### Quatization of gauge theory

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## 1 Path Integral Quantization of Gauge Theory

Canonical quantization of gauge theory is difficult because the gauge invariance implies that not all components of gauge fileds are real physical degree of freedom. To eliminate those components which are dependent, it is easier to use path integral quantization

To see the difficulty, consider for simplicity, SU(2) Yang-Mills fields,

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \qquad a = 1, 2, 3$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon^{abc} A^b_\mu A^c_\nu$$

We can write the generating functional as

$$W[J] = \int [dA_{\mu}] e^{i \int d^4 x [\mathcal{L} + \vec{J}_{\mu} \cdot \vec{A}^{\mu}]}$$

The free-field part is then

$$W_0[J] = \int [dA_\mu] \exp\{i \int d^4x [\mathcal{L}_0 + \overrightarrow{J}_\mu \cdot \overrightarrow{A}^\mu]\}$$

Write the free Lagrangian part as,

$$\int d^4x \mathcal{L}_o(x) = -\frac{1}{4} \int d^4x (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) (\partial_\mu A^{a\nu} - \partial_\nu A^{a\mu})$$
$$= \frac{1}{2} \int d^4x A^a_\mu(x) (g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu) A^a_\nu(x)$$

The general formula for the Gaussian integral is of the form,

$$\int [d\phi] exp[-\frac{1}{2} \left\langle \phi K \phi \right\rangle + \left\langle J \phi \right\rangle] \sim \frac{1}{\sqrt{detK}} exp\left\langle J K^{-1} J \right\rangle$$

However, in our case the operator K

$$K_{\nu\mu}(x-y) = (g^{\mu\nu}\partial^2 - \partial^{\mu}\partial^{\nu})\delta^4(x-y)$$

has the property of the projection operator, i.e.

$$\int d^4 y K_{\mu\nu}(x-y) K^{\nu}{}_{\lambda}(y-z) \propto K_{\nu\lambda}(x-z)$$

and has no inverse. This means that the Gaussian intergral diverges. The reason that  $W_0(J)$  is singular is due to the gauge invariance which projects out the transverse gauge fields. In the path integeral for  $W_0(J)$ we have summed over all field configurations, including "orbits" that are related by gauge transformation. This over-counting is the root of the divergent integral. Thus we have to remove this "volume" of the orbit in the quantization.

## Volume factor in gauge theory

### 1.0.1 Simple example

We shall use a 2-dimensional integral to illustrate the strategy to factor out the volume factor. Take a simple integral of the form,

$$W = \int dx dy e^{iS(x,y)} = \int d^2 r e^{iS(\vec{r})}$$
<sup>(1)</sup>

(3)

where  $\vec{r} = (r, \theta)$ . Suppose  $S(\vec{r})$  is invariant under rotation,

$$S(\vec{r}) = S(\vec{r}_{\phi}), \quad \text{with} \quad \vec{r}_{\phi} = (r, \theta + \phi)$$

$$\tag{2}$$

Thus  $S(\vec{r})$  is constant over (circular) orbit and the integral W is proportional to the length of the orbit. So if we only wish to sum over contribution from inequivalent  $S(\vec{r})'s$  we can simply divide out the volume factor corresponding to polar integration  $\int d\theta = 2\pi$ . We will use a more complicate prodedure which can be generalized to more general cases. Insert an identity,

$$1 = \int d\phi \delta \left(\theta - \phi\right)$$

into W given in Eq(1)

$$W = \int d\phi \int d^2 r e^{iS(\vec{r})} \delta\left(\theta - \phi\right) = \int d\phi W_{\phi}$$

Use the invariant property  $S(\vec{r}) = S(\vec{r}_{\phi})$ , we see that

$$W_{\phi} = W_{\phi'}, \qquad \Longrightarrow \qquad W_{\phi} \text{ is independent of } \phi$$

and

$$W = \int d\phi W_{\phi} = W_{\phi} \int d\phi = 2\pi W_{\phi}$$

 $g(\vec{r}) = 0$ 

We can impose more complicate constraint,



which intersects each orbit only once. We need to compute  $\left[\Delta_g\left(\overrightarrow{r}\right)\right]$  defined by

$$1 = \int d\phi \, \left[ \Delta_g \left( \vec{r} \right) \right] \delta \left[ g(\vec{r}_{\phi}) \right]$$

Write

$$\left[\Delta_g\left(\overrightarrow{r}\right)\right]^{-1} = \int d\phi \ \delta\left[g(\overrightarrow{r}_{\phi})\right]$$

We can show that  $\Delta_{g}(r)$  is rotational invariant,

$$\left[\Delta_g\left(\overrightarrow{r}_{\phi'}\right)\right]^{-1} = \int d\phi \ \delta\left[g(\overrightarrow{r}_{\phi+\phi'})\right] = \int d\phi" \ \delta\left[g(\overrightarrow{r}_{\phi"})\right] = \left[\Delta_g\left(\overrightarrow{r}\right)\right]^{-1}$$

Integrating over  $\phi$ , we get

$$\Delta_g\left(\vec{r}\right) = \left.\frac{\partial g(\vec{r})}{\partial \theta}\right|_{g=0} \tag{4}$$

