Quantum Field Theory

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Weak Interaction

Classification

Because strong interaction are difficult to compute reliably, we distinguish processes involving hadrons from leptons.

1. Leptonic weak interactions
   Here all particles are leptons and we can calculate these processes accurately.
   Examples:
   $$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e \qquad \nu_\mu + e \to \nu_\mu + e$$
   $$\tau^- \to \mu^- + \nu_\tau + \bar{\nu}_\mu \qquad \tau^- \to e^- + \nu_\tau + \bar{\nu}_e$$

2. Semi-leptonic interactions
   These involve both leptons and hadrons. Since we can calculate leptonic parts reliabley, these can be used to study the properties of hadrons.
   Examples :
   $$\pi^- \to \mu^- + \nu_\mu \qquad K^+ \to \pi^0 + e^+ + \nu_e$$
   $$n \to p + e^- + \bar{\nu}_e \qquad \bar{\nu}_\mu + p \to \mu^+ + n$$

3. Non-leptonic interactions
   Here all particles are hadrons and they are the most difficult reactions to study. This class of reactions differ from normal strong interaction in the slower decay rates and smaller crossections.
   Example :
   $$K^+ \to \pi^+ + \pi^0 \qquad K^0 \to \pi^+ + \pi^- + \pi^0$$
   $$\Sigma^+ \to P + \pi^0 \qquad \Lambda \to P + \pi^-$$

Selection Rules in Weak Interaction

1. Leptonic Interaction

Two neutrino experiments: ν from β-decay and ν from π decay are different
If they were the same then,

$$n \longrightarrow p + e + \nu$$

$$\nu + p \longrightarrow \mu^+ + n$$

However, only e^+ is observed in the final product and no μ^+. A simple explanation ν_e from β-decayis different from ν_μ in π-decay accompanied by μ and there is also muon number and electron number conservation

$$\begin{array}{ll} e^-, \nu_e & L_e = 1 \\ e^+, \overline{\nu}_e & L_e = -1 \end{array}$$

Similarly, for the muon number L_μ

$$\begin{array}{ll} \mu^-, \nu_\mu & L_\mu = 1 \\ \mu^+, \overline{\nu}_\mu & L_\mu = -1 \end{array}$$

As a consequenc, the reaction $\mu^\pm \longrightarrow e^\pm + \gamma$ are forbidden and experimentally this is indeed the case. Lepton number conservations seem to hold up very well until neutrino oscillations have been observed recently.

1. The $\Delta S \neq 0$ reactions are about a factor of 10 or so smaller than , $\Delta S = 0$. reactiond
2. Hadrons in the strangeness changing decays satisfy the selection rule

$$\Delta S = \Delta Q$$

For examples,

$$K^+ \longrightarrow \pi^0 \mu^+ \nu_\mu, \qquad \text{but} \quad K^+ \nrightarrow \pi^+ \pi^+ e^- \bar{\nu}_e$$

3. Absence of $\Delta S = 1$ neutral currents
For example,

$$\frac{\Gamma\left(K_L \to \mu^+ \mu^-\right)}{\Gamma\left(K^+ \to \mu^+ \nu\right)} \leq 10^{-9}$$

4. No $\Delta S = 2$ transistion been obsverbed
For example,

$$\Xi^- \nrightarrow n e^- \bar{\nu}_e$$

When quark model was developed, all these properties can be accormodated by writing the hadronic weak current as

$$J_{\mu}^{had} = \left[\bar{u}\gamma_{\mu}\left(1-\gamma_{5}\right)d\cos\theta_{c} + \bar{u}\gamma_{\mu}\left(1-\gamma_{5}\right)s\sin\theta_{c}\right]$$

where $\theta_c \approx 0.25$ is the Cabbibo angle.

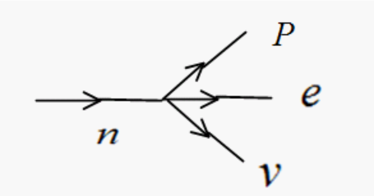
3. Non-leptonic interactions
   Here we havethe $\Delta I = 1/2$ rule,

$$\frac{\Gamma\left(K^{+} \longrightarrow \pi^{+}\pi^{0}\right)}{\Gamma\left(K_{s} \longrightarrow \pi^{+}\pi^{-}\right)} \simeq 1.5 \times 10^{-3}$$

This rule is very difficult to explain because the strong interaction effect.

Milestones of Weak Interaction

1. **Neutrino and Nuclear β decay**,
   The e^- from nuclei decay,
   $$(A, Z) \longrightarrow (A, Z+1) + e^-$$
   was observed to have continuous energy spectrum. If basic mechanism for e^- emission were
   $$n \longrightarrow p + e^-$$
   the energy momentum conservation will require e^- to hav a single energy. Pauli (1930) postulated the presence of **neutrino** which carries away energy and momentum in nuclear β -decay,
   $$n \longrightarrow p + e^- + \overline{\nu}_e$$
2. **Fermi Theory**
   Fermi (1934) proposed to explain the β decay by making analogy with QED to write weak interaction in the form,
   $$\mathcal{L}_F = \frac{G_F}{\sqrt{2}}[\bar{p}(x)\gamma_\mu n(x)][\bar{e}(x)\gamma^\mu \nu_e(x)] + h.c. \quad G_F : \text{ Fermi coupling constant}$$
   Fitting nuclear β decay reates give
   $$G_F \simeq \frac{10^{-5}}{M_p^2}, \qquad M_p \text{ is the proton mass}$$
   This works very well for $\Delta J = 0$, β decays of many nuclei.



