Quantum Field Theory

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Group Theory

The tool for studying symmetry is the group theory. Will give a simple discussion

Elements of group theory

group G :collection of elements (a, b, c$\cdots$) with a multiplication laws satisfies;

1. Closure. If $a, b \in G$, $c = ab \in G$
2. Associative $a(bc) = (ab)c$
3. Identity $\exists e \in G \ni a = ea = ae \quad \forall a \in G$
4. Inverse For every $a \in G$, $\exists a^{-1} \ni aa^{-1} = e = a^{-1}a$

Examples

1. **Abelian group** —– group multiplication commutes, i.e. $ab = ba \quad \forall a, b \in G$
   e.g. cyclic group of order n, Z_n, consists of $a, a^2, a^3, \cdots, a^n = E$
2. **Orthogonal group** —— $n \times n$ orthogonal matrices, $RR^T = R^T R = 1$, $R : n \times n$ matrix
   e. g. the matrices representing rotations in 2-dimesions,

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

3. **Unitary group** ——— $n \times n$ unitary matrices,

Built larger groups from smaller ones by direct product:
Direct product group —– Given two groups , $G = \{g_1, g_2 \cdots\}, \quad H = \{h_1, h_2 \cdots\}$ define a direct product group is defined as $G \times H = \{g_i h_j\}$ with multiplication law

$$(g_i h_j)(g_m h_n) = (g_i g_m)(h_j h_n)$$

Theory of Representation

group $G = \{g_1 \cdots g_n \cdots\}$. If for each group element $g_i \longrightarrow D(g_i)$,$n \times n$ matrix such that

$$D(g_1)D(g_2) = D(g_1 g_2) \qquad \forall \;\; g_1, g_2 \in G$$

then $D's$ a representation of the group G (n-dimensional representation). If a non-singular matric M such that matrices can be transformed into block diagonal form,

$$MD(a)M^{-1} = \begin{pmatrix} D_1(a) & 0 & 0 \\ 0 & D_2(a) & 0 \\ 0 & 0 & \ddots \end{pmatrix} \qquad \textit{for all } a \in G.$$

$D(a)$ is called reducible representation. Otherwiseit is irreducible representation (irrep)
Continuous group: groups parametrized by continuous parameters
Example: Rotations in 2-dimensions can be parametrized by $0 \leq \theta < 2\pi$

SU(2) group

Set of 2×2 unitary matrices with determinant 1 is called $SU(2)$ group.
In general, $n \times n$ unitary matrix U can be written as

$$U = e^{iH} \qquad H : n \times n \ \text{ hermitian matrix}$$

From

$$\det U = e^{iTrH}$$

$$TrH = 0 \qquad if \qquad \det U = 1$$

Thus $n \times n$ unitary matrices U can be written in terms of $n \times n$ traceless Hermitian matrices.

Note that Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a complete set of 2×2 hermitian traceless matrices.
Define $J_i = \frac{\sigma_i}{2}$ then

$$[J_1, J_2] = iJ_3 \ , \ [J_2, J_3] = iJ_1 \ , \ [J_3, J_1] = iJ_2$$

Lie algebra of $SU(2)$ symmetry. same as commutators of angular momentum.
To construct the irrep of $SU(2)$ algebra, define

$$J^2 = J_1^2 + J_2^2 + J_2^3 \ , \ \text{with property} \ [J^2, J_i] = 0 \ , \ i = 1, 2, 3$$

Also define

$$J_\pm \equiv J_1 \pm iJ_2 \quad \text{then} \quad J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2 \quad \text{and} \quad [J_+, J_-] = 2J_3$$

choose simultaneous eigenstates of J^2, J_3,

$$J^2|\lambda, m\rangle = \lambda|\lambda, m\rangle \quad , \quad J_3|\lambda, m\rangle = m|\lambda, m\rangle$$

