Note 7 Grand Unification

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1 Grand Unification Theory

The standard model of electroweak and strong interactions based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ seems to work quite well. Since these interactions are all based on gauge theory, it is desirable to have more unified theory which can combine all these interactions as components of a single force; a theory with only one guage coupling.

1974 Georgi and Glashow proposed a $SU(5)$ model which is the simplest unification model.

1.1 SU(5) Model

A general representation under an $SU(5)$ transformation may be expressed in tensor notation as,

$$\psi^{ij\cdots}_{kl\cdots} \longrightarrow U^i_m U^j_n U^s_k U^t_l \cdots \psi^{mn\cdots}_{st\cdots}$$

where indices run from 1 to 5 and

$$[U]^i_m = [\exp(i\alpha^a \lambda^a/2)]^i_m$$

is a 5×5 unitary matrix and $\{\lambda^a\}, a = 0, 1, 2, \cdots, 23$ is a set of 5×5 hermitian traceless matrices with normalization,

$$T(\lambda_a \lambda_b) = 2\delta_{ab}$$

For example,

$$\lambda^3 = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}, \qquad \lambda^0 = \frac{1}{\sqrt{15}}\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

To obtain the $SU(3)_C \times SU(2)_L$ content of a representation, we identify first 3 of $SU(5)$ indices as color indices and the other 2 as $SU(2)_L$ indices,

$$i = (\alpha, r), \qquad \text{with } \alpha = 1, 2, 3 \qquad r = 1, 2 \tag{1}$$

Fermion content

In the standard model with one generation, the fermion content with respect to $SU(3)_C \times SU(2)_L$ are given by,

$$(\nu_e, e)_L \sim (\mathbf{1}, \mathbf{2}), \qquad e_L^+ \sim (\mathbf{1}, \mathbf{1}),$$

$$(u_\alpha, d_\alpha)_L \sim (\mathbf{3}, \mathbf{2}), \qquad u_L^{c\alpha} \sim (\mathbf{3}^*, \mathbf{1}), \qquad d_L^{c\alpha} \sim (\mathbf{3}^*, \mathbf{1})$$

where we have used the relaitons

$$\psi^c = C\gamma^0\psi^*, \qquad (\psi_R)^c = (\psi^c)_L \equiv \psi_L^c$$

The $SU(3)_C \times SU(2)_L$ contents of the simple $SU(5)$ representations are ;

lowest rep	ψ_i	$\mathbf{5} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$
lowest conjugate	ψ^i	$\mathbf{5}^* = (\mathbf{3}^*, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$
anti-symm	ψ_{ij}	$\mathbf{10} = (\mathbf{3}^*, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{1}, \mathbf{2})$

Comparing these with first generation fermions, we see that they can be accomodated as $\mathbf{5}^*$+$\mathbf{10}$ representation of $SU(5)$,

$$\mathbf{5}^* : \left(\psi^i\right)_L = \left(d^{c1}, d^{c2}, d^{c3}, e^-, \nu\right)_L$$

and

$$10 : \left(\chi_{ij}\right)_L = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u^{c3} & -u^{c2} & u_1 & d_1 \\ -u^{c3} & 0 & u^{c1} & u_2 & d_2 \\ u^{c2} & -u^{c1} & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & -0 \end{pmatrix}$$

Charge Quantization

One immediate consequence of the $SU(5)$ unification is a simple explanation for the experimentally observed charge quantization. In general, if the unification group is simple, the charge quantization follows. This is because the eigenvalues of a non-Abelian group are discrete while those corresponding to the Abelian $U(1)$ group are continuous. Note that in $SU(3)_C \times SU(2)_L$ model the electric charge operator can be written as

$$Q = T_3 + \frac{Y}{2}$$

It is useful to express this relation in terms of the generators of $SU(5)$. Write

$$Q = T_3 + cT_0$$

It is straightforward to see that

$$c = -\sqrt{\frac{5}{3}}$$

1.2 Gauge bosons

The $SU(5)$ adjoint representation A_i^j has dimension $5^2 - 1 = 24$ and the $SU(3)_C \times SU(2)_L$ decomposition is

$$\mathbf{24} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{3}^*, \mathbf{2})$$