Later Gamow-Teller interaction was added

$$\mathcal{L}_{GT} = \frac{G_F}{\sqrt{2}}[\bar{p}(x)\gamma_\mu\gamma_5 n(x)][\bar{e}(x)\gamma^\mu\gamma_5\nu_e(x)] + h.c.$$

to account for $\Delta J = 1$ nuclear β decays.

3. Parity violation and V - A theory
   $\theta - \tau$ puzzle
   In 1950's, it was observed that there are two decays

$$\theta \to \pi^+ + \pi^-, \qquad \text{(even parity)}$$

$$\tau \to \pi^+ + \pi^- + \pi^0, \qquad \text{(odd parity)}$$

while θ and τ have same mass, charge and spin. It is difficult to understand these if the parity is a good symmetry.

1956 : Lee and Yang proposed that parity is not conserved.
1957 : C. S. Wu showed that e^- in ^{60}Co decay has the property,

$$\left\langle \vec{\sigma} \cdot \vec{p} \right\rangle \neq 0, \qquad \vec{\sigma},\ \vec{p} \text{ spin and momentum of } e^-$$

This implies that the parity is violated in this decay.
V-A theory (1958 Feynman and Gell-Mann, Sudarshan and Marshak, Sakurai)
As a result of parity violation, weak interaction was casted in term of $V-A$ currents,

$$L_{eff} = \frac{G_F}{\sqrt{2}} J_\mu^\dagger J^\mu + h.c.$$

where

$$J_\lambda(x) = J_{l\,\lambda}(x) + J_{h\,\lambda}(x)$$

$$J_l^\lambda(x) = \bar{\nu}_e \gamma^\lambda (1-\gamma_5) e + \bar{\nu}_\mu \gamma^\lambda (1-\gamma_5)\mu, \qquad \text{leptonic current} \tag{1}$$

and

$$J_h^\lambda(x) = \bar{u}\gamma^\lambda(1-\gamma_5)(\cos\theta_c d + \sin\theta_c s) \qquad \text{hadronic current}$$

$$\theta_c \ : \ \text{Cabibbo angle}$$

Note that in V-A form the fermion fields are all left-handed.
Define

$$\psi_L \equiv \frac{1}{2}(1-\gamma_5)\psi$$

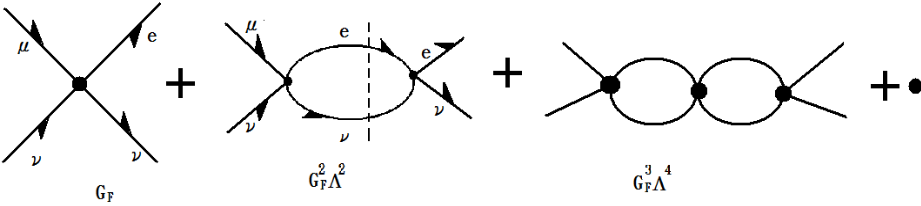
Then we can simplify the weak currents,

$$J_l^\lambda(x) = 2\bar{\nu}_{eL}\gamma^\lambda e_L + 2\bar{\nu}_{\mu L}\gamma^\lambda \mu_L + \ldots$$

Difficulties:

(1) **Not renormalizable**

In Fermi theory, 4 fermions interaction has dimension 6 and is not renormalizable. The higher order graphs are more and more divergent. For example, in μ decay,



(2) **Violate unitarity**

The tree amplitude for $\nu_\mu + e \longrightarrow \mu + \nu_e$ has only J = 1 partial wave at high energies and cross section has the form,

$$\sigma(\nu_\mu e) \approx G_F^2 S, \qquad S = 2m_e E$$

On the other hand, unitarity for J=1 cross section is

$$\sigma(J = 1) < \frac{1}{S}$$

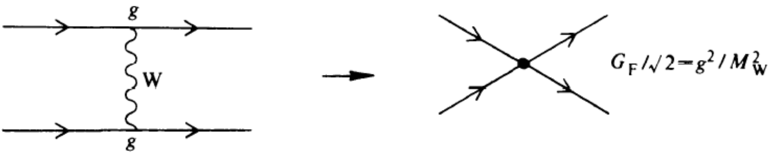
Thus $\sigma(\nu_\mu e)$ violates unitarity for $E \geq 300 GeV$. Since unitarity comes from conservation of probablity, this violation is unacceptable.

Intermidate Boson Theory(IVB)

In analogy with QED, introduce vector boson W to couple to the V-A current

$$\mathcal{L}_W = g(J_\mu W^\mu + h.c.)$$

For example, the μ decay is now mediated by W-exchange.