From

$$[J_+, J_3] = -J_+$$

we get

$$(J_+J_3 - J_3J_+)|\lambda, m\rangle = -J_+|\lambda, m\rangle$$

Or

$$J_3(J_+|\lambda, m\rangle) = (m+1)(J_+|\lambda, m\rangle)$$

Thus J_+ is called *raising operator*. Similarly, J_- lowers m to $m-1$,

$$J_3(J_-|\lambda, m\rangle) = (m-1)(J_-|\lambda, m\rangle)$$

Since

$$J^2 \geq J_3^2 \quad , \quad \lambda - m^2 \geq 0$$

m is bounded above and below. Let j be the largest value of m, then

$$J_+|\lambda, j\rangle = 0$$

Then

$$0 = J_- J_+ |\lambda, j\rangle = (J^2 - J_3^2 - J_3)|\lambda, j\rangle = (\lambda - j^2 - j)|\lambda, j\rangle$$

and

$$\lambda = j(j+1)$$

Similarly, let j' be the smallest value of m, then

$$J_-|\lambda, j'\rangle = 0 \qquad \lambda = j'(j'-1)$$

Combining these 2,

$$j(j+1) = j'(j'-1) \quad \Rightarrow \quad j' = -j \;\; \textit{and} \;\; j - j' = 2j = \textit{integer}$$

use j, m to label the states. Assume the states are normalized,

$$\langle jm|jm'\rangle = \delta_{mm'}$$

Write

$$J_\pm|jm\rangle = C_\pm(jm)|j, m \pm 1\rangle$$

Then

$$\langle jm|J_-J_+|jm\rangle = |C_+(j,m)|^2 \rightarrow$$

$$LHS = \langle j,m|(J^2 - J_3^2 - J_3)|jm\rangle = j(j+1) - m^2 - m$$

Then

$$C_+(j,m) = \sqrt{(j-m)(j+m+1)}$$

Similarly

$$C_-(j,m) = \sqrt{(j+m)(j-m+1)}$$

Summary: eigenstates $|jm\rangle$ have the properties

$$J_3|j,m\rangle = m|j,m\rangle \quad J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}|jm\pm 1\rangle \quad , \quad J^2|j,m\rangle = j(j+1)jm\rangle$$

$|j,m\rangle$, $m=-j,-j+1,\cdots,j$ are the basis for irreducible representation of SU(2) group. From these we can construct the representation matrices.
Example: $j=\frac{1}{2}$, $m=\pm\frac{1}{2}$

$$J_3 = \left|\frac{1}{2},\pm\frac{1}{2}\right\rangle = \pm\frac{1}{2}|\frac{1}{2},\pm\frac{1}{2}\rangle$$

$$J_+|\frac{1}{2},\frac{1}{2}\rangle = 0 \quad , \quad J_+|\frac{1}{2},-\frac{1}{2} >= |\frac{1}{2},\frac{1}{2}\rangle \quad , \quad J_-|\frac{1}{2},\frac{1}{2} >= |\frac{1}{2},-\frac{1}{2}\rangle \quad , \quad J_-|\frac{1}{2},-\frac{1}{2}\rangle = 0$$

If we write

$$|\frac{1}{2},\frac{1}{2}\rangle = \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\frac{1}{2},-\frac{1}{2}\rangle = \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then we can represent $J's$ by matrices,

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Taking linear combinations,

$$J_1 = \frac{1}{2}(J_+ + J_-) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad J_2 = \frac{1}{2i}(J_+ - J_-) = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

these are just Pauli matrices.

Product representation

Let α be the spin-up and β the spin-down states. For 2 spin $\frac{1}{2}$ particles, the total wavefunction is , $\alpha_1\alpha_2, \alpha_1\beta_2 \cdots$
Define $\vec{J}^{(1)}$ acts only on particle 1 and $\vec{J}^{(2)}$ on particle 2.

$$\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$$

Use

$$J_3 = J_3^{(1)} + J_3^{(2)} \quad , \quad J_3(\alpha_1\alpha_2) = (J_3^{(1)} + J_3^{(2)})(\alpha_1\alpha_2) = (\alpha_1\alpha_2)$$

From

$$\vec{J}^2 = (\vec{J}^{(1)} + \vec{J}^{(2)})^2 = (\vec{J}^{(1)})^2 + (\vec{J}^{(2)})^2 + 2[\frac{1}{2}(J_+^{(1)}J_-^{(2)} + J_-^{(1)}J_+^{(2)} + J_3^{(1)}J_3^{(2)}]$$

$$\vec{J}^2(\alpha_1\alpha_2) = (\frac{3}{4} + \frac{3}{4} + \frac{2}{4})|\alpha_1\alpha_2\rangle = 2|\alpha_1\alpha_2\rangle$$

