1 Deep Inelastic Scattering

Introduction

Many important development in high energy physics comes from the studies of the properties of the proton which has mass $M_p = 938.3\ Mev/c^2$. The difficulty in the study of proton comes from the fact the proton is a hadron where the strong interaction is very hard to study. The reason is that in strong interaction the coupling is intrinsically strong and can not be handled by perturbation theory which has been quite successful in the electromagnetic interaction. Futhermore in earlier days we do not even know what the right theory looks like. In the 60's there were many attempts to abandon the field theory framework and advocate S-matrix theory which is not as simple as field theory. It is remarkable that a series of experiments in late 60's and early 70'on elecron proton scatterings has led to the formulation of strong interaction in the form of QCD. Even though QCD works quite well in high energies, it is still hampered by large coupling constants at low erengies.

1.1 Structure of proton

Electron proton scattering

One of the most useful tools to study the structure of proton is the electron proton scattereing where we can probe the unknown structure of proton with electron which can be described very well by Quantum Electrodynamics. It is clear that to reveal the structure of the proton on some scale depends on the wavelength or energy of the probe, in our case is the electron. The higher the energy the finer the structure probed. Here we list in the in the order of increasing enegy the description used to describe this reaction.

1. Rutherford formula

   Here the electron energy is low enough that it can be treated as non-relativistic particle. Also the proton can be treated as a point particle and we can neglect the recoil of the proton. The differential cross section is of the simple form,

   $$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Rutherford}} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}$$

   E : incident energy. θ : scattering angle. α: fine structure constant

   This is derived from classcial mechanics. In fact, Rutherford used the deviation from this formula to infer that the atom is not a point particle but has a structure.

2. Mott fromula

   Take into account the spin of electron and relativistic nature of electron, we get Mott cross section

   $$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \left(\frac{d\sigma}{d\Omega}\right)_{Rutherford}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)$$

   Here proton is still treated as a point particle with no structure.

3. <u>Rosenbluth formula</u>

As the energy of electron is large enough we need to take into account the strong interaction of the proton. In this simple situation, actually we can parametrize the strong interaction effect of the proton in terms of form factors because the electromagnetic current of proton is local and the initial and final state are simple. We will describe this as follows.

If proton were a pointed particle, the interaction of proton with photon is simply,

$$\langle p'|J_\mu^{em}|p\rangle = \bar{u}(p')\gamma_\mu u(p)$$

If we include the strong interaction of the proton, we can parametrize this interaction as

$$\langle p'|J_\mu^{em}|p\rangle = \bar{u}(p')\left[\gamma_\mu F_1(q^2) + i\frac{\sigma_{\mu\nu}q^\nu}{2m}F_2(q^2)\right]u(p)$$

where we have used the Lornetz covariance and current conservation to deduce thie simple form. Here $q = p - p'$, and $F_1(q^2)$, $F_2(q^2)$ are functions of q^2 which parametrize the strong interaction effect are usuall called $form\ factors$. As was described in Note 6 , the differential cross section can be written as,

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\left[\frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1+\tau} + 2\tau G_M^2(Q^2)\tan^2\frac{\theta}{2}\right] ,$$

This is usually called the Rosenbluth formula. Here $\tau = \frac{Q^2}{4M^2}$ and $Q^2 = -q^2$. In this formula, the combinations

$$G_E(q^2) = F_1 + \tau F_2$$

$$G_M(q^2) = F_1 + F_2$$

are called *electric* and *magnetic* form factors repectively and they satisfy

$$G_E(0) = F_1(0) = 1 \quad \text{total charge}$$

$$G_M(0) = F_1(0) + F_2(0) = 1 + F_2(0) \quad \text{magnetic moment}$$

Experimental measurements give

$$G_M^p(0) = 2.79\mu_N, \quad G_M^n(0) = -1.91\mu_N, \quad \text{with } \mu_N = \frac{e}{2M_p} \quad \textit{nuclear magneton}$$

which are usually referred to as the anomalous magnetic moments of the nucleons,

$$G_E^p(Q^2) = \frac{G_M^P(q^2)}{2.79} = \frac{G_n^M(Q^2)}{-1.91} \approx \frac{1}{(1 - q^2/0.7Gev^2)^2}$$

These behaviors are known as the dipole form factor. Form factor $F_1\left(q^2\right)$ can be related to Fourier transform of charge distribution,

$$F(q^2) = \int e^{i\vec{q}\cdot\vec{x}}\rho(x)d^3x \longrightarrow \rho(x) = \int \frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{x}}F(q^2)$$

The measurement of form factor will give information about the charge distribution. For spherical charge distribution, we can write

$$F(q^2) = 1 - \frac{1}{6}q^2\langle r^2\rangle + \cdots$$

where

$$\langle r^2\rangle = 4\pi \int_0^\infty r^2 f(r) r^2 dr, \qquad \text{charge radius,}$$

For proton

$$\langle r^2\rangle_{proton} \simeq (0.86\ fm)^2$$

Note that for $-q^2$ large, form factors decrease very fast $\sim \dfrac{1}{q^4}$

Summary:

1. Proton is not a point particle and has structure
2. The structure of proton can be described by two form factors $F_1\left(q^2\right)$, $F_2\left(q^2\right)$
3. The charge distribution of proton gives charge radius about 0.86 fm.

1.2 Deep Inelastic ep scattering

As the energy of electron gets larger and larger, the inelastic channels become more and more important. As the number of particles in the final state increase in the inelastic channels the analysis become very complicate and the form factors approach is no longer useful. It is very remarkable development that the description becomes simple when we add up all the hadronic final states in the inelastic channels and eventually leads to the formulation of strong interaction in the form of QCD. To describe this, wrtie the inelastic scattering as