Since weak interaction is short range, we need $M_W \neq 0$. Use W-boson propagator in the form

$$\frac{-g^{\mu\nu} + \frac{k^\mu k^\nu}{M_W^2}}{k^2 - M_W^2} \to \frac{g^{\mu\nu}}{M_W^2} \quad \text{when} \quad |k_\mu| \ll M_W$$

This reproduces 4-fermion interaction with $\dfrac{g^2}{M_W^2} = \dfrac{G_F}{\sqrt{2}}$

In this theory, $\nu_\mu + e \rightarrow \mu + \nu_e$ no longer violates unitarity. But the violation of unitarity shows up in

$$\nu + \bar{\nu} \rightarrow W^+ + W^-$$

and the theory is still non-renormalizable.

Construction of SU(2)×U(1) model

Choice of group

In IVB theory, interaction is

$$\mathcal{L}_W = g(J_\mu W^\mu + h.c)$$

For simplicity neglect all other fermions excepts ν, e and write

$$J_\mu = \bar{\nu}\gamma_\mu(1-\gamma_5)e$$

In electromagnetic interaction, we have

$$\mathcal{L}_{em} = eJ_\mu^{em}A^\mu, \qquad \text{where} \qquad J_\mu^{em} = \bar{e}\gamma_\mu e$$

Define eletromagnetic and weak charges as the intergals

$$T_+ = \frac{1}{2}\int d^3x J_0(x) = \frac{1}{2}\int d^3x \nu^\dagger(1-\gamma_5)e \qquad T_- = (T_+)^\dagger$$

$$Q = \int d^3x J_0^{em}(x) = -\int d^3x e^\dagger e$$

We can compute the commutator $[T_+, T_-] = 2T_3$ and

$$T_3 = \frac{1}{4}\int d^3x[\nu^\dagger(1-\gamma_5)\nu - e^\dagger(1-\gamma_5)e] \neq Q$$

These 3 charges, T_+, T_- and Q don't form a $SU(2)$ algebra. Note weak charges $T_\pm$ have $V - A$ form while the em charge Q is pure vector.
At this point, there are 2 alternatives:

1. Introduce another guage boson coupled to T_3. This leads to group $SU(2) \times U(1)$. This is the choice we will adapt eventually.
2. Add new fermions such that T_+, T_- and Q do form a SU(2) algebra (Georgi and Glashow 1972) e.g.

$$\frac{1}{2}(1-\gamma_5)\begin{pmatrix} E^+ \\ \nu_e \cos\alpha + N \sin\alpha \\ e^- \end{pmatrix}$$

$$\frac{1}{2}(1+\gamma_5)\begin{pmatrix} E^+ \\ N \\ e^- \end{pmatrix}$$

and a singlet

$$\frac{1}{2}(1+\gamma_5)(N\cos\alpha - \nu_e \sin\alpha)$$

The weak charge is

$$\begin{aligned} T_+ &= \frac{1}{2}\int d^3x \left[E^+(1-\gamma_5)(\nu_e\cos\alpha + N\sin\alpha)\right] \\ &+ (\nu_e\cos\alpha + N\sin\alpha)(1-\gamma_5)e + E^+(1+\gamma_5)N + N^\dagger(1+\gamma_5)e \end{aligned}$$

We can verify that

$$[T_+, T_-] = 2Q$$

with

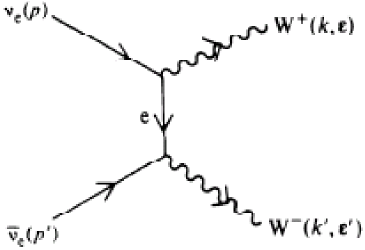
$$Q = \int d^3x \left[E^\dagger E - e^\dagger e\right]$$

Clearly, here only electromagnetic current is neutral and is ruled out by the discoveries of neutral weak current reactions in 1973.

Unitarity argument

Equivalently we can argue from unitarity that we needs to introduce either new leptons or a new guage boson.

Consider in IVB theory the reaction $\nu + \bar{\nu} \longrightarrow W_L^+ + W_L^-$. The lowest order amplitude from following graph is



$$
\begin{aligned}
T\left(\nu\bar{\nu} \longrightarrow W^{+}W^{-}\right) &= -i\bar{v}\left(p'\right)\left(-ig\not{\varepsilon}'\right)\left(1-\gamma_5\right)\frac{i}{\not{p}-\not{k}-m_e}\left(-ig\not{\varepsilon}\right)\left(1-\gamma_5\right)u\left(p\right) \quad (2)\\
&= -2g^2\bar{v}\left(p'\right)\frac{\not{\varepsilon}'\left(\not{p}-\not{k}\right)\not{\varepsilon}\left(1-\gamma_5\right)}{\left(p-k\right)^2-m_e^2}u\left(p\right)
\end{aligned}
$$

W at rest, polarization vectors are

$$
\varepsilon_\mu^{(i)}\left(k\right) \quad \text{with} \quad \varepsilon^{(i)}\cdot\varepsilon^{(j)} = -\delta_{ij} \quad \text{and} \quad k\cdot\varepsilon^{(i)} = 0
$$