$|1,1\rangle = \alpha_1\alpha_2$ These means that $|\alpha_1\alpha_2\rangle$ is a $j=1$ state. Use lowering operator to get other $j=1$ states

$$J_-(\alpha_1\alpha_2) = (J_-^{(1)} + J_-^{(2)})(\alpha_1\alpha_2) = (\beta_1\alpha_2 + \alpha_1\beta_2)$$

On the other hand

$$J_-(\alpha_1\alpha_2) = J_-|11\rangle = \sqrt{(1+1)(1-1+1)}|1,0\rangle = \sqrt{2}|1,0\rangle$$

Thus

$$|1,0\rangle = \frac{1}{\sqrt{2}}(\beta_1\alpha_2 + \alpha_1\beta_2)$$

Clearly $|1,0\rangle = \beta_1\beta_2$The The only state left-over is

$$\frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2)$$

This is a $|0,0\rangle$ state
Summary:

1. Among the generator only J_3 is diagonal, — SU(2) is a rank-1 group
2. Irreducible representation is labeled by j and the dimension is $2j+1$
3. Basis states $|j,m\rangle$ $m = j, j-1\cdots(-j)$ representation matrices can be obtained from

$$J_3|j,m\rangle = m|j,m\rangle \qquad J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$$

SU(2) and rotation group

The generators of $SU(2)$ group are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;,\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let $\overrightarrow{r} = (x, y, z)$ be arbitrary vector in R_3 (3 dimensional coordinate space). Define a 2×2 matrix h by

$$h = \vec{\sigma} \cdot \overrightarrow{r} = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

h has the following properties

1. $h^+ = h$
2. $Trh = 0$
3. $det \;\; h = -(x^2 + y^2 + z^2)$

Let U be a 2×2 unitary matrix with $detU = 1$. Consider the transformation

$$h \longrightarrow h' = UhU^\dagger$$

Then we have

1. $h'^+ = h'$
2. $Trh' = 0$
3. $\det h' = \det h$

Properties (1)&(2) imply that h' can also be expanded in terms of Pauli matrices

$$h' = \vec{r}' \cdot \vec{\sigma} \;\; \overrightarrow{r} = (x', y', z')$$

$$\det h' = \det h \quad \Rightarrow \quad x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$$

Thus relation between $\overrightarrow{r}$ and $\overrightarrow{r}'$ is a rotation.
An arbitrary 2×2 U induces a rotation in R_3. This is a connection between $SU(2)$and $O(3)$ groups. Note that U and $-U$ correspon to the same rotation.

Rotation group & QM

Rotation in R_3 can be represented as

$$\vec{r} = (x, y, z) = (r_1, r_2, r_3) \quad , \quad r_i \rightarrow r_i' = R_{ij} X_j \qquad RR^T = 1 = R^T R$$

Consider an arbitary function of coordinates, $f(\vec{r}) = f(x, y, z)$. Under the rotation, the change in f

$$f(r_i) \rightarrow \; f(R_{ij} r_j) = f'(r_i)$$

If $f = f'$ we say f is invariant under rotation, e.g. $f(\vec{r}) = f(r)$, $r = \sqrt{x^2 + y^2 + z^2}$
In QM, implement the rotation by

$$|\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle, \quad O \rightarrow O' = UOU^{\dagger}$$

so that

$$\langle\psi'|O'|\psi'\rangle = \langle\psi|O|\psi\rangle$$

If $O'^{+} = O$, we say the operator O is invariant under rotation

$$UO = OU \quad [O, U] = 0$$

In terms of infinitesimal generators

$$U = e^{-i\theta\vec{n}\cdot\vec{J}/\hbar}$$

this implies $[J_i, O] = 0$, $i = 1, 2, 3$. If O is the Hamiltonian H, this gives $[J_i, H] = 0$.

Let $|\psi\rangle$ be an eigenstate of H with eigenvaule E,

$$H|\psi\rangle = E|\psi\rangle$$

then

$$(J_i H - H J_i)|\psi\rangle = 0 \quad \Rightarrow \quad H(J_i|\psi\rangle) = E(J_i|\psi\rangle)$$

$i.e$ $\quad |\psi\rangle$ & $J_i|\psi\rangle$ are degenerate. For example, let $|\psi\rangle = |j,m\rangle$ the eigenstates of angular momentum, then $J_\pm|j.m\rangle$ are also eigenstates if $|\psi\rangle$ is eigenstate of H. This means for a given j , the degeneracy is $(2j+1)$.