Use the index convention given in Eq(1), we can interpret this as

$$\begin{array}{ccc} A^{\alpha}_{\beta} & (\mathbf{8},\mathbf{1}) & SU(3)_C \text{ color gluons} \\ A^{r}_{s} & (\mathbf{1},\mathbf{3}) & SU(2)_L \text{ weak bosons} \\ A^{\alpha}_{\alpha}-A^{r}_{r} & (\mathbf{1},\mathbf{1}) & U(1) \text{ weak boson} \\ A^{r}_{\alpha} & (\mathbf{3},\mathbf{2}) & \text{Lepto-quark} \\ A^{\alpha}_{r} & (\mathbf{3}^{*},\mathbf{2}) & \text{Lepto-quark} \end{array}$$

The leptoquark

$$A^{r}_{\alpha} = (X_{\alpha}, Y_{\alpha}), \qquad A^{\alpha}_{r} = \begin{pmatrix} X^{\alpha} \\ Y^{\alpha} \end{pmatrix}$$

have fractional charges

$$Q(X) = -\frac{4}{3}, \qquad Q(Y) = -\frac{1}{3}$$

and will play an important role in the Baryon number violation. If we put all $SU(5)$ gauge bosons in 5×5 matrix

$$A_{\mu} = \sum_{a=0}^{23} A^{a}_{\mu} \frac{\lambda^{a}}{2}$$

we get

$$A = \begin{pmatrix} G^{1}_{1} & G^{1}_{2} & G^{1}_{3} & X_{1} & Y_{1} \\ G^{2}_{1} & G^{2}_{2} & G^{2}_{3} & X_{2} & Y_{2} \\ G^{3}_{1} & G^{3}_{2} & G^{3}_{3} & X_{3} & Y_{3} \\ X^{1} & X^{2} & X^{3} & W^{3}+B & W^{+} \\ Y^{1} & Y^{2} & Y^{3} & W^{-} & -W^{3}+B \end{pmatrix} \tag{2}$$

1.3 Spontaneous symmetry breaking

The spontaneous symmetry breaking is supposed to take place in two stages, characterized by two mass scales, v_1 and v_2

$$SU(5) \xrightarrow{v_1} SU(3)_C \times SU(2)_L \times U(1)_Y \xrightarrow{v_2} SU(3)_C \times U(1)_{EM}$$

where $v_1 \gg v_2$. This corresponds to X, Y masses being superheavy, $M_{X,Y} \gg M_{W,Z}$. This can be achieved with scalars inin adjoint $\left(H^{i}_{j}\right)$ and vector (ϕ_i)representations. The general $SU(5)$ −invariant 4-th order potential is,

$$V(H,\phi) = V_1(H) + V_2(\phi) + \lambda_4 tr\left(H^2\right)\left(\phi^{\dagger}\phi\right) + \lambda_5\left(\phi^{\dagger}H^2\phi\right)$$

with

$$V_1(H) = -m_1^2 tr\left(H^2\right) + \lambda_1\left[tr\left(H^2\right)\right]^2 + \lambda_2 tr\left(H^4\right)$$

$$V_2(\phi) = -m_2^2\left(\phi^{\dagger}\phi\right) + \lambda_3\left(\phi^{\dagger}\phi\right)^2$$

Here H is 5×5 a traceless hermitian matrix and we have imposed a discrete symmetry $H \to -H$ and $\phi \to -\phi$ to get rid of various cubic terms. For simplicity, we first minimize the potential $V_1(H)$. It can be shown that, for $\lambda_2 > 0$ and $\lambda_1 > -\frac{7}{30}$, $V_1(H)$ has an extremum at $H = \langle H \rangle$ with

$$\langle H \rangle = v_1 \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

where

$$v_1^2 = \frac{m_1^2}{[60\lambda_1 + 14\lambda_2]}$$

With VEV the pattern for the symmetry breaking is

$$SU(5) \longrightarrow SU(3) \times SU(2) \times U(1)$$

and gauge bosons X, and Y obtain masses $\propto v_1$. As we will see later v_1 could be of order of 10^{15} Gev or so.

The fact H develops VEV also effects the ϕ system through the cross couplings λ_4, λ_5. The color triplet $\phi_t : (\mathbf{3}, \mathbf{1})$ and flavor doublet $\phi_d : (\mathbf{1}, \mathbf{2})$ components of $\phi = (\phi_t, \phi_d)$ acquires respective mass terms,

$$m_t^2 = -m_2^2 + (30\lambda_4 + 4\lambda_5) v_1^2 \tag{3}$$

$$m_d^2 = -m_2^2 + (30\lambda_4 + 9\lambda_5) v_1^2 \tag{4}$$

Thus after the first stage of symmetry breaking all particle masses are expected to be of order of v_1 which should be superheavy. For the second stage of symmetry breaking we need a $SU(2)$ doublet to break the symmetry at energy of order of 250 Gev. Here we assume that "somehow" the m_d^2 in Eq (4) is much smaller than v_1^2 and will survive to low energy(~ 250 Gev) as the superheavy particle (with masses of order v_1) decouple. The relevant physics is described by the effective potential,

$$V_{eff}(\phi_d) = -m_d^2 \left(\phi_d^\dagger \phi_d\right) + \lambda_3 \left(\phi_d^\dagger \phi_d\right)^2$$

which produce symmetry breaking

$$SU(2) \times U(1) \longrightarrow U(1)$$

$$\phi_d = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \qquad v_2 = \sqrt{\frac{m_d^2}{\lambda_3}} \sim 250\ Gev$$