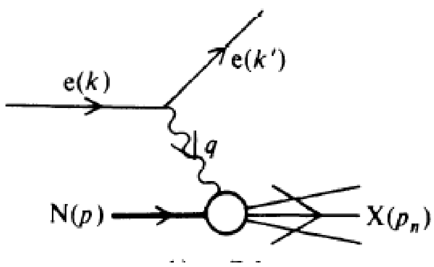
$$e + p \rightarrow e + X$$

where X denotes some generic final state which contain one or more particles. The cross section where the final states are summed over is called the **inclusive cross section**. For example, the inclusive differential cross section is of the form,

$$\frac{d^2\sigma}{d\Omega dE'}(inclusive) = \sum_X \frac{d^2\sigma}{d\Omega dE'}(e + p \rightarrow e + X)$$

where E' is the energy of the final state electron. Denote the momenta of this reaction as

$$e\,(k) + p\,(p) \rightarrow e\,(k) + X\,(p_n)$$



Deifne kinematic variables by

$$q = k - k', \qquad \nu = \frac{p \cdot q}{M}, \qquad W^2 = p_n^2 = (p+q)^2$$

In the lab-frame, we have

$$p_\mu = (M, 0, 0, 0), \qquad k_\mu = (E, \vec{k}), \qquad k_\mu^{'} = (E', \vec{k}^{'})$$

Then

$$\nu = E - E'$$

is the energy lost of the lepton and, when the lepton mass is neglected

$$q^2 = (k - k')^2 = -4EE' \sin^2 \frac{\theta}{2} \leq 0, \qquad Q^2 = -q^2$$

where θ is the scattering angle. The scattering amplitude can be written as,

$$T_n = e^2 \overline{u}(k', \lambda') \gamma^\mu u(k, \lambda) \frac{1}{q^2} \left\langle n \left| J_\mu^{em} \right| p, \sigma \right\rangle$$

where we have used the hadronic electromagnetic current operator J_μ^{em} to denote the interaction of photon with hadronic states. From the Feynman rule for QED, summing over spins we get the unpolarized differential cross section,

$$\begin{aligned} d\sigma_n &= \frac{1}{\left|\vec{v}\right|} \frac{1}{2M} \frac{1}{2E} \frac{d^3 k'}{(2\pi)^3 2k_0'} \prod_{i=1}^{n} \left[\frac{d^3 p_i}{(2\pi)^3 2p_{i0}} \right] \\ &\quad \times \frac{1}{4} \sum_{\sigma\lambda\lambda'} |T_n|^2 (2\pi)^4 \delta^4 (p + k - k' - p_n) \end{aligned}$$

where $p_n = \sum_{i=1}^{n} p_i$. If we sum over all possible hadronic final states n, we get the inclusive cross section

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2}{q^4} \left(\frac{E'}{E} \right) l^{\mu\nu} W_{\mu\nu}$$

The leptonic tensor $l^{\mu\nu}$ is of the form,

$$l_{\mu\nu} = \frac{1}{2} tr \left(\not{k}' \gamma_\mu \not{k} \gamma_\nu \right) = 2 \left(k_\mu k_\nu' + k_\mu' k_\nu + \frac{q^2}{2} g_{\mu\nu} \right)$$

and the hadronic tensor $W^{\mu\nu}$ can be written as

$$\begin{aligned} W_{\mu\nu}(p, q) &= \frac{1}{4M} \sum_\sigma \sum_n \int \prod_{i=1}^{n} \left[\frac{d^3 p_i}{(2\pi)^3 2p_{i0}} \right] \left\langle p, \sigma \left| J_\mu^{em} \right| n \right\rangle \left\langle n \left| J_\nu^{em} \right| p, \sigma \right\rangle (2\pi)^3 \delta^4 (p_n - q - p) \\ &= \frac{1}{4M} \sum_\sigma \int \frac{d^4 x}{2\pi} e^{iq \cdot x} \left\langle p, \sigma \left| J_\mu^{em}(x) J_\nu^{em}(0) \right| p, \sigma \right\rangle \end{aligned}$$

where we have used completeness in last step. It is more convenient to cast this in the form of matrix element of a commutator. To achieve this we note that the term with two current operators in reverse order can be written in the form,

$$\int \frac{d^4 x}{2\pi} e^{iq \cdot x} \left\langle p, \sigma \left| J_\nu^{em}(0) J_\mu^{em}(x) \right| p, \sigma \right\rangle = \sum_n (2\pi)^3 \delta^4 (p_n + q - p) \left\langle p, \sigma \left| J_\nu^{em} \right| n \right\rangle \left\langle n \left| J_\mu^{em} \right| p, \sigma \right\rangle$$

The $\delta - function$ reqires the intermediate state $|n\rangle$ to have enegy with $E_n = M - q_0$ in order to have nonzero result. But since $q_0 > 0$ and the proton is stable, we can not satisfy the $\delta - function$ constraint and this matrix element is zero. We can therefore write the structure functions in terms of commutator of currents,

$$W_{\mu\nu}(p,q) = \frac{1}{4M}\sum_{\sigma}\int \frac{d^4x}{2\pi} e^{iq\cdot x}\left\langle p,\sigma\left|\left[J_{\mu}^{em}(x)\,,\ J_{\nu}^{em}(0)\right]\right|p,\sigma\right\rangle$$

From current conservation $\partial^{\mu}J_{\mu}^{em} = 0$, we get

$$q^{\mu}\left\langle n\left|J_{\mu}^{em}\right|p,\sigma\right\rangle = 0$$

which implies that

$$q^{\mu}W_{\mu\nu}(p,q) = q^{\nu}W_{\mu\nu}(p,q) = 0$$

From the fact that $W_{\mu\nu}$ is a second rank Lorentz tensor and depends on momenta p, q, one can deduce its covariant decomposition as,,

$$W_{\mu\nu}(p,q) = \left[-W_1\left(g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}\right) + \frac{W_2}{M^2}\left(p_{\mu} - \frac{p\cdot q}{q^2}q_{\mu}\right)\left(p_{\nu} - \frac{p\cdot q}{q^2}q_{\nu}\right)\right]$$

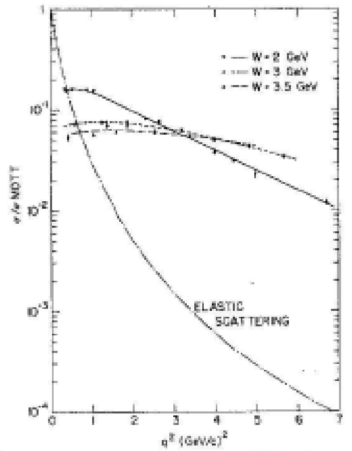
where $W_1(q^2,\nu)$, $W_2(q^2,\nu)$ are Lorentz invariant structure functions of the target proton. It is straightforward to compute the differential cross section in terms of the structure functions,

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2}{4E'^2\sin^4\frac{\theta}{2}}\left(2W_1\sin^2\frac{\theta}{2} + W_2\cos^2\frac{\theta}{2}\right)$$

By measuring differential cross section at different angles and energies, we can extract the structure functions, W_1 and W_2.In the laboratory frame, $\nu = E - E'$ is the energy loss of lepton

1. Bjorken scatting
   As we mentioned before the elastic ep scattering falls off very rapidly as the momentum transfer $-q^2$ increases due to the compositness of proton. If this feature persists for other hadronic final state, we would expect the total inelastic cross section to fall off rapidly as well. The surprise is that experimentally these cross section seem to be quite sizable instead of

falling off rapidly for large q^2.