The integral is then

$$W = \int d\phi W_{\phi} \qquad \text{with} \quad W_{\phi} = \int d^2 r e^{iS(\vec{r})} \delta\left[g(\vec{r}_{\phi})\right] \Delta_g\left(\vec{r}\right)$$
(5)

Again,  $W_{\phi}$  is rotational invariant and we can remove the voulume factor in Eq(5),

$$W_{\phi'} = \int d^2 r e^{iS(\vec{r}\,)} \delta\left[g(\vec{r}_{\phi'})\right] \Delta_g\left(\vec{r}\right)$$
  
=  $\int d^2 r' e^{iS(\vec{r}\,')} \delta\left[g(\vec{r}_{\phi'})\right] \Delta_g(\vec{r'})$  with  $\vec{r'} = (r, \phi')$ 

### 1.0.2 Volume factor in Gauge Theories

In the case of gauge theory the situation is much mor complicate. But the principle is the same and it is useful to think of the local gauge symmetry as the generalization of the rotational symmetry in the simple example we describe before.

Under the gauge transformation we have  $\overrightarrow{A}_{\mu} \rightarrow \overrightarrow{A}_{\mu}^{\theta}$ , where

$$\vec{A}_{\mu} \cdot \frac{\vec{\tau}}{2} \longrightarrow \vec{A}_{\mu}^{\theta} \cdot \frac{\vec{\tau}}{2} = U(\theta) [(\vec{A}_{\mu} \cdot \frac{\vec{\tau}}{2}) + \frac{1}{ig} U^{-1}(\theta) \partial_{\mu} U(\theta)] U^{-1}(\theta)$$

where

$$U( heta) = \exp[rac{-iec{ heta}\cdotec{ au}}{2}]$$

This is analogous to the rotational transformation given in Eq(2). We restrict the path integration to hypersurface which intersects each orbit once. If we choose the hypersurface as

$$f_a(\vec{A}_\mu) = 0, \quad a = 1, 2, 3$$
 (6)

so that the equation

$$f_a(\overrightarrow{A}^{\theta}_{\mu}) = 0$$

has a unique solution for  $\theta$  for a given  $\overrightarrow{A}_{\mu}$ . This is analogous to Eq (3). In the neighborhood of identity, we can write

$$U(\theta) = 1 + i\frac{\overrightarrow{\theta} \cdot \overrightarrow{\tau}}{2} + O(\theta^2)$$

The integration over group space can be chosen as

$$[d\theta] = \prod_{a=1}^{3} d\theta_a$$

Define

$$\Delta_f^{-1}[\vec{A}_{\mu}] = \int [d\theta(x)]\delta[f_a(\vec{A}_{\mu})]$$

then

$$\Delta_f[A_f] = \det M_f \qquad \text{where} \quad (M_f)_{ab} = \frac{\delta f_a}{\delta \theta_b}$$

This is the generalization of the formula,

$$\int dx \,\delta\left(f\left(x\right)\right) = \left.\frac{1}{df/dx}\right|_{f=0}$$

Recall that the infinitesimal gauge transformation is of the form,

$$A^{\theta a}_{\mu} = A^{a}_{\mu} + \epsilon^{abc} \theta^{b} A^{c}_{\mu} - \frac{1}{g} \partial_{\mu} \theta^{a}$$

and the responce of the function f is written as

$$f_a(\vec{A}_{\mu}) = f_a(\vec{A}_{\mu}) + \int d^4 y [M_f(x, y]_{ab} \theta_b(y) + O(\theta^2)]$$

Again  $\Delta_f[\vec{A}_{\mu}]$  is gauge invariant, as illustrated by the following simple calculation. From

$$\Delta_f^{-1}[\vec{A}_\mu] = \int [d\theta'(x)] \delta[f_a(\vec{A}_\mu)]$$

we get

$$\begin{split} \Delta_f^{-1}[\overrightarrow{A}_{\mu}^{\theta}] &= \int [d\theta'(x)]\delta[f_a(\overrightarrow{A}_{\mu}^{\theta\theta'})] = \int [d(\theta(x)\theta'(x)]\delta[f_a(\overrightarrow{A}_{\mu}^{\theta\theta'})] \\ &= \int [d\theta''(x)]\delta[f_a(\overrightarrow{A}_{\mu}^{\theta''})] = \Delta_f^{-1}[\overrightarrow{A}_{\mu}] \end{split}$$

The path integral is then

$$\int [\vec{dA_{\mu}}] \exp\{i \int \mathcal{L}(x) d^{4}x\} = \int [d\theta(x)] \left[\vec{dA_{\mu}}\right] \Delta_{f}(\vec{A}_{\mu}) \delta[f_{a}(\vec{A}_{\mu})] \exp\{i \int \mathcal{L}(x) d^{4}x\}$$
$$= \int [d\theta(x)] \left[\vec{dA_{\mu}}\right] \Delta_{f}(\vec{A}_{\mu}) \delta[f_{a}(\vec{A}_{\mu})] \exp\{i \int \mathcal{L}(x) d^{4}x\}$$

We can now drop the "volume factor"  $\int [d\theta(x)]$  to write the generating functional as

$$W_f[\vec{J}] = \int [\vec{dA_{\mu}}] (\det M_f) \delta[f_a(\vec{A}_{\mu})] \exp\{i \int d^4x [\mathcal{L}(x) + \vec{J}_{\mu} \cdot \vec{A}^{\mu}]\}$$

This is calles Faddeev-Popov ansatz and the factor det  $M_f$  is called the Faddeev-Popov determinant. This is the path integral suitable for quantization.