Gauge Theory

Abelian gauge theory(QED)

Maxwell Equation

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad , \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad , \quad \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J}$$

Source free equations can be solved by

$$\vec{B} = \nabla \times \vec{A} \quad , \quad \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

In Minkowski space

$$\partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu} \qquad F^{ij} \sim \epsilon^{ijk} B_h \quad F^{0i} \sim E^i$$

$\overrightarrow{E}$ and $\overrightarrow{B}$ are unchanged under the transformation

$$\phi \rightarrow \phi - \frac{\partial \alpha}{\partial t}, \qquad \vec{A} \rightarrow \vec{A} + \vec{\nabla} \alpha$$

Or

$$A^\mu \rightarrow A^\mu - \partial^\mu \alpha, \qquad \text{where} \qquad A^\mu = (\frac{\phi}{c}, \vec{A})$$

This is called **gauge invariance**.
Schrodinger Equation for a charged particle

$$[\frac{1}{2m}(\frac{\hbar}{i}\vec{\nabla} - e\vec{A})^2 - e\phi]\psi = i\hbar\frac{\partial\psi}{\partial t}$$

To get same physics, need to transform ψ

$$\psi \longrightarrow e^{ie\alpha/\hbar}\psi \qquad \alpha = \alpha(x)$$

This provides a **connection between gauge transformation with symmetry transformation**.

More general construction, start with a free electron field Lagrangian,

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$$

This has global U(1) symmetry,

$$\psi(x) \to \psi = e^{-i\alpha}\psi(x) \qquad \alpha : \text{constant}$$

$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)e^{i\alpha}$$

Suppose

$$\alpha = \alpha(x) \quad \psi' = e^{-i\alpha(x)}\psi(x) \;\;,\;\; \bar{\psi}'(x) = \bar{\psi}(x)e^{i\alpha(x)}$$

transformation of derivative

$$\bar{\psi}(x)\partial_\mu\psi(x) \;\to\; \bar{\psi}'(x)\partial_\mu\psi'(x) = \bar{\psi}(x)\partial_\mu\psi(x) - i(\partial_\mu\alpha)(\bar{\psi}\psi) \quad \text{not invariant}$$

Introduce gauge field $A_\mu(x)$ to form **covariant derivative**

$$D_\mu\psi \equiv (\partial_\mu + igA_\mu)\psi(x)$$

So that $D_\mu\psi$ transforms same way asψ,

$$(D_\mu\psi)' = e^{-i\alpha(x)}(D_\mu\psi)$$

This requires that

$$(\partial_\mu + igA'_\mu)\psi' = e^{-i\alpha}(\partial_\mu + igA_\mu)\psi$$

and

$$A'_\mu = A_\mu - \frac{1}{g}\partial_\mu\alpha$$

Then

$$\mathcal{L} \;=\; \bar{\psi}i\gamma^\mu(\partial_\mu + igA_\mu)\psi - m\bar{\psi}\psi$$

is invariant under local symmetry transformation (local symmetry)
Lagrangian for gauge field is ,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This is invariant under gauge transformation.
We can write $F_{\mu\nu}$ in terms of covariant derivative,

$$\begin{aligned} D_\mu D_\nu\psi &= (\partial_\mu + igA_\mu)(\partial_\nu + igA_\nu)\psi \\ &= \partial_\mu\partial_\nu\psi - g^2 A_\mu A_\nu\psi + ig(A_\mu\partial_\nu + A_\nu\partial_\mu)\psi + ig(\partial_\mu A_\nu)\psi \end{aligned}$$

The antisymmetric combination is

$$(D_\mu D_\nu - D_\nu D_\mu)\psi = ig(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi = ig(F_{\mu\nu})\psi$$