M.Breidenbach et al.,
Phys. Rev. Lett. **23** (1969) 935

Define the dimensionless scaling variable

$$x = \frac{-q^2}{2Mv} = \frac{Q^2}{2Mv} \ , \quad Q^2 = -q^2$$

The range for x is

$$0 \leq x < 1$$

coming from the fact that the invariant mass of final hadronic state is

$$W^2 = (p+q)^2 = q^2 + 2M\nu + M^2 \geq M^2$$

Also define

$$y = \frac{\nu}{E} = 1 - \frac{E'}{E}$$

the fraction of initial energy transfered to the hadrons. Define

$$MW_1(Q^2, \nu) = F_1(x, q^2/M^2)$$

$$\nu W_2(Q^2, \nu) = F_2(x, q^2/M^2)$$

We can write the inclusive cross section as

$$\frac{d^2\sigma}{dxdy} = \frac{8\pi\alpha^2}{MEx^2y^2}\left[xy^2F_1 + \left(1 - y - \frac{M}{2E}xy\right)F_2\right]$$

Bjorken scaling is the statement that in the large Q^2 limit the $F_i's$ are functions of x only, ,It turns out that all structure functions have the limit behavior

$$\lim_{|q^2| \to \infty,\ x\ fixed} F_i(x, q^2/M^2) = F_i(x)$$

Experimentally for $Q^2 \geq 2GeV^2$ Bjorken scaling seems to be a good approximation. This seems to suggest that there are point-like constituents inside the proton.

Neutrino-nucleon scattering

Here we consider a very similar reaction,

$$\nu_l(k) + N(p) \longrightarrow l^-(k') + X(p_n)$$

where instead of photon we have weak interaction with

$$\mathcal{L}_{eff} = -\frac{G_F}{\sqrt{2}} J_\lambda J^\lambda + h.c.$$

where G_F is the Fermi constant. The charged weak current J^λ cna be separated into leptonic and hadronic parts,

$$J^\lambda = J_l^\lambda + J_h^\lambda$$

The leptonic part is

$$J_l^\lambda = \bar{\nu}_e \gamma^\lambda (1-\gamma_5) e + \bar{\nu}_\mu \gamma^\lambda (1-\gamma_5) \mu$$

It is straightforward to work out the differential cross sections

$$\frac{d^2\sigma^{(\nu)}}{d\Omega dE'} = \frac{G_F^2}{2\pi} E'^2 \left(2\sin^2\frac{\theta}{2} W_1^{(\nu)} + \cos^2\frac{\theta}{2} W_2^{(\nu)} - \frac{(E+E')}{M} \sin^2\frac{\theta}{2} W_3^{(\nu)} \right)$$

$$\frac{d^2\sigma^{(\bar{\nu})}}{d\Omega dE'} = \frac{G_F^2}{2\pi} E'^2 \left(2\sin^2\frac{\theta}{2} W_1^{(\bar{\nu})} + \cos^2\frac{\theta}{2} W_2^{(\bar{\nu})} + \frac{(E+E')}{M} \sin^2\frac{\theta}{2} W_3^{(\bar{\nu})} \right)$$

where the structure functions are defined as

$$\begin{aligned} W_{\mu\nu}^{(\nu)}(p,q) &= \frac{1}{4M} \sum_\sigma \int \frac{d^4x}{2\pi} e^{iq\cdot x} \left\langle p,\sigma \middle| \left[J_{h\mu}(x),\ J_{h\nu}^\dagger(0) \right] \middle| p,\sigma \right\rangle \\ &= -W_1^{(\nu)} g_{\mu\nu} + \frac{W_2^{(\nu)} p_\mu p_\nu}{M^2} - iW_3^{(\nu)} \frac{\varepsilon_{\alpha\beta\mu\nu} p^\alpha q^\beta}{M^2} + \frac{W_4^{(\nu)} q_\mu q_\nu}{M^2} \\ &\quad + \frac{W_5^{(\nu)}(p_\mu q_\nu + q_\mu p_\nu)}{M^2} + i\frac{W_6^{(\nu)}(p_\mu q_\nu - q_\mu p_\nu)}{M^2} \end{aligned}$$

Here we have more structure functions because the $V-A$ current is not conserved and violates parity. Bjorken Scaling for these structure functions are in the form,

$$MW_1^{(\nu)}(q^2,\nu) \longrightarrow F_1^{(\nu)}(x)$$

$$\nu W_2^{(\nu)}(q^2,\nu) \longrightarrow F_2^{(\nu)}(x)$$

$$\nu W_3^{(\nu)}(q^2,\nu) \longrightarrow F_3^{(\nu)}(x)$$

It is useful to use the structure functions with definite helicities. In the laboratory frame, choose the z-axis such that

$$p_\mu = (M,0,0,0), \qquad q_\mu = (q_0,0,0,q_3)$$

The longitudinal polarization of the virtual photon is then

$$\varepsilon_\mu^{(s)} = \frac{1}{\sqrt{-q^2}} (q_3,0,0,q_0)$$

and the corresponding structure function is

$$W_s = \varepsilon_\mu^{(s)*} W^{\mu\nu} \varepsilon_\mu^{(s)} = -W_1 - \frac{q_3^2}{q^2 W_2} = \left(1 - \frac{\nu^2}{q^2}\right) W_2 - W_1$$

The right- and left-handed polarization vectors are

$$\varepsilon_\mu^{(R)} = \frac{1}{\sqrt{2}}\left(0, 1, i, 0\right), \qquad \varepsilon_\mu^{(L)} = \frac{1}{\sqrt{2}}\left(0, 1, -i, 0\right)$$

and their structure functions are

$$W_R = W_1 + \frac{1}{2M}\sqrt{\nu^2 - q^2} W_3, \qquad W_L = W_1 - \frac{1}{2M}\sqrt{\nu^2 - q^2} W_3$$