This feature where $v_1 \gg v_2$ is usually called the **gauge hierachy**.

1.4 Coupling constant unification

The standard model describes elecromagnetic, weak and strong interactions for energies $\leq 10^2 Gev$ with 3 different coupling constants: g_s, g, and g' for the gauge groups, $SU(3)_C, SU(2)_L$ and $U(1)_Y$ respectively. The grand unified theory unifies these into one coupling constant corresponding to unified gauge group.

The possibility of different couplings for the various subgroups arises because of spontaneous symmetry breaking; X, Y gauge bosons of $SU(5)$ acquire masses and decouple from the coupling constant renormalization. Note that since the energy dependence of coupling constants is only logarithmic, and since in the energy region $\simeq 10^2 Gev$, g, g',and g_s are quite different the unification scale M_X is expected to be many orders of magnitude larger than $10^2 Gev$.

The covariant derivative for $SU(5)$ is

$$D_\mu = \partial_\mu + ig_5 \sum_{a=0}^{23} A_\mu^a \frac{\lambda^a}{2}$$

and for $SU(3)_C \times SU(2)_L \times U(1)_Y$

$$D_\mu = \partial_\mu + ig_s \sum_{a=1}^{8} G_\mu^a \frac{\lambda^a}{2} + ig \sum_{r=1}^{3} W_\mu^r \frac{\lambda^r}{2} + ig' B_\mu \frac{Y}{2}$$

The definition of coupling constants depends on the normalization of the generators. All non-Abelian groups here are normalized as $Tr(\lambda^a \lambda^b) = 2\delta^{ab}$ and we have

$$g_5 = g_3 = g_2 = g_1 \tag{5}$$

with

$$g_3 = g_s, \qquad g_2 = g$$

The coupling g_1 is that of the Abelian $U(1)$ subgroup. Thus

$$ig_1 \lambda^0 A_\mu^0 = ig' Y B_\mu$$

and A_μ^0 is identified with B_μ gauge field. Note that

$$Y = \begin{pmatrix} -\frac{2}{3} & & & & \\ & -\frac{2}{3} & & & \\ & & -\frac{2}{3} & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$$

From this we get

$$Y = -\sqrt{\frac{5}{3}}\lambda^0, \qquad g' = -\sqrt{\frac{3}{5}} g_1$$

The weak mixing angle is then

$$\sin^2 \theta_W = \frac{g'^2}{g^2 + g'^2} = \frac{3}{8} \tag{6}$$

The relations given Eqs (5,6) are valid at unification scale. To compare them with experimental data we need to evolve them down to low energy $\sim 10^2 Gev$. The evolution of the $SU(n)$ coupling constant is of the form,

$$\frac{dg_n}{d(\ln \mu)} = -b_n g_n^3$$

where

$$b_n = \frac{1}{48\pi^2}(11n - 2N_F) \qquad \text{for} \quad n \geq 2$$

$$b_1 = -\frac{N_F}{24\pi^2}$$

Then we get

$$b_n - b_1 = \frac{11n}{48\pi^2}$$

In our case, the solution for the effective coupling constants are

$$\frac{1}{g_1^2(\mu)} = \frac{1}{g_1^2(\mu_0)} + 2b_1 \ln\left(\frac{\mu}{\mu_0}\right)$$

$$\frac{1}{g_2^2(\mu)} = \frac{1}{g_2^2(\mu_0)} + 2b_2 \ln\left(\frac{\mu}{\mu_0}\right)$$

$$\frac{1}{g_3^2(\mu)} = \frac{1}{g_3^2(\mu_0)} + 2b_3 \ln\left(\frac{\mu}{\mu_0}\right)$$