A simple choice

$$
\varepsilon_0^{(i)} = 0, \qquad \varepsilon_j^{(i)} = \delta_{ij}
$$

For a moving W with $k_\mu = (E, 0, 0, k)$ and $k = \sqrt{E^2 - M_w^2}$, make a Lorentz transformation along $z - axis$. The transverse polarizations do not change while the longitudinal polarization becomes $\varepsilon_\mu^{(3)} = \dfrac{1}{M_W}(k, 0, 0, E)$. In the high energy limit with $k = E - \dfrac{M_W^2}{2E} + \cdots$, we see

$$
\varepsilon_\mu^{(3)} = \frac{k_\mu}{M_W} + O\left(\frac{M_W}{E}\right)
$$

Then W_L bosons, scattering amplitude in Eq(2) becomes,

$$T \approx -\frac{2g^2}{k^2 - 2p\cdot k}\bar{v}\left(p'\right)\frac{\not{k}'}{M_W}\left(\not{p} - \not{k}\right)\frac{\not{k}}{M_W}\left(1-\gamma_5\right)u\left(p\right) \tag{3}$$

$$\approx \frac{2g^2}{M_W^2}\bar{v}\left(p'\right)\not{k}'\left(1-\gamma_5\right)u\left(p\right)$$

To show that this is a pure $J = 1$ partial wave, take

$$p_\mu = \left(E, 0, 0, E\right), \qquad p'_\mu = \left(E, 0, 0, -E\right)$$

$$k_\mu = \left(E, k\vec{e}\right), \qquad k'_\mu = \left(E, -k\vec{e}\right), \qquad \text{with} \quad \vec{e} = \left(\sin\theta, 0, \cos\theta\right)$$

Since ν and $\bar{\nu}$ have opposite helicities, we have

$$u\left(p\right) = \sqrt{E}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E} \end{pmatrix}\chi_{-1/2} = \sqrt{E}\begin{pmatrix} 1 \\ \sigma_z \end{pmatrix}\chi_{-1/2}$$

$$\bar{v}\left(p'\right) = \sqrt{E}\chi_{1/2}^{\dagger}\left(\frac{\vec{\sigma}\cdot\vec{p}'}{E}, -1\right) = \sqrt{E}\chi_{1/2}^{\dagger}\left(-\sigma_z, -1\right)$$

where

$$\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then the combination in Eq(3) becomes,

$$\bar{v}\left(p'\right) \not{k}' \left(1-\gamma_5\right) u\left(p\right) = E\chi^{\dagger}_{1/2}\left(-1,-1\right)\begin{pmatrix} E & k\vec{\sigma}\cdot\vec{e} \\ -k\vec{\sigma}\cdot\vec{e} & -E \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\chi_{-1/2}$$
$$= -4E\chi^{\dagger}_{1/2}\left(E-k\vec{\sigma}\cdot\vec{e}\right)\chi_{-1/2} = 4Ek\sin\theta$$

We have then

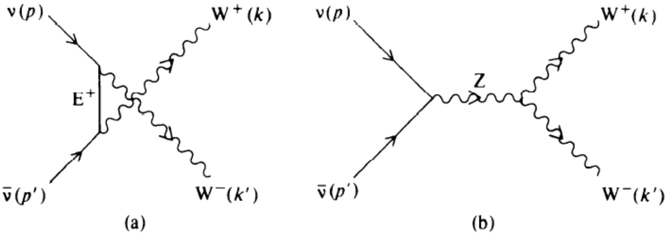
$$T \approx G_F E^2 \sin\theta \qquad \text{as} \qquad E\longrightarrow\infty \tag{4}$$

The partial wave expansin for the helicity amplitude is,

$$T_{\lambda_3\lambda_4,\lambda_1\lambda_2}\left(E,\theta\right) = \sum_{J=M}^{\infty}\left(2J+1\right)T^{J}_{\lambda_3\lambda_4,\lambda_1\lambda_2}\left(E\right)d^{J}_{\mu\lambda}\left(\theta\right)$$

where $\lambda_1 = -\lambda_2 = 1/2$ and $\lambda_3 = \lambda_4 = 0$ are the helicities of the initial and final particles with $\lambda = \lambda_1 - \lambda_2 = 1$, $\mu = \lambda_3 - \lambda_4$ and $M = \max\left(\lambda,\mu\right) = 1$. $d^{J}_{\mu\lambda}\left(\theta\right)$ is the usual rotation matrix with $d^{1}_{10}\left(\theta\right) = \sin\theta$. It is clear that T in Eq(4) corresponds to a pure $J=1$ partial wave and violates the unitarity bound of $T^{J=1}\left(E\right) \leq$ constant at high energies. To cancel this bad high energy behavior we need other diagrams. There are 2 possibilities: s-channel or u-channel diagrams.

1. heavy lepton alternative;
   u-channel exchange from diagram(a) yields amplitude,



$$
\begin{aligned}
T_u\left(\nu\bar{\nu} \longrightarrow W^+W^-\right) &= -2g'^2\bar{v}\left(p'\right)\frac{\not{\varepsilon}(\not{p}-\not{k})\not{\varepsilon}\left(1-\gamma_5\right)}{\left(p-k'\right)^2-m_E^2}u\left(p\right) \\
&= \frac{-2g'^2}{M_W^2}\bar{v}\left(p'\right)\not{k}'\left(1-\gamma_5\right)u\left(p\right)
\end{aligned}
$$

If $g^2 = g'^2$, this will cancel bad behavior given in Eq(3)