From

$$[(D_\mu D_\nu - D_\nu D_\mu)\psi]' = e^{-i\alpha}(D_\mu D_\nu - D_\nu D_\mu)\psi$$

we see

$$F'_{\mu\nu} = F_{\mu\nu}$$

is gauge invariant.
The complete Lagrangian

$$\mathcal{L} = \bar{\psi} i\gamma^\mu(\partial_\mu + igA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

is invariant under gauge transformation

$$\psi(x) \to \psi'^{-i\alpha(x)}\psi(x)$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x)$$

Remarks:

1. $A_\mu A^\mu$ term is not gauge invariant $\Rightarrow$ field massless.
2. $D_\mu\psi = (\partial_\mu + igA_\mu)\psi$ the coupling is universal
3. no gauge self coupling

Recipe for the construction of theory with local symmetry

1. Write down a Lagrangian with local symmetry

2. Replace $\partial_\mu \phi$ by covariant derivative $D_\mu \phi \sim \left(\partial_\mu - igA_\mu^a t^a\right) \phi$ where guage fields A_μ^a have been introduced.
3. Use $\left(D_\mu D_\nu - D_\nu D_\mu\right) \phi \sim F_{\mu\nu}^a \phi$ to construct the field tensor $F_{\mu\nu}^a$ and add $-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}$ to the Lagrangian density

Non-Abelian symmetry-Yang Mills fields

1954: Yang-Mills generalized U(1) local symmetry to SU(2) local symm.
Consider isospin doublet

$$\psi = \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right)$$

Under SU(2) transformation

$$\psi(x) \to \psi'(x) = \exp\{-\frac{i\vec{\tau}\cdot\vec{\theta}}{2}\}\psi(x),$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are Pauli matrices, with

$$[\frac{\tau_i}{2}, \frac{\tau_j}{2}] = i\epsilon_{ijk}(\frac{\tau_R}{2})$$

Start with free Lagrangian which is invariant under $SU(2)$ symm,

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi$$

Under local symmetry transformation,

$$\psi(x) \to \psi'(x) = U(\theta)\psi(x) \quad \text{with} \quad U(\theta) = exp\{-\frac{i\vec{\tau}\theta(\vec{x})}{2}\}$$

As usual for local symmetry, the derivative term does not transform linearly,

$$\partial_\mu \psi(x) \to \partial_\mu \psi'(x) = U \partial_\mu \psi + (\partial_\mu U)\psi$$

Introduce gauge fields $\vec{A}_\mu$ to form covariant derivative,

$$D_\mu \psi(x) \equiv (\partial_\mu - ig \frac{\vec{\tau} \cdot \vec{A}_\mu}{2})\psi$$

Require that $D_\mu \psi$ transforms as $\psi(x)$

$$[D_\mu \psi]' = U[D_\mu \psi]$$

which requires

$$(\partial_\mu - ig \frac{\vec{\tau} \cdot \vec{A}_\mu{}'}{2})(U\psi) = U(\partial_\mu - ig \frac{\vec{\tau} \cdot \vec{A}_\mu}{2})\psi$$

Or

$$-ig(\frac{\vec{\tau} \cdot \vec{A}_\mu{}'}{2})U + \partial_\mu U = U(-ig \frac{\vec{\tau} \cdot \vec{A}_\mu}{2})$$

$$\frac{\vec{\tau} \cdot \vec{A}_\mu{}'}{2} = U(\frac{\vec{\tau} \cdot \vec{A}_\mu}{2})U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}$$

Use covariant derivatives to construct field tensor

$$D_\mu D_\nu \psi = (\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A_\mu}}{2})(\partial_\nu - ig\frac{\vec{\tau}\cdot\vec{A_\nu}}{2})\psi = \partial_\mu\partial_\nu\psi - ig(\frac{\vec{\tau}\cdot\vec{A_\mu}}{2}\partial_\nu\psi + \frac{\vec{\tau}\cdot\vec{A_\nu}}{2}\partial_\mu\psi)$$

$$-ig\partial_\mu(\frac{\vec{\tau}\cdot\vec{A_\nu}}{2})\psi + (-ig)^2(\frac{\vec{\tau}\cdot\vec{A_\mu}}{2})(\frac{\vec{\tau}\cdot\vec{A_\nu}}{2})\psi$$