In terms of more familiar parameters,

$$\frac{g_1^2(\mu)}{4\pi} = \left(\frac{5}{3}\right)\frac{\alpha(\mu)}{\cos^2\theta_W}, \qquad \frac{g_2^2(\mu)}{4\pi} = \frac{\alpha(\mu)}{\sin^2\theta_W}, \qquad \frac{g_3^2(\mu)}{4\pi} = \alpha_s(\mu)$$

we get

$$\frac{1}{\alpha_s(\mu)} = \frac{1}{\alpha_5} + 8\pi b_3 \ln\left(\frac{\mu}{M_X}\right)$$

$$\frac{\sin^2\theta_W}{\alpha(\mu)} = \frac{1}{\alpha_5} + 8\pi b_2 \ln\left(\frac{\mu}{M_X}\right)$$

$$\frac{3}{5}\frac{\cos^2\theta_W}{\alpha(\mu)} = \frac{1}{\alpha_5} + 8\pi b_1 \ln\left(\frac{\mu}{M_X}\right)$$

where we have used

$$g_1(M_X) = g_2(M_X) = g_2(M_X) = g_5, \qquad \text{and} \quad \frac{g_5^2}{4\pi} = \alpha_5$$

Taking a linear combination to eliminate $\ln\left(\frac{\mu}{M_X}\right)$ to get

$$\frac{2}{\alpha_s} - \frac{3}{\alpha}\sin^2\theta_W + \frac{3}{5\alpha}\cos^2\theta_W = 8\pi\left[2(b_3 - b_1) - 3(b_2 - b_1)\right]\ln\left(\frac{\mu}{M_X}\right) = 0$$

This implies

$$\sin^2\theta_W = \frac{1}{6} + \frac{5\alpha(\mu)}{9\alpha_s(\mu)}$$

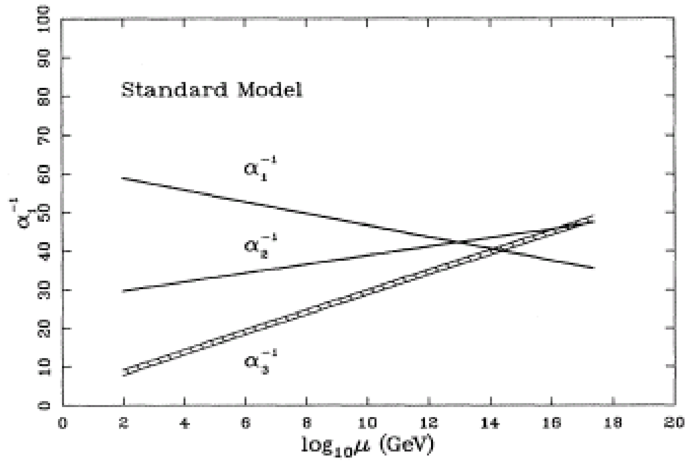
Using the measured values of $\alpha(\mu)$ and $\alpha_s(\mu)$ we get

$$\sin^2\theta_W \simeq .21$$

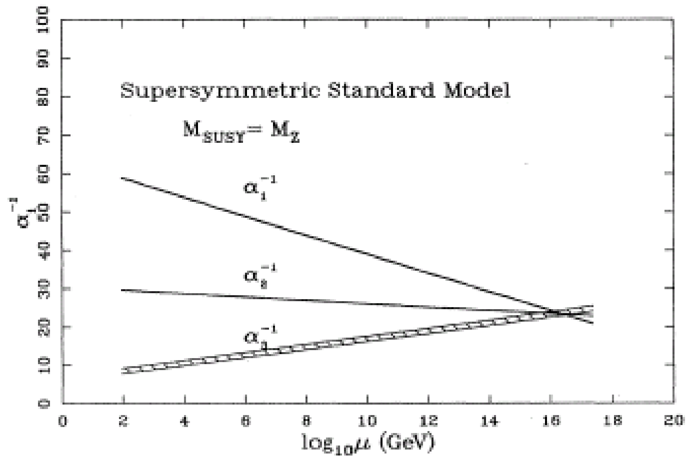
and

$$M_X \simeq 4 \times 10^{14} Gev$$

Or using the measured values of $\sin^2\theta_W$, we get



This shows that these 3 coupling constants do not unify very well. It turns out that if we use supersymmetric version of unification we get



This has become one of the motivation for supersymmetry.