In the scaling limit we get,

$$2MW_s \longrightarrow F_S = \frac{1}{x}F_2 - 2F_1$$

$$MW_L \longrightarrow F_2 - \frac{1}{2}F_3$$

$$MW_R \longrightarrow F_2 + \frac{1}{2}F_3$$

The differential cross sections can be written as

$$\frac{d^2\sigma^{(\nu)}}{dxdy} = G_F^2 \frac{MEx}{\pi}\left[\left(1-y\right) F_S^{(\nu)} + F_L^{(\nu)} + \left(1-y\right)^2 F_R^{(\nu)}\right]$$

$$\frac{d^2\sigma^{(\bar{\nu})}}{dxdy} = G_F^2 \frac{MEx}{\pi}\left[\left(1-y\right) F_S^{(\bar{\nu})} + F_R^{(\bar{\nu})} + \left(1-y\right)^2 F_L^{(\bar{\nu})}\right]$$

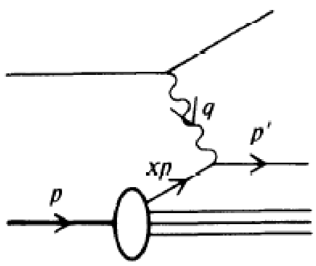
Note that the cross sections increase linearly with energy.

Parton model

Feynman (1969) suggests that deep inelastic scattering can be viewed as due to incoherent elastic scattering from point-like constitents inside the neucleon : Parton.

Assuming that parton has spin 1/2 and carries a fraction of proton momentum, ξp with $0 \leq \xi \leq 1$. Then the contribution to hadronic tensor is

$$\begin{aligned} K_{\mu\nu}\left(\xi\right) &= W_{\mu\nu}\left(p,q\right) = \frac{1}{4\xi M}\sum_{\sigma\sigma'}\int\left[\frac{d^3p'}{\left(2\pi\right)^3 2p_0'}\right]\left\langle \xi p,\sigma\left|J_\mu^{em}\right|p',\sigma'\right\rangle\left\langle p',\sigma'\left|J_\nu^{em}\right|\xi p,\sigma\right\rangle\left(2\pi\right)^3\delta^4\left(p'-q-\xi p\right) \\ &= \frac{1}{4\xi M}\sum_{\sigma\sigma'}\bar{u}\left(\xi p,\sigma\right)\gamma_\mu u\left(p',\sigma'\right)\bar{u}\left(p',\sigma'\right)\gamma_\nu u\left(\xi p,\sigma\right)\delta\left(p_0'-q_0-\xi p_0\right)/2p_0' \end{aligned}$$



The δ-function can be written as

$$\begin{aligned}
\delta\left(p_0' - q_0 - \xi p_0\right)/2p_0' &= \theta\left(p_0'\right)\delta\left[p'^2 - \left(q - \xi p\right)^2\right] \\
&= \theta\left(q_0 + \xi p_0\right)\delta\left(2M\nu\xi + q^2\right) = \theta\left(q_0 + \xi p_0\right)\frac{\delta\left(\xi - x\right)}{2M\nu}
\end{aligned}$$

For the spin sum, we have

$$\begin{aligned}
& \frac{1}{2}\sum_{\sigma\sigma'}\bar{u}\left(\xi p, \sigma\right)\gamma_\mu u\left(p', \sigma'\right)\bar{u}\left(p', \sigma'\right)\gamma_\nu u\left(\xi p, \sigma\right) \\
= & \frac{\xi}{2} tr\left[\not{p}\gamma_\mu\left(\xi\not{p} + \not{q}\right)\gamma_\nu\right] \\
= & 2\xi\left[p_\mu\left(\xi p + q\right)_\nu + p_\nu\left(\xi p + q\right)_\mu - p\cdot\left(\xi p + q\right)g_{\mu\nu}\right] \\
= & 4M^2\xi^2\left(\frac{p_\mu p_\nu}{M^2}\right) - 2M\nu\xi g_{\mu\nu} + \cdots
\end{aligned}$$

where we have neglected the parton mass. The parton tensor is then,

$$K_{\mu\nu}\left(\xi\right) = \delta\left(\xi - x\right)\left(\frac{\xi p_\mu p_\nu}{M^2\nu} - \frac{1}{2M}g_{\mu\nu} + \cdots\right)$$

Let $f\left(\xi\right)d\xi$ be the number of partons with momentum between ξ and $\xi + d\xi$ (weighted by the squared charges). Then hadronic tensor is

$$\begin{aligned}
W_{\mu\nu} &= \int_0^1 f\left(\xi\right)K_{\mu\nu}\left(\xi\right)d\xi \\
&= \frac{xf\left(x\right)}{\nu}\frac{p_\mu p_\nu}{M^2} - \frac{f\left(x\right)}{2M}g_{\mu\nu} + \cdots
\end{aligned}$$

From this we can read out the structure functions,

$$MW_1 \to F_1\left(x\right) = \frac{1}{2}f\left(x\right) \tag{1}$$

$$\nu W_2 \to F_2\left(x\right) = xf\left(x\right) \tag{2}$$

Thus the scaling functons $F_{1,2}$ are the measures of momentum distribution of the partons insdie the target proton.

Note that Eqs (1,2) implies that

$$2xF_1\left(x\right) = F_2\left(x\right)$$

which is known as **Callan-Gross** relation and is a direct consequence of the assumption that partion has spin $\frac{1}{2}$.

Note that for the case of spin 0 parton, we would have

$$\begin{aligned}
K_{\mu\nu} &\propto \left\langle xp\left|J_\mu^{em}\right|xp + q\right\rangle\left\langle xp + q\left|J_\nu^{em}\right|xp\right\rangle \\
&\propto \left(2xp + q\right)_\mu\left(2xp + q\right)_\nu
\end{aligned}$$

Since there is no $g_{\mu\nu}$ term, this implies

$$F_1\left(x\right) = 0$$

In terms of helicity structure functions these imply

$$\begin{aligned}
F_S = 0 &\qquad \text{for a spin 1/2 parton} \\
F_T = 0 &\qquad \text{for a spin 0 parton}
\end{aligned}$$

There is a simple explanation for this.