Antisymmetrization

$$(D_\mu D_\nu - D_\nu D_\mu)\psi \equiv ig(\frac{\vec{\tau}\cdot\vec{F_{\mu\nu}}}{2})\psi \qquad \frac{\vec{\tau}\cdot\vec{F_{\mu\nu}}}{2} = \frac{\vec{\tau}}{2}\cdot(\partial_\mu\vec{A_\nu} - \partial_\nu\vec{A_\mu}) - ig[\frac{\vec{\tau}\cdot\vec{A_\mu}}{2}, \frac{\vec{\tau}\cdot\vec{A_\nu}}{2}]$$

Or

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk}A^i_\mu A^k_\nu$$

The the term quadratic in A is new in Non-Abelian symmetry.
Under gauge transformation.

$$\vec{\tau}\cdot\vec{F_\mu}\nu' = U(\vec{\tau}\cdot\vec{F_\mu}\nu)U^{-1}$$

<u>Infinitesmal transformation</u> $\theta(x) \ll 1$

$$A^{i/\mu} = A^\mu + \epsilon^{ijk}\theta^j A^k_\mu - \frac{1}{g}\partial_\mu\theta^i$$

$$F^{/i}_{\mu\nu} = F^i_{\mu\nu} + \epsilon^{ijk}\theta^j F^k_{\mu\nu}$$

Remarks

1. Again $A^a_\mu A^{a\mu}$ is not gauge invariant$\Rightarrow$gauge boson massless$\Rightarrow$long range force
2. A^a_μ carries that symmetry charge (e.g. color —)
3. $F^{a\mu\nu} \sim \partial A - \partial A + gAA \longrightarrow$ term responsible for Asymptotic freedom.

Spontaneous symmetry breaking

Spontaneous symmetry breaking—-symm of ground state $\neq$ symmetry of the Hamiltonian
$\Rightarrow$If symmetry is continuous, $\Longrightarrow$ massless scalar fields–Goldstone boson
Example:ferromagnetism
$T > T_c$(Curie temp) all dipoles are randomly oriented——rotational invariant
$T < T_c$ all dipoles are oriented in some direction

Ginzburgh-Landau theory

Free energy as function of magnetization $\vec{M}$ (averaged)

$$\mu(\vec{M}) = (\partial_t \vec{M})^2 + \alpha_1(T)\vec{M}\cdot\vec{M} + \alpha_2(\vec{M}\cdot\vec{M})^2$$

take $\alpha_2 > 0$ so that free energy is positive for large M and $\alpha_1(T) = \alpha(T - T_c) \quad \alpha > 0$ so that there is a transition going through T_c. Ground state is governed by

$$\vec{M}(\alpha_1 + 2\alpha_2\vec{M}\cdot\vec{M}) = 0$$

For $T > T_c$ only solution is $\vec{M} = 0$ and $T < T_c$ non-trivial sol $|\vec{M}| = +\sqrt{\frac{\alpha_1}{2\alpha_2}} \neq 0$
$\Rightarrow$ ground state with $\vec{M}$ in some direction is no longer rotational invariant.

Nambu-Goldstone theorem

Noether's theorem: a continuous symmetry $\Longrightarrow$ conserved charge Q. Suppose 2 local operators A, B with property

$$[Q, B] = A \qquad Q = \int d^3x\, j_0(x) \quad \text{indep of time}$$

Suppose $\langle 0|A|0\rangle = v \neq 0$ (symmetry breaking condition)

$$0 \neq \langle 0|[Q,B]|0\rangle = \int d^3x \ \langle 0|[j_0(x),B]|0\rangle$$

$$= \sum_n (2\pi)^3 \delta^3(\vec{P}_n)\{\langle 0|j_0(0)|n\rangle\langle n|B|0\rangle e^{-iE_n t} - \langle n|B|0\rangle\langle 0|j_0(0)|n\rangle e^{-iE_n t}\} = v$$

Since $V \neq 0$ and time-independent, we need to a state such that

$$E_n \to 0 \qquad for \ \ \vec{P}_n = 0$$

massless excitation. For relativistic particle $E = \sqrt{\vec{P}^2 + m^2}$,this implies massless particle— Goldstone boson.