1.5 Baryon number violation

The gauge couplings of $\mathbf{5}^*+\mathbf{10}$ fermions $\left(\psi^i,\chi_{ij}\right)$, coming from their covariant derivative, are of the form, Using the gauge boson matrix in Eq (2)

$$\begin{aligned} g\bar{\psi}\gamma^\mu A_\mu^T\psi + trg\bar{\chi}\gamma^\mu\{A_\mu,\chi\} &= -\sqrt{\frac{1}{2}}gW_\mu^\dagger\left(\bar{\nu}\gamma^\mu e+\bar{u}_\alpha\gamma^\mu e_\alpha\right) \\ &\quad +\sqrt{\frac{1}{2}}gX_{\mu\alpha}^a[\varepsilon^{\alpha\beta\gamma}\bar{u}_\alpha\gamma^\mu q_{\beta a}+\varepsilon^{ab}(\bar{q}_{\alpha b}\gamma^\mu e^+ - \bar{l}_b\gamma^\mu d_\alpha^c)] \end{aligned}$$

Note that X bosons couple to two-fermion channels with different baryon numbers. In one case, they couple to quarks and leptons ($B=\frac{1}{3}$); in other case

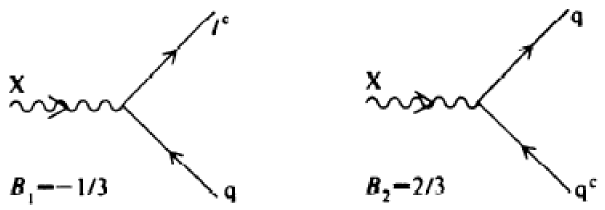


FIG. 14.4. X bosons as leptoquarks and diquarks.

they transform quarks to antiquark ($B = \frac{2}{3}$). Consequently , through mediation of $X - boson$, a $B = -\frac{1}{3}$ channel can be converted into a $B = \frac{2}{3}$ one, a baryon number violation process. Since M_X is very heavy we can write down the effective 4-fermion local interaction for the baryon violating processes,

$$\mathcal{L}_{\Delta B=1} = \frac{g^2}{2M_X^2}\varepsilon^{\alpha\beta\gamma}\varepsilon^{ab}(\bar{u}^c_\alpha\gamma^\mu q_{\beta a})(\bar{d}^c_\alpha\gamma_\mu l_b + \bar{e}^+\gamma_\mu q_{\alpha b})$$

This will give rise to the following decays of the proton,

$$p \longrightarrow e^+\pi^0, \qquad e^+\omega, \qquad \cdots$$

To calculate these decay rates one needs to renormalize these effective Lagrangian from M_X down to energy of order of 1 Gev and use some hadronic model to compute the hadronic matrix elements. One of the most important feature here is the factor M_X^2 in the denominator which the decay rates very small because $M_X \sim 10^{15} Gev$ or more.

In the 80's the proton decay experiments have been actively pursued and none has been found. The best limit is

$$\tau(p \longrightarrow e^+\pi^0) \geq 1.6 \times 10^{33}\ years$$

Baryon number asymmetry in the universe

Observationally, the universe seems to be made out of mostly matter and very little anti-matter. In the standard hot Big Bang Model, the matter and anti-matter are produced in equal amount. Question is then how this matter-antimatter symmetric situation can evolve into matter- antimatter asymmetric universe we observe today. One quantitative measure of this asymmetry is the ratio of baryon number density n_B to the Cosmic Background Radiation (CMB) photon density n_γ,

$$\eta = \frac{n_B}{n_\gamma} \simeq (6.1 \pm 0.2) \times 10^{-10}$$

Saharov has studied this problem and came up with 3 conditions needed to generate this asymmetry,

1. Baryon number violation
   If the baryon number were conserved by all processes, then the initial situation of $n_B = 0$ of hot Big Bang model can not change as the universe evolves.

2. <u>C and CP violations</u>
   For a baryon violating reaction involving baryon, $X \longrightarrow qq$, there will also be a mirror processes $\bar{X} \longrightarrow \bar{q}\,\bar{q}$ for the corresponding anti-baryon that can creat exactly negative amount of n_B, no net baryon number can be generated if these two processes can occur with same rate. Thus C and CP violations are needed to get a different rates for the particle and anti-particle processes, i.e., $\Gamma\left(X \longrightarrow qq\right) \neq \Gamma\left(\bar{X} \longrightarrow \bar{q}\,\bar{q}\right)$.

3. <u>Out of thermal equilibrium</u>
   Heuristically, we can understand this by recalling that CPT invariance requires particle and anti-particle to have the same mass, hence to be equally weighted in the Boltzmann distribution; thus no CPT invariant interactions can generate a non-zero baryon number density.

GUTs, such as $SU(5)$, togather with expansion of the universe can satisfy all these conditions. Unfortunately the CP violation in the Standard Model is not large enough to account for observed aymmetry $\eta \sim 10^{10}$.