2. neutral vector boson alternative;
the s-channel exchange in the diagram (b) gives

$$T_s\left(\nu\bar{\nu}\longrightarrow W^+W^-\right) = -i\bar{v}\left(p'\right)\left(-if\gamma_\beta\right)\left(1-\gamma_5\right)u\left(p\right)L_{\alpha\mu\nu}\varepsilon'^\mu\left(k'\right)\varepsilon^\nu\left(k\right) \times i\left[-g^{\alpha\beta}+\frac{\left(k+k\right)^\alpha\left(k+k\right)^\beta}{M_Z^2}\right]\left[\frac{1}{\left(k+k\right)^2-M_Z^2}\right]$$

Choose ZWW coupling to have Yang-Mills structure

$$L_{\alpha\mu\nu}=-if'\left[\left(k'-k\right)_\alpha g_{\mu\nu}-\left(2k'+k\right)_\nu g_{\mu\alpha}+\left(k'+2k\right)_\mu g_{\alpha\nu}\right]$$

we get

$$\begin{aligned} L_{\alpha\mu\nu}\varepsilon'^\mu\left(k'\right)\varepsilon^\nu\left(k\right) &= -if'\left[\left(k'-k\right)_\alpha\varepsilon\cdot\varepsilon'-\left(2k'\cdot\varepsilon\right)\varepsilon'_\alpha+\left(2k\cdot\varepsilon'\right)\varepsilon_\alpha\right] \\ &\approx \frac{if'}{M_W^2}\left[\left(k'-k\right)_\alpha\left(k\cdot k'\right)\right] \end{aligned}$$

and

$$T_s\simeq-\frac{ff'}{M_W^2}\bar{v}\left(p'\right)\not{k}'\left(1-\gamma_5\right)u\left(p\right)$$

Thus if $ff'=2g^2$, this will also cancel amplitude in Eq(3). This corresponds to adding another $U\left(1\right)$ symmetry
In fact if one demands that all amplitudes which violate unitarity be canceled out, one get renormalizable Lagrangian which is the same as the one derived from the algebraic approach.

Now choose gauge group to be $SU(2) \times U(1)$. The Lagrangian for the gauge fields is

$$L = -\frac{1}{4} F^{i\mu\nu} F^{i}_{\mu\nu} - \frac{1}{4} G^{\mu\nu} G_{\mu\nu}$$

where

$$F^{i}_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\nu + g\epsilon^{ijk} A^j_\mu A^k_\nu \qquad SU(2) \quad \text{gauge fields}$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\mu B_\mu \qquad U(1) \quad \text{gauge field}$$

Fermions

Clearly, from structure of weak charged current given in Eq(1) ν, e form a doublet under SU(2),

$$l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

Then

$$T_+ = \int (\nu_L^+ e_L) d^3x, \quad T_- = \int (e_L^+ \nu_L) d^3x, \quad Q = \int (e_L^+ e_L + e_R^+ e_R)$$

Note that

$$Q - T_3 = \int [-\frac{1}{2}(\nu_L^+ \nu_L + e_L^+ e_L) - e_R^+ e_R] d^3x$$

We can show that

$$[Q - T_3, T_i] = 0 \,, \quad i = 1, 2, 3$$

Take $Q - T_3$ to be $U(1)$ charge $Y \equiv 2(Q - T_3)$, called **weak hypercharge**. The Y charges for fermions are

$$l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \qquad Y = -1, \qquad e_R \;\; Y = -2$$

Lagrangian for gauge coupling is

$$\mathcal{L}_2 = \bar{l}_L i\gamma^\nu D_\nu l_l + \bar{l}_R i\gamma^\nu D_\nu l_R \tag{5}$$

where

$$D_\nu \psi = (\partial_\nu - ig\frac{\vec{\tau}\cdot\vec{A}_\nu}{2} - ig'\frac{Y}{2}B_\nu)\psi$$

For example,

$$D_\nu l_L = (\partial_\nu - ig\frac{\vec{\tau}\cdot\vec{A}_\nu}{2} - ig'\frac{Y}{2}B_\nu)l_L$$

Spontaneous Symmetry Breaking

Symmetry braking pattern we want is $SU(2) \times U(1) \rightarrow U(1)_{em}$. Choose scalar fields in SU(2) doublet with hypercharge $Y = 1$,

$$\phi = \begin{pmatrix} \phi^{\dagger} \\ \phi^{0} \end{pmatrix}, \; Y = 1$$

Lagrangian contanging ϕ is,

$$\mathcal{L}_3 = (D_\mu \phi)^{\dagger} (D^\mu \phi) - V(\phi)$$

where

$$D_\mu \phi = (\partial_\mu - \frac{ig}{2} \vec{\tau} \cdot \vec{A}_\mu - \frac{ig^{'}}{2} B_\mu) \phi$$

and

$$V(\phi) = -\mu^2 \phi^{\dagger} \phi + \lambda (\phi^{\dagger} \phi)^2$$

Coupling between leptons and scalar field ϕ,

$$\mathcal{L}_4 = f \bar{L}_L \phi e_R + h.c.$$

Spontaneous symmetry breaking is generated by the vaccum expectation value

$$< \phi >_0 = \langle 0 | \, \phi \, | 0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \qquad \text{with} \quad v = \sqrt{\frac{\mu^2}{\lambda}}$$