1.3 Sum rules and application of parton model

It is tempting to identiy the parton with the quarks which as we will discuss later are believed to be the constituents of the proton. The quarks are supposed to be bound togather by interacting with gluons. Suppose we start with a primitive model of 3 free quarks inside the proton, the structure function is essentially a delta function at $x = 1/3$, $f(x)\,\tilde{}\,\delta\left(x - \frac{1}{3}\right)$. As we turn on the interacton this distribution will be smeared and the gluons can produce $q\bar{q}$ pairs and quarks can bremsstrahlung gluons. All these processes will produce a "$q\bar{q}$" at small x. Working with quark model. Working with quark with only 3 light quarks, we write th electromagnetic current as

$$J_\mu^{em} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s$$

Then the structure function is

$$F_1^{ep}(x) = \frac{4}{9}(u + \bar{u}) + \frac{1}{9}(d + \bar{d}) + \frac{1}{9}(s + \bar{s})$$

here $q_i(x)$ denotes the probability of finding a parton with longitudinal momentum fraction x carrying the quantum member of quark q in the proton. (Parton distribution function). From isospin symmetry, we get en structure functions by $u \leftrightarrow d$

$$F_1^{en}(x) = \frac{4}{9}(d + \bar{d}) + \frac{1}{9}(u + \bar{u}) + \frac{1}{9}(s + \bar{s})$$

These parton distribution functions are contraineed by the quantum numbers of the proton. For example for isospin we get

$$\text{Isospin:} \qquad \frac{1}{2}\int_0^1 \left\{\left[u(x) - \bar{u}(x)\right] - \left[d(x) - \bar{d}(x)\right]\right\} dx = \frac{1}{2}$$

$$\text{Strangeness:} \qquad \int_0^1 \left[s(x) - \bar{s}(x)\right] = 0$$

$$\text{Charge:} \qquad \int_0^1 \frac{2}{3}\left[u(x) - \bar{u}(x)\right] - \frac{1}{3}\int_0^1 \left[d(x) - \bar{d}(x)\right] - \frac{1}{3}\int_0^1 \left[s(x) - \bar{s}(x)\right] dx = 1$$

Neutrino deep inelastic scattering

$$\nu_\mu + N \to \mu + X$$

$$\nu_e + N \to e + X$$

$$J_\mu^W \approx \cos\theta_c \bar{u}\gamma^\mu(1 - \gamma_5)d + \sin\theta_c \bar{u}\gamma^\mu(1 - \gamma_5)s + \cdots$$

Here structure functions can also be expressed in terms of parton distribution functions $q_i(x)$

All data are consistent with partons carrying quark quantum numbers.

2 Light-cone Singularity and Bjorke Scaling

It turns out that the Bjorken scaling is intimately connecte with the light-cone behavior in field theory. Recall that the hadronic tensors can be written as

$$W_{\mu\nu}(p,q) = \frac{1}{4M}\sum_{\sigma}\int \frac{d^4x}{2\pi}e^{iq\cdot x}\left\langle p,\sigma\left|\left[J_\mu^{em}(x),\ J_\nu^{em}(0)\right]\right|p,\sigma\right\rangle \tag{3}$$

The scalar product in the exponential can be written as

$$q\cdot x = \frac{(q_0+q_3)}{\sqrt{2}}\frac{(x_0-x_3)}{\sqrt{2}} + \frac{(q_0-q_3)}{\sqrt{2}}\frac{(x_0+x_3)}{\sqrt{2}} - \vec{q}_T\cdot\vec{x}_T$$

where $\vec{q}_T = (q_1,q_2)$, $\vec{x}_T = (x_1,x_2)$. In the rest frame of the nucleon,

$$p_\mu = (M,0,0,0), \qquad q_\mu = (\nu,0,0,\sqrt{\nu^2-q^2})$$

In the scaling limit $-q^2$, $\nu\to\infty$ with $-q^2/2M\nu$ fixed we see that

$$q_0+q_3 \sim 2\nu, \qquad q_0-q_3 \sim \frac{q^2}{2\nu}$$

We expect that the dominant contribution to the integral in Eq(3) comes from regions with less rapid oscillations i.e. $q\cdot x = O(1)$, which implies that

$$x_0-x_3 \sim O\left(\frac{1}{\nu}\right), \qquad \text{and} \qquad x_0+x_3 \sim O\left(\frac{1}{xM}\right)$$

Or

$$x_0^2-x_3^2 \sim O\left(\frac{1}{-q^2}\right)$$

Thus $x^2 = x_0^2-x_3^2-\vec{x}_T^{\,2} \le x_0^2-x_3^2 \sim O\left(\frac{1}{-q^2}\right)$ which vanishes as $-q^2\to\infty$. In other words, in the scaling limitwe are probing the current product near the light cone.

2.1 Free Field Light-cone Singularity

1) Product of fields

In free field theory, the product of fields, such as commutator or propagator are singular on the light-cone $(x^2\approx 0)$ and the leading singularities are indepenet of masses. Consider the free propagator of a scalar field,

$$\langle 0\,|T(\phi(x)\,\phi(0)|\,0\rangle = i\Delta_F(x) = i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2-m^2+i\varepsilon}$$

It is possible to carry out the integration to give

$$\Delta_F(x) = \frac{-1}{4\pi}\delta\left(x^2\right)+\frac{m}{8\pi\sqrt{x^2}}\theta\left(x^2\right)\left[J_1\left(m\sqrt{x^2}\right)-iN_1\left(m\sqrt{x^2}\right)\right]-\frac{im}{4\pi^2\sqrt{x^2}}\theta\left(-x^2\right)K_1\left(m\sqrt{-x^2}\right)$$

where J_n, N_n and K_n are Bessel functions. For $x^2\approx 0$, we have

$$\Delta_F(x) = \frac{i}{4\pi^2}\frac{1}{(x^2-i\varepsilon)}+O\left(m^2x^2\right)$$

One can also show that for the commutator we have

$$\begin{aligned}
[\phi(x), \phi(0)] &= i\Delta(x) = \frac{1}{(2\pi)^3}\int d^4k e^{-ik\cdot x}\varepsilon(k_0)\,\delta\left(k^2-m^2\right) \\
&= \frac{-i}{2\pi}\varepsilon(x_0)\,\delta\left(x^2\right) \qquad \text{for} \qquad x^2 \approx 0
\end{aligned}$$

