right] u'_{Li} + \bar{d}'_{Li}\gamma_\mu \left[-\frac{1}{2} + \sin^2\theta_W\left(\frac{1}{3}\right)\right] d'_{Li} \\ &\quad + \bar{u}'_{Ri}\gamma_\mu \left[-\sin^2\theta_W\left(\frac{2}{3}\right)\right] u'_{Ri} + \bar{d}'_{Ri}\gamma_\mu \left[\sin^2\theta_W\left(\frac{1}{3}\right)\right] d'_{Ri} \end{aligned}$$

Since weak eigen states q'_{iL} and mass eigen states q'_{iL} are related by unitary matries,

$$u'_{Li} = U(u_L)_{ij}\, u_{Lj}, \qquad \cdots$$

We see that these unitary matrices cancel out in the combination, $\bar{u}'_{Li}u'_{Li}$ so that the neutral current in terms of mass eigenstates has the same form as the one in terms of weak eigenstates. Thus it conserves all quark flavor. Note this feature is due to the fact that all quarks with same helicity and electric charge have the same quantum number with repect to $SU(2) \times U(1)$ gauge group. In other words, if there are quarks in representations other than $SU(2)$ doublets, then there will be flavor changing neutral current if the new quarks have the same eletric charge as either u or d quark.