Discrete symmetry case

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \ , \qquad \phi \to -\phi \ \ \textit{symmetry}$$

The Hamiltonian density

$$H = \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

Effective energy

$$\mu(\phi) = \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi) \ \ , \ \ V(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

For $\mu^2 < 0$ the ground state has $\phi = \pm\sqrt{\frac{-\mu^2}{\lambda}}$ classically.
This means the quantum ground state $|0\rangle$ will have the property

$$\langle 0|\phi|0\rangle = v \neq 0 \qquad \text{symmetry breaking condition}$$

Define quantum field ϕ' by $\phi' = \phi - v$

$$\textit{then} \quad \mathcal{L} = \frac{1}{2}(\partial_\mu \phi'^2 - (-\mu^2)\phi'^2 - \lambda v \phi'^3 - \frac{\lambda}{4}\phi'^4$$

No Goldstone boson—-discrete symmetry

Abelian symmetry case

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2] - V(\sigma^2 + \pi^2)$$

with

$$V(\sigma^2 + \pi^2) = -\frac{\mu^2}{2}(\sigma^2 + \pi^2) + \frac{\lambda}{4}(\sigma^2 + \pi^2)^2$$

This system has O(2) symmetry,

$$\begin{pmatrix} \sigma \\ \pi \end{pmatrix} \to \begin{pmatrix} \sigma' \\ \pi' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix}$$

The minimum is located at

$$\sigma^2 + \pi^2 = \frac{\mu^2}{\lambda} = v^2$$

This is a circle *in* $\sigma - \pi$ *plane*. For convenience choose $\langle 0|\sigma|0\rangle = v \quad \langle 0|\pi|0\rangle = 0$.
New quantum field

$$\sigma' = \sigma - v \quad , \quad \pi' = \pi$$

The new Lagrangian is

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \sigma'^2 + (\partial_\mu \pi)^2] - \mu^2 \sigma'^2 - \lambda v \sigma'(\sigma'^2 + \pi'^2) - \frac{\lambda}{4}(\sigma'^2 + \pi'^2)^2$$

Note that there is no π'^2 term, $\Rightarrow$ π' massless Goldstone boson

Non-Abelian case

$\sigma-$model

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \sigma^2 + (\partial_\mu \vec{\pi})^2] + \bar{N} i \gamma^\mu \partial_\mu N + g\bar{N}(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)N - V(\sigma^2 + \vec{\pi}^2)$$

with

$$V(\sigma^2 + \vec{\pi}^2) = -\frac{\mu^2}{2}(\sigma^2 + \vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2$$

It has the following symmetries, in the infinitesmal forms,

$$\begin{cases} \sigma \longrightarrow \sigma' = \sigma \\ \overrightarrow{\pi} \longrightarrow \overrightarrow{\pi}' = \overrightarrow{\pi} + i\overrightarrow{\alpha} \times \overrightarrow{\pi} \\ N \longrightarrow N' = N - i\dfrac{\overrightarrow{\alpha}\cdot\overrightarrow{\tau}}{2}N \end{cases}$$

and

$$\begin{cases} \sigma \longrightarrow \sigma' = \sigma + \overrightarrow{\beta}\cdot\overrightarrow{\pi} \\ \overrightarrow{\pi} \longrightarrow \overrightarrow{\pi}' = \overrightarrow{\pi} - \overrightarrow{\beta}\sigma \\ N \longrightarrow N' = N - i\dfrac{\overrightarrow{\beta}\cdot\overrightarrow{\tau}}{2}\gamma_5 N \end{cases}$$

where $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ are arbitrary parameters. The symmetry is of the form, $SU(2) \times SU(2)$. This Lagrangian has been used in 1960's to describe the interaction between pions and nucleons.

Note that the nucleon is massless due to the symmetry under the axial transformation. As for the spontaneous symmetry breaking, it is easy to see that the minimum is located at

$$\sigma^2 + \vec{\pi}^2 = v^2 = \frac{\mu^2}{\lambda}$$

If we choose

$$\langle \sigma \rangle = v \quad , \quad \langle \vec{\pi} \rangle = 0$$

The $\vec{\pi}$ are Goldstone bosons. The symmetry is broken from $SU(2) \times SU(2)$ to $SU(2)$. Note that the spontaneous symmetry also give mass to the nucleon,

$$M_N = gv.$$

Higgs Phenomena

Combine spontaneous symmetry breaking with local symmetry$\Longrightarrow$ *Higgs* Phenomena.