Setting $m^2 \to 0$, we get the relation,

$$i\int d^4k e^{-ik\cdot x}\varepsilon(k_0)\,\delta\left(k^2\right) = (2\pi)^2\,\varepsilon(x_0)\,\delta\left(x^2\right)$$

Thus the light-cone singularties of the commutator $\Delta(x)$ and that of the propagator function $\Delta_F(x)$ are directly related,

$$\Delta(x) = 2\varepsilon(x_0)\,\mathrm{Im}\left(i\Delta_F(x)\right)$$

This reflects the singular function identity

$$\frac{1}{-x^2+i\varepsilon} - \frac{1}{-x^2-i\varepsilon} = -2\pi i\varepsilon(x_0)\,\delta\left(x^2\right)$$

which is a special case of the general identity

$$\left(\frac{1}{-x^2+i\varepsilon}\right)^n - \left(\frac{1}{-x^2-i\varepsilon}\right)^n = -\frac{2\pi i}{(n-1)!}\varepsilon(x_0)\,\delta^{(n-1)}\left(x^2\right)$$

In the following calculation we shall obtain the commuator singularities from those of propagators by the replacement,

$$\left(\frac{1}{-x^2+i\varepsilon}\right)^n \longrightarrow \frac{2\pi i}{(n-1)!}\varepsilon(x_0)\,\delta^{(n-1)}\left(x^2\right)$$

For the fermions the results are summarized as

$$\left\{\psi_\alpha(x), \bar{\psi}_\beta(y)\right\} = iS_{\alpha\beta}(x-y), \qquad S_{\alpha\beta}(x) = (i\gamma\cdot\partial + m)_{\alpha\beta}\,\Delta(x)$$

$$\left\langle 0\left|T(\psi_\alpha(x), \bar{\psi}_\beta(y)\right|0\right\rangle = iS^F_{\alpha\beta}(x-y), \qquad S^F_{\alpha\beta}(x) = (i\gamma\cdot\partial + m)_{\alpha\beta}\,\Delta^F(x)$$

For $x^2 \approx 0$, we have

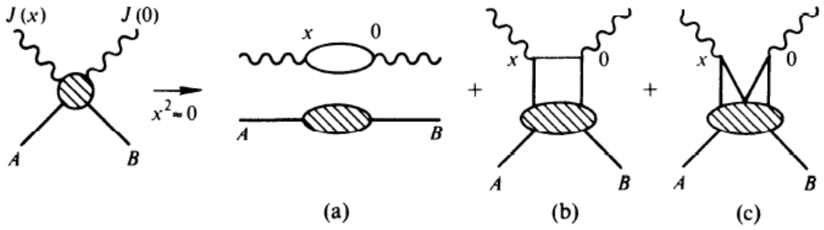
$$S_{\alpha\beta}(x) \approx (i\gamma\cdot\partial)_{\alpha\beta}\left[\frac{1}{2\pi}\varepsilon(x_0)\,\delta\left(x^2\right)\right]$$

$$S^F_{\alpha\beta}(x) \approx (i\gamma\cdot\partial)_{\alpha\beta}\left[\frac{1}{2\pi}\frac{1}{x^2-i\varepsilon}\right]$$

2) Product of scalar currents

Now we consider the composite operators like currents. For simplicity we consider the scalr current of the form,

$$J(x) =: \phi^2(x):$$



Note the normal ordering is to remove the singularities which occur in the product in the product $\phi(x+\zeta)\phi(x-\zeta)$ as $\zeta^\mu \to 0$. The singularities in the product of the currents can be worked out by using Wick's therem,

$$\begin{aligned} T(J(x)J(0)) &= T(:\phi^2(x)::\phi^2(0):) = 2\left[\langle 0|T(\phi(x)\phi(0))|0\rangle\right]^2 \\ +4\langle 0|T(\phi(x)\phi(0))|0\rangle &: \phi(x)\phi(0):+:\phi^2(x)\phi^2(0): \\ &= -2\left[\Delta_F(x,m)\right]^2 + 4i\Delta_F(x,m):\phi(x)\phi(0):+:\phi^2(x)\phi^2(0): \end{aligned}$$

Hence for $x^2 \approx 0$, we get

$$T(J(x)J(0)) \approx \frac{1}{8\pi^4(x^2-i\varepsilon)^2} - \frac{:\phi(x)\phi(0):}{\pi^2(x^2-i\varepsilon)} + :\phi^2(x)\phi^2(0):$$

Note that in this expansion the singularitis as $x^2 \approx 0$ are all contained in the c-number functions which are independent of the initial or final states. If we take this between 2 arbitrary states,

$$\langle A|T(J(x)J(0))|B\rangle \approx \frac{\langle A|B\rangle}{8\pi^4(x^2-i\varepsilon)^2} - \frac{\langle A|:\phi(x)\phi(0):|B\rangle}{\pi^2(x^2-i\varepsilon)} + \langle A\left|:\phi^2(x)\phi^2(0):\right|B\rangle$$

which corresponds to diagrams below.

Free Field Singularities and Scaling

Consider the electromagnetic current given by

$$J_\mu(x) =: \bar{\psi}(x)\gamma_\mu Q\psi(x):$$

where Q is the electric charge operator. We will first calculate the time-ordered product by Wick's theorem,

$$\begin{aligned} T(J_\mu(x)J_\nu(0)) &= T\left(:\bar{\psi}(x)\gamma_\mu Q\psi(x)::\bar{\psi}(0)\gamma_\nu Q\psi(0):\right) \qquad (4) \\ &= Tr\left[iS_F(-x)\gamma_\mu iS_F(x)\gamma_\nu Q^2\right] + :\bar{\psi}(x)\gamma_\mu QS_F(x)\gamma_\nu Q\psi(0): \\ + \quad &: \bar{\psi}(0)\gamma_\nu QS_F(-x)\gamma_\mu Q\psi(x):+:\bar{\psi}(x)\gamma_\mu Q\psi(x)\bar{\psi}(0)\gamma_\nu Q\psi(0): \end{aligned}$$