Write the scalar field in the form

$$\phi(x) = U^{-1}(\vec{\xi}) \begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix} \quad \text{where} \quad U(\vec{\xi}) = \exp[\frac{i\vec{\xi}(x)\cdot\vec{\tau}}{v}] \tag{6}$$

Gauge Transformation

We can then simplify the form of scalar by a gauge transformation

$$\phi' = U(\vec{\xi})\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+\eta(x) \end{pmatrix}$$

$$\frac{\vec{\tau}\cdot\vec{A}'_\mu}{2} = U(\vec{\xi})\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}U^{-1}(\vec{\xi}) - \frac{i}{g}(\partial_\mu U)U^{-1}$$

field $\vec{\xi}(x)$ disappears from Lagrangian because of gauge invariance. From $\mathcal{L}_4$ (Yukawa coupling), VEV of the scalar field gives

$$\mathcal{L}_4 = f\frac{1}{\sqrt{2}}(\bar{l}_L < \phi > e_R + h.c.) + f\frac{\eta(x)}{\sqrt{2}}(\bar{e}_L e_R + h.c.)$$

the electron is now massive withs

$$m_e = \frac{f}{\sqrt{2}}v$$

Mass spectrum

We now list the mass spectrum of the theory after the spontaneous symmetry breaking:

1. Fermion mass

$$m_e = \frac{fv}{\sqrt{2}}$$

2. Scalar mass(Higgs)

$$V(\phi') = \mu^2\eta^2 + \lambda v\eta^3 + \frac{\lambda}{4}\eta^4 \qquad \rightarrow \quad m_\eta = \sqrt{2}\mu$$

3. Gauge boson masses
From covariant derivative in $\mathcal{L}_3$

$$\mathcal{L}_3 = \frac{v^2}{2}\chi^\dagger(g\frac{\vec{\tau}\cdot\vec{A}'_\mu}{2} + \frac{g' B'_\mu}{2})(g\frac{\vec{\tau}\cdot\vec{A}'^\mu}{2} + \frac{g' B'^\mu}{2})\chi + \cdots, \qquad \chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we get the mass terms for the gauge bosons,

$$\begin{aligned}\mathcal{L}_3 &= \frac{v^2}{8}\{g^2[(A^{\mathbf{1}}_\mu)^2 + (A^{\mathbf{2}}_\mu)^2] + (gA^{\mathbf{3}}_\mu - g' B_\mu)^2\} + \cdots \\ &= M_W^2 W^{+\mu} W^-_\mu + \frac{1}{2}M_Z^2 Z^\mu Z_\mu + \cdots\end{aligned}$$

where

$$W^+_\mu = \frac{1}{\sqrt{2}}(A^{\mathbf{1}}_\mu - iA^{\mathbf{2}}_\mu), \qquad M_W^2 = \frac{g^2 v^2}{4}$$

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g' A^3 - gB_\mu), \quad M_Z^2 = \frac{g^2 + g'^2}{4} v^2$$

The field

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g' A_\mu^3 + gB_\mu)$$

is massless photon.
For convience we define

$$\tan\theta_W = \frac{g'}{g} \quad \theta_W : \text{Weinberg angle or weak mixing angle}$$

Then we can write

$$Z_\mu = \cos\theta_W A_\mu^3 - \sin\theta_W B_\mu \quad M_Z^2 = \frac{g^2 v^2}{4} \sec^2\theta_W$$

$$A_\mu = \sin\theta_W A_\mu^3 - \cos\theta_W B_\mu$$

Note that there is a relation of the form,

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1$$

which is a consequence of the doublet nature of the scalar fields.

The weak interactions mediated by W and Z bosons can be read out from Eq(5)

1. Charged current

$$\mathcal{L}_{cc} = \frac{g}{\sqrt{2}}(J_\mu^\dagger W^{\dagger\mu} + h.c.) \quad J_\mu^\dagger = J_\mu^1 + iJ_\mu^2 = \frac{1}{2}\bar{\nu}\gamma_\mu(1-\gamma_5)e$$

Again to get 4-fermion interaction as low energy limit, we require

$$\frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}$$

which implies that

$$v = \sqrt{\frac{\sqrt{2}}{G_F}} \approx 246\,Gev$$

This is usually referred to as the **weak scale**.

2. Neutral Current

The Largrangian for the neutral currents is

$$\mathcal{L}_{NC} = gJ^3_\mu A^{\mathbf{3}\mu} + \frac{g^{'}}{2} J^Y_\mu B^\mu = eJ^{em}_\mu A^\mu + \frac{g}{\cos\theta_W} J^Z_\mu Z^\mu$$

where

$$e = g\sin\theta_W,$$

and

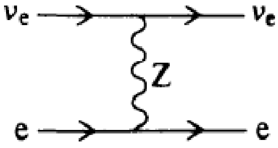
$$J^Z_\mu = J^{\mathbf{3}}_\mu - \sin^2\theta_W J^{em}_\mu$$

is the weak neutral current. We can define the weak neutral charge as

$$Q^Z = \int J^Z_0 d^3x = (T_3 - \sin^2\theta_W Q)$$

So the coupling strength of fermions to Z-boson is proportional to $T_3 - \sin^2\theta_W Q$. In particluar, Z boson can contribute to the scattering

$$\nu_e + e \longrightarrow \nu_e + e$$



The measurement of this corss section in the 1970's give $\sin^2\theta_W \approx 0.22$. This yields $M_W \approx 80$ GeV and $M_Z \approx 90$ GeV.