### 1.0.3 Faddeev-Popov Ghost

The factor det  $M_f$  can be written as

$$(\det M_f) \sim \int [dc] [dc^+] exp\{i \int d^4x d^4y \sum c_a^+(x) [M_f(x,y)]_{ab} c_b(y)\}$$

where  $c_a, c_b^{\dagger}$  are Grassman fields and are called Faddeev-Popov ghost, because they are not real physical degrees of freedoms. In this form, we can treat the Faddeev-Popov determinant as an additional term in the Lagrangian and adaptable for the perturbative calculation. We also want to convert  $\delta[f_a(A_{\mu})]$  into some effective Lagrangian form. Suppose we choose the gauge fixing term to be instead of Eq(6),

$$[f_a(\vec{A}_\mu)] = B_a(x)$$

where  $B_a(x)$  is some arbitrary function. Then the integral

$$\int [d\theta(x)] \Delta_f[\vec{A}_{\mu}] \delta[f_a(\vec{A}_{\mu}) - B_a(x)] = 1$$

will give the same  $\Delta_f[A_\mu]$  as before. Note that

$$\int [dB_a(x)]exp\{-\frac{i}{2\xi}\overrightarrow{B}^2(x)\} \sim constant, \qquad \xi \text{ is arbitrary}$$

We can then write

$$W[J] = \int [dA^{a}_{\mu}] [dB_{a}(x)] (\det M_{f}) \delta[f_{a}(\vec{A}_{\mu}) - B_{a}] \exp\{i \int d^{4}x [\mathcal{L}(x) - \vec{J}^{\mu} \cdot \vec{A}_{\mu} - \frac{1}{2\xi} \vec{B}^{2}(x)]\}$$
  
=  $\int [dA^{a}_{\mu}] (\det M_{f}) \exp\{i \int d^{4}x [\mathcal{L}(x) - \vec{J}^{\mu} \cdot \vec{A}_{\mu} - \frac{1}{2\xi} [f^{a}(A_{\mu})]^{2}]\},$ 

Put all these together we can write

$$W[J] = \int [dA^{a}_{\mu}][dc(x)][dc^{\dagger}(x)] \exp\{iS_{eff}[\vec{J}]\}$$

where the effective action is of the form,

$$S_{eff}[\vec{J}] = S[\vec{J}] + S_{gf} + S_{FPG}$$

Here  $S_{gf}$  is the gauge fixing term,

$$S_{gf} = \frac{1}{2\xi} \int d^4x \{ f_a[A_\mu(x)] \}^2$$

and  $S_{FPG}$  is the Faddev-Popov ghost term,

$$S_{FPG} = \int d^4x d^4y \sum_{a,b} c_a^{\dagger}(x) [M_f(x,y)]_{ab} c_b(y)$$

# Covariant gauge

One of the most common choice of the gauge fixing term is that which leads to covariant gauge

$$f_a(A_\mu) = \partial^\mu A^a_\mu = 0$$

We can compute the Faddev-Popov determinan as follows. Under infinitesimal gauge transformation,

$$U(\theta(x)) = 1 + \frac{i\vec{\theta}\cdot\vec{\tau}}{2} + O(\theta^2)$$

we get

$$A^{a\theta}_{\mu} = A^{a}_{\mu} + \epsilon^{abc}\theta^{b}(x)A^{c}_{\mu}(x) - \frac{1}{g}\partial_{\mu}\theta^{a}$$

Then

$$f^{a}(A^{\theta}_{\mu}) = f^{a}(A_{\mu}) + \partial^{\mu}[\epsilon^{abc}\theta^{b}(x)A^{c}_{\mu}(x) - \frac{1}{g}\partial_{\mu}\theta^{a}(x)] = f^{a}(A_{\mu}) + \int d^{4}y[M_{f}(x,y]_{ab}\theta^{b}(y)$$

$$[M_f(x,y)]_{ab} = -\frac{1}{g}\partial^{\mu}[\delta^{ab}\partial_{\mu} - g\epsilon^{abc}A^c_{\mu}]\delta^4(x-y)$$

Then

$$S_{gf} = -\frac{1}{2\xi} \int d^4 x (\partial^\mu A_\mu)^2$$
$$S_{FPG} = \frac{1}{g} \int d^4 x \sum_{a,b} c_a^+(x) \partial^\mu [\delta_{ab}\partial_\mu - g\epsilon_{abc}A^c_\mu] c_b(x)$$

In this form we can generate Feynman rule and do the calculation perturbatively if applicable.