Using the identity

$$\gamma_\mu\gamma_\nu\gamma = (S_{\mu\nu\lambda\rho} + i\varepsilon_{\mu\nu\lambda\rho})\gamma^\rho, \qquad \text{where} \qquad S_{\mu\nu\lambda\rho} = g_{\mu\nu}g_{\lambda\rho} + g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}$$

we can write Eq(4) in the limit $x^2 \approx 0$ as

$$\begin{aligned}
T\left(J_\mu(x) J_\nu(0)\right) \quad &\approx \quad \left(trQ^2\right) \frac{\left(x^2 g_{\mu\nu} - x_\mu x_\nu\right)}{\pi^4 \left(x^2 - i\varepsilon\right)^4} + \frac{i x^\alpha}{2\pi^2 \left(x^2 - i\varepsilon\right)^2} \\
& \left\{S_{\mu\alpha\nu\beta}\left[V^\beta(x,0) - V^\beta(0,x)\right] + i\varepsilon_{\mu\alpha\nu\beta}\left[A^\beta(x,0) - A^\beta(0,x)\right]\right\} \\
+ \quad &: \quad \bar{\psi}(x)\gamma_\mu Q\psi(x)\bar{\psi}(0)\gamma_\nu Q\psi(0) :
\end{aligned}$$

where

$$V^\beta(x,y) =: \bar{\psi}(x)\gamma^\beta Q^2 \psi(y) :$$

$$A^\beta(x,y) =: \bar{\psi}(x)\gamma^\beta \gamma_5 Q^2 \psi(y) :$$

If we write

$$\frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{\left(x^2 - i\varepsilon\right)^4} = \frac{2}{3}\frac{g_{\mu\nu}}{\left(x^2 - i\varepsilon\right)^3} - \frac{1}{12}\partial_\mu\partial_\nu \frac{1}{\left(x^2 - i\varepsilon\right)^2}$$

and

$$\frac{x^\alpha}{\left(x^2 - i\varepsilon\right)^2} = -\frac{1}{2}\partial^\alpha\left(\frac{1}{x^2 - i\varepsilon}\right)$$

we get for the commutator,

$$\begin{aligned}
\left[J_\mu(x), J_\nu(0)\right] \quad &\approx \quad \frac{itrQ^2}{\pi^3}\left\{\frac{2}{3}g_{\mu\nu}\delta"\left(x^2\right)\varepsilon(x_0) + \frac{1}{6}\partial_\mu\partial_\nu\left[\delta'\left(x^2\right)\varepsilon(x_0)\right]\right\} \qquad (5) \\
& + \left\{S_{\mu\alpha\nu\beta}\left[V^\beta(x,0) - V^\beta(0,x)\right] + i\varepsilon_{\mu\alpha\nu\beta}\left[A^\beta(x,0) - A^\beta(0,x)\right]\right\}\partial^\alpha \frac{\left[\delta\left(x^2\right)\varepsilon(x_0)\right]}{2\pi} \\
&: \quad \bar{\psi}(x)\gamma_\mu Q\psi(x)\bar{\psi}(0)\gamma_\nu Q\psi(0) :
\end{aligned}$$

We can then apply these to the cross sections of e^+e^- annihilation and inelastic eN scattering.

1. <u>**$\mathbf{e^+e^-} \rightarrow$ hadrons**</u>
   Following the same procedure as in the discussion of inelastic eN scattering, it is straightforward to show that the total hadronic cross section for $\mathbf{e^+e^-}$ annihilation can be written as a current commutator,

   $$\sigma\left(e^+e^- \rightarrow hadrons\right) = \frac{8\pi^2\alpha^2}{3\left(q^2\right)^2}\int d^4x e^{iq\cdot x}\left\langle 0\left|\left[J_\mu(x), J^\mu(0)\right]\right|0\right\rangle$$

   The most singular light-cone term comes from the first term on the right-handed side of Eq (5) and we get from this term

   $$\sigma\left(e^+e^- \rightarrow hadrons\right) \approx \frac{8\pi^2\alpha^2 i\left(trQ^2\right)}{3\pi^3\left(q^2\right)^2}\int d^4x e^{iq\cdot x}\left\{\frac{8}{3}\delta"\left(x^2\right)\varepsilon(x_0) + \frac{1}{6}\partial^2\left[\delta'\left(x^2\right)\varepsilon(x_0)\right]\right\}$$

   in the large q^2 limit. Using the identity

   $$i\int d^4x e^{-iq\cdot x}\varepsilon\left(q^0\right)\delta\left(q^2\right) = (2\pi)^2\varepsilon\left(x^0\right)\delta\left(x^2\right)$$

we get

$$\begin{aligned}
\sigma\left(e^+e^- \to hadrons\right) &\approx \frac{8\pi^2\alpha^2 i\left(trQ^2\right)}{3\pi^3\left(q^2\right)^2}\left[\frac{8}{3}\frac{q^2}{4}-\frac{q^2}{6}\right]\varepsilon\left(q^0\right)\delta\left(q^2\right) \\
&= \frac{4\pi\alpha^2}{3q^2}tr\left(Q^2\right)
\end{aligned}$$

Recall that

$$\sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{4\pi\alpha^2}{3q^2}$$

Thus we get the simple result

$$\frac{\sigma\left(e^+e^- \to hadrons\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = tr\left(Q^2\right)$$

This justify the simple naive picture that in the deep inelastic limit, $q^2 \to \infty$, the virtual photon will first produce quarks where is coupling is point like and then quarks trun into hadrons through some strong interaction which is difficult to compute.

2. **Lepton-hadron scattering**

For deep inelastic lN scattering the first term on the right-handed side of Eq (5) will not contribute since it is a c-number and the non-trivial leading singular term will be the second term which for convenience is written in the form,

$$\left[J_\mu\left(\frac{x}{2}\right), J_\nu\left(-\frac{x}{2}\right)\right] \approx \left\{\begin{array}{c} S_{\mu\alpha\nu\beta}\left[:\bar{\psi}\left(\frac{x}{2}\right)\gamma^\beta Q^2\psi\left(-\frac{x}{2}\right):-:\bar{\psi}\left(-\frac{x}{2}\right)\gamma^\beta Q^2\psi\left(\frac{x}{2}\right):\right] \\ +i\varepsilon_{\mu\alpha\nu\beta}\left[:\bar{\psi}\left(\frac{x}{2}\right)\gamma^\beta\gamma_5 Q^2\psi\left(-\frac{x}{2}\right):-:\bar{\psi}\left(-\frac{x}{2}\right)\gamma^\beta\gamma_5 Q^2\psi\left(\frac{x}{2}\right):\right]\end{array}\right\}\partial^\alpha \left[\delta\left(x\right.\right. \tag{6}$$