Generalization to more than one family.
From 4-fermion and IVB theory, the form of weak currents of leptons and hadrons gives the following multiplets structure,

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \quad e_R, \mu_R \quad \begin{pmatrix} u \\ d_\theta \end{pmatrix}_L \quad u_R, d_R, s_R$$

where

$$d_\theta = \cos\theta_C d + \sin\theta_C s$$

The neutral current in the down quark sector is

$$\begin{aligned}\mathcal{L}_{NC} &= [\bar{d}_\theta \gamma_\mu (-\frac{1}{2} + \sin^2\theta_C \frac{1}{3}) d_{\theta L} - \sin^2\theta_W \frac{1}{3}(\bar{d}_R \gamma_\mu d_R + \bar{s}_R \gamma_\mu s_R)] Z^\mu \\ &= (-\frac{1}{2} + \sin^2\theta_W \frac{1}{3})[(\bar{d}_L \gamma_\mu d_L + \bar{s}_L \gamma_\mu s_L) + \sin\theta_W \cos\theta_W (\bar{d}_L \gamma_\mu s_L + \bar{s}_L \gamma_\mu d_L) + \ldots\end{aligned}$$

The term $\left(\bar{d}_L \gamma_\mu s_L + \bar{s}_L \gamma_\mu d_L\right)$ gives rise to $\Delta S = 1$ neutral current processes, e.g. $K_L \to \mu^+ + \mu^-$ with same order of magnitude as charged current interaction. But experimentally,

$$R = \frac{\Gamma(K_L \to \mu^+ + \mu^-)}{\Gamma(K^+ \to \mu + \nu)} \leq 10^{-8}$$

Thus we can not have $\Delta S = 1$ neutral curent process at the same order of magnitude as the charged current process.
GIM mechanism

Glashow, Iliopoulos and Maiani (1970) suggested a 4-th quark, the charm quark c, which couples to the orthogonal combination $s_\theta = -\sin\theta_c d + \cos\theta_c s$ so that

$$\begin{pmatrix} u \\ d_\theta \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s_\theta \end{pmatrix}_L$$

As a result , the $\Delta S = 1$, neutral current is canceled out. The new current is of the form

$$\bar{d}_\theta(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W)\gamma_\mu d_\theta + \bar{s}_\theta(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W)\gamma_\mu s_\theta = (-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W)(\bar{d}\gamma_\mu d + \bar{s}\gamma_\mu s)$$

which conserves the strangeness. This avoids the conflict with exp on $K_L \longrightarrow \mu^+\mu^-$

Quark mixing

Before spontaneous symmetry braking, fermions are all massless becauseψ_L and ψ_R have different quantum numbers under $SU(2)\times U(1)$ i.e. mass term $(\bar{\psi}_L\psi_R + h.c.)$ is not invariant under $SU(2)\times U(1)$. For more than one doublets, ψ_{iR}, ψ_{iL} have same quantum numbers under $SU(2)\times U(1)$ group we call "weak eigenstates". After spontaneous symmetry breaking, fermions obtain their masses through Yukawa coupling.

$$\mathcal{L}_Y = (f_{ij}\bar{g}_{iL}u_{Rj} + f'_{ij}\bar{g}_{iL}d_{Rj})\phi + h.c.$$

Renormalizability requires all possible terms consistent with $SU(2)\times U(1)$ symmetry. Since f_{ij},, f'_{ij} are arbitrary, fermion mass matrices are not diagonal.

$$\text{mass eigenstates} \neq \text{weak eigenstates}$$

The mass matrices in up and down sectors are

$$m_{ij}^{(u)} = f_{ij}\frac{v}{\sqrt{2}} \quad m_{ij}^{(d)} = f'_{ij}\frac{v}{\sqrt{2}}$$

These matrices which are sandwiched between left and right handed fields can be diagonalized by bi-unitary transformations, i.e. given a mass matrix m_{ij},there exits unitary matrices S and T such that

$$S^{\dagger} m T = m_d$$

is diagonal. S is the unitary matrix which diagnoalizes the hermitian combination mm^{+}, i. e.

$$S^{\dagger}(mm^{\dagger})S = m_d^2$$

Biunitary transformation

Write

$$m_d^2 = \begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix}$$

Define

$$m_d = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix}$$

and

$$H = S m_d S^{\dagger} \qquad \text{hermitian}$$

Define a matrix V by

$$V \equiv H^{-1} m$$

Then

$$VV^{\dagger} = H^{-1} m m^{\dagger} H^{-1} = H^{-1} S m_d^2 S^{\dagger} H^{-1} = H^{-1} H^2 H^{-1} = 1$$

So V is unitary and we have

$$S^{\dagger} H S = m_d, \qquad \Longrightarrow \qquad S^{\dagger} m V^{\dagger} S = m_d$$

Or

$$S^{\dagger} m T = m_d, \qquad \text{with} \qquad T = V^{\dagger} S$$

————————-

If we write the doublets, (weak eigenstates) as

$$q_{1L} = \begin{pmatrix} u' \\ d' \end{pmatrix}_L \quad q_{2L} = \begin{pmatrix} c' \\ s' \end{pmatrix}_L$$