Discovered in the $60's$ by Higgs, Englert & Brout, Guralnik, Hagen & Kibble independently

Abelian case

Consider the Lagrangian

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) + \mu^2\phi\phi^\dagger - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where

$$D^\mu\phi = (\partial^\mu - igA^\mu)\phi \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This is invariant under the local gauge transformation

$$\phi(x) \rightarrow \phi' = e^{-i\alpha(x)}\phi(x)$$

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x)$$

The spontaneous symm. breaking is generated by the potential

$$V(\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$$

minimum at

$$\phi^\dagger\phi = \frac{v^2}{2} = \frac{1}{2}(\frac{\mu^2}{\lambda})$$

In quantum theory, choose

$$|\langle 0|\phi|0\rangle| = \frac{\nu}{\sqrt{2}}$$

Write

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

With the choice

$$\langle\phi_1\rangle = \nu \quad , \quad \langle\phi_2\rangle = 0$$

ϕ_2 corresponds to Goldstone boson. Define quantum fields by

$$\phi_1' = \phi_1 - \nu \quad , \quad \phi_2' = \phi_2$$

Covariant derivative terms gives

$$(D_\mu\phi)^+(D^\mu\phi) = [(\partial_\mu + igA_\mu)\phi^+][(\partial^\mu - igA^\mu)\phi]$$

$$\frac{-1}{2}(\partial_\mu\phi_1' + gA_\mu\phi_2')^2 + \frac{1}{2}(\partial_\mu\phi_2' - gA_\mu\phi_1')^2 + \underline{\frac{g^2\nu^2}{2}A^\mu A_\mu} + \cdots$$

Here we have mass terms for A^μ. Write the scalar field as

$$\phi(x) = \frac{1}{\sqrt{2}}(\nu + \eta(x))e^{i\xi(x)/\nu}$$

"Gauge" transformation:

$$\phi \longrightarrow \phi' = e^{-i\xi(x)/v}\phi(x) \quad , \quad B_\mu = A_\mu(x) - \frac{1}{gv}\partial_\mu \xi$$

$\xi(x)$ disappears from the Lagrangian
Roughly speaking, massless gauge field A_μ combine with Goldstone boson $\xi(x)$ to become massive gauge boson. As a consequence, two long range forces (from Goldstone boson $\xi(x)$ and $A_\mu(x)$) disappear.

Non-Abelian case

SU(2) group: $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ doublet

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$V(\phi) = -\mu^2(\phi^\dagger\phi) + \lambda(\phi^\dagger\phi)^2$$

Spontaneous symmetry breaking:

$$\langle\phi\rangle_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \nu \end{pmatrix} \qquad \nu = \sqrt{\frac{\mu^2}{\lambda}}$$

Define $\phi' = \phi - \langle\phi\rangle_0$
From covariant derivative

$$(D_\mu\phi)^\dagger(D^\mu\phi) = [\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}(\phi' + \langle\phi\rangle_0)]^\dagger[\partial^\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}(\phi' + \langle\phi\rangle_0)]$$

$$\rightarrow \frac{1}{4}g^2\langle\phi\rangle_0(\vec{\tau}\cdot\vec{A}_\mu)(\vec{\tau}\cdot\vec{A}^\mu)\langle\phi\rangle_0 = \frac{1}{2}(\frac{gv}{2})^2\vec{A}_\mu\cdot\vec{A}^\mu$$

All gauge bosons get masses

$$M_A = \frac{1}{2}g\nu$$

The symmetry is completely broken. Write

$$\phi(x) = \exp\{\frac{i\vec{\tau}\cdot\vec{\xi}(x)}{\nu}\}\left(\begin{array}{c} 0 \\ \frac{\nu+\eta(x)}{\sqrt{2}} \end{array}\right)$$

Use "gauge" transformation

$$\phi'(x) = U(x)\phi(x) = \frac{1}{\sqrt{2}}\left(\begin{array}{c} 0 \\ \nu+\eta(x) \end{array}\right)$$

$$\frac{\vec{\tau}\cdot\vec{B}_\mu}{2} = U\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}U^{-1} - \frac{i}{g}[\partial_\mu U]U^{-1}$$

$$\text{where} \quad U(x) = \exp\{\frac{\vec{\tau}\cdot\vec{\xi}}{\nu}\}$$

to transform away $\vec{\xi}(x)$.