We can expand the bilocal operator

$$\begin{aligned}
\bar{\psi}\left(\frac{x}{2}\right)\psi\left(-\frac{x}{2}\right) &= \bar{\psi}(0)\left[1+\overleftarrow{\partial}_{\mu_1}\frac{x^{\mu_1}}{2}+\frac{1}{2!}\overleftarrow{\partial}_{\mu_1}\overleftarrow{\partial}_{\mu_2}\frac{x^{\mu_1}}{2}\frac{x^{\mu_2}}{2}+\cdots\right]\times \\
&\quad \left[1-\frac{x^{\nu_1}}{2}\overrightarrow{\partial}_{\nu_1}+\frac{1}{2!}\frac{x^{\nu_1}}{2}\frac{x^{\nu_2}}{2}\overrightarrow{\partial}_{\nu_1}\overrightarrow{\partial}_{\nu_2}+\cdots\right]\psi(0) \\
&= \sum_n \frac{1}{n!}\frac{x^{\mu_1}}{2}\frac{x^{\mu_2}}{2}\cdots\frac{x^{\mu_n}}{2}\bar{\psi}(0)\overleftrightarrow{\partial}_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\cdots\overleftrightarrow{\partial}_{\mu_n}\psi(0)
\end{aligned}$$

to get

$$\begin{aligned}
\left[J_\mu\left(\frac{x}{2}\right), J_\nu\left(-\frac{x}{2}\right)\right] &= \sum_{n=odd}\frac{1}{n!}\frac{x^{\mu_1}}{2}\frac{x^{\mu_2}}{2}\cdots\frac{x^{\mu_n}}{2}O^{(n+1)}_{\beta\mu_1\mu_2\cdots\mu_n}(0)\,S_{\mu\alpha\nu\beta}\partial^\alpha\frac{\left[\delta\left(x^2\right)\varepsilon\left(x_0\right)\right]}{2\pi} \\
&\quad + \sum_{n=even}\frac{1}{n!}\frac{x^{\mu_1}}{2}\frac{x^{\mu_2}}{2}\cdots\frac{x^{\mu_n}}{2}O'^{(n+1)}_{\beta\mu_1\mu_2\cdots\mu_n}(0)\,i\varepsilon_{\mu\alpha\nu\beta}\partial^\alpha\frac{\left[\delta\left(x^2\right)\varepsilon\left(x_0\right)\right]}{2\pi}
\end{aligned}$$

where

$$O^{(n+1)}_{\beta\mu_1\mu_2\cdots\mu_n}(0) = \bar{\psi}(0)\overleftrightarrow{\partial}_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\cdots\overleftrightarrow{\partial}_{\mu_n}\gamma_\beta Q^2\psi(0)$$

$$O'^{(n+1)}_{\beta\mu_1\mu_2\cdots\mu_n}(0) = \bar{\psi}(0)\overleftrightarrow{\partial}_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\cdots\overleftrightarrow{\partial}_{\mu_n}\gamma_\beta\gamma_5 Q^2\psi(0)$$

To calculate the structure function we write,

$$\frac{1}{2}\sum_{\sigma}\left\langle p\sigma\left|O^{(n+1)}_{\beta\mu_1\mu_2\cdots\mu_n}(0)\right|p\sigma\right\rangle = A^{(n+1)}p^{\beta}p_{\mu_1}p_{\mu_2}\cdots p_{\mu_n} + trace\ terms$$

where $A^{(n+1)}$ is some constant and trace terms contain one or more $g_{\mu_i\mu_j}$ factors. Also $O'^{(n+1)}$ tetm will not contribute to the spin average structure functions due to the antisymmetric property of $\varepsilon_{\mu\alpha\nu\beta}$. We then have for the structure function,

$$W_{\mu\nu}(p,q) \approx \frac{1}{2M}\int\frac{d^4x}{2\pi}e^{iq\cdot x}\sum_{odd\,n}^{\infty}\left(\frac{x\cdot p}{2}\right)^n\frac{p^{\beta}}{n!}A^{(n+1)}S_{\mu\alpha\nu\beta}\partial^{\alpha}\frac{\left[\delta\left(x^2\right)\varepsilon\left(x_0\right)\right]}{2\pi}$$

Define

$$\sum_{odd\,n}^{\infty}\left(\frac{x\cdot p}{2}\right)^n\frac{A^{(n+1)}}{n!} = \int d\xi e^{ix\cdot\xi p}f(\xi)$$

then

$$W_{\mu\nu}(p,q) \approx \frac{1}{2M}\int\frac{d^4x}{2\pi}e^{iq\cdot x}\int d\xi e^{ix\cdot\xi p}f(\xi)\,S_{\mu\alpha\nu\beta}\left(q+\xi p\right)^{\alpha}p^{\beta}\frac{\left[\delta\left(x^2\right)\varepsilon\left(x_0\right)\right]}{2\pi}$$

Using the identity,

$$i\int\frac{d^4x}{2\pi}e^{ix\cdot(q+\xi p)}\delta\left(x^2\right)\varepsilon\left(x_0\right) = \delta\left(\left(q+\xi p\right)^2\right)\varepsilon\left(q_0+\xi p_0\right)$$

we have

$$\begin{aligned} W_{\mu\nu}(p,q) &\approx \frac{1}{M}\int d\xi f(\xi)\,\delta\left(q^2+2M\nu\xi\right)\left(g_{\mu\alpha}g_{\beta\nu}+g_{\mu\beta}g_{\alpha\nu}-g_{\mu\nu}g_{\alpha\beta}\right)\left(q+\xi p\right)^{\alpha}p^{\beta} \\ &= \frac{1}{2M^2\nu}\int d\xi f(\xi)\,\delta\left(\xi+\frac{q^2}{2M\nu}\right)\left(-M\nu g_{\mu\nu}+2\xi p_{\mu}p_{\nu}+\cdots\right) \\ &= f(x)\left[-\frac{g_{\mu\nu}}{2M}+\frac{x}{\nu}\frac{p_{\mu}p_{\nu}}{M^2}+\cdots\right] \end{aligned}$$

for $x = -\dfrac{q^2}{2M\nu}$. Thus we recover the parton modle results

$$MW_1 \longrightarrow F_1(x) = \frac{1}{2}f(x)$$

$$\nu W_2 \longrightarrow F_2(x) = xf(x)$$

This implies that the assumption of canonical free-field light-cone structure is equivalent to that of parton model.