These weak eigenstates are related to mass eigenstates by unitary transformations,

$$\begin{pmatrix} u' \\ c' \end{pmatrix} = S_u \begin{pmatrix} u \\ c \end{pmatrix}, \qquad \begin{pmatrix} d' \\ s' \end{pmatrix} = S_d \begin{pmatrix} d \\ s \end{pmatrix}$$

Note that in the coupling to charged gauge boson $W^{\pm}$, we have

$$\mathcal{L}_W = \frac{g}{\sqrt{2}} W_\mu [\bar{q}_{1L}\gamma^\mu \tau^\dagger q_{1L} + \bar{q}_{2L}\gamma^\mu \tau^\dagger q_{2L}] + h.c.$$

and is invariant under unitary transformation in q_{1L}, q_{2L} space, i.e.

$$\begin{pmatrix} q'_{1L} \\ q'_{2L} \end{pmatrix} = V \begin{pmatrix} q_{1L} \\ q_{2L} \end{pmatrix} \quad VV^\dagger = 1 = V^\dagger V$$

We can use this feature to put all mixing in the down quark sector,

$$q'_{iL} = \begin{pmatrix} u \\ d'' \end{pmatrix}_L, \begin{pmatrix} c \\ s'' \end{pmatrix}_L, \qquad \text{where} \quad \begin{pmatrix} d'' \\ s'' \end{pmatrix} = U \begin{pmatrix} d \\ s \end{pmatrix}$$

Here U is a 2×2 unitary matrix. Clearly, we can extend this to 3 generations with result

$$q_{iL} : \begin{pmatrix} u \\ d'' \end{pmatrix}, \begin{pmatrix} c \\ s'' \end{pmatrix}, \begin{pmatrix} t \\ b'' \end{pmatrix}, \qquad \begin{pmatrix} d'' \\ s'' \\ b'' \end{pmatrix} = U \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

Now U is a 3×3 unitary matrix, usually called the Cabibbo-Kobayahsi-Maskawa (CKM) matrix.

CP violation Phase

CP violetion can come form complex coupling to gauge bosons. The coupling of $W^{\pm}$ to quarks is governed by the 3×3 unitary matrix U . This unitary matrix U can have many complex entries. However, in diagonalzing mass matrices, $S^{\dagger}(mm^{\dagger})S = m_d^2$. There is arbitrariness in the matrix S, in the form of diagonal phases i.e. if S diagonalizes the mass matrix, so does $S^{'}$

$$S^{'} = S \begin{pmatrix} e^{i\alpha_1} & \dots & \dots \\ \vdots & \ddots & \vdots \\ \vdots & \dots & e^{\alpha_n} \end{pmatrix}$$

We can then redefine the quark fields to get rid of some phases in U. It turns out that for n$\times$n unitary matrix, number of independent physical phases left over is

$$\frac{(n-1)(n-2)}{2}$$

Thus to get CP violetion we need 3 generations or more (Kobayashi Mskawa). Here we give a constructive proof of this statement. Start with a first doublet written as,

$$q_{1L} = \begin{pmatrix} u \\ U_{11}d + U_{12}s + U_{13}b \end{pmatrix}$$

If U_{11} has phase δ,

$$U_{11} = R_{11}e^{i\delta}, \qquad R_{11} \qquad \text{real}$$

then δ can be absorbed in the redefiniton of the $u-$quark

$$u \longrightarrow u' = ue^{-i\delta}$$

and we can write

$$q_{1L} = e^{i\delta} \begin{pmatrix} u' \\ R_{11} d + U'_{12} s + U'_{13} b \end{pmatrix}$$

Similarly, we can factor out the complex phases of U_{21} and U_{31} by redefinition of c and t quark fields. These overall phases are immaterial. Finally we can absorb two more phases of U_{12} and U_{13} by a redefinition of the s and b fields. The doublets now take the form

$$\begin{pmatrix} u' \\ R_{11} d + R_{12} s + R_{13} b \end{pmatrix}_L, \qquad \begin{pmatrix} c' \\ R_{21} d + R_{22} e^{i\delta_1} s + R_{23} e^{i\delta_2} b \end{pmatrix}_L,$$

$$\begin{pmatrix} t' \\ R_{31} d + R_{32} e^{i\delta_3} s + R_{33} e^{i\delta_4} b \end{pmatrix}_L,$$

Now we have reduced the number of parameters to 13. The normalization conditons of each down-like state gives 3 real conditions and orthogonality conditions among different states give 6 real conditions on the parameters, Now we are down to 4 parameters. Since we need 3 parameters for the real orthogonal matrix, we end up with one independent phase.

Flavor consevation in neutral current interaction

The coupling of neutral Z boson to the fermions conserve flavors. This can be illustrated as follows. Write the neutral currents in terms of weak eigenstates,

$$J_\mu^Z = \sum_i \bar{\psi}_i \gamma_\mu \left[T_3 \left(\psi_i \right) - \sin^2 \theta_W Q \left(\psi_i \right) \right] \psi_i$$

Separate into left- and right-handed fields and distingush the up and down components,

$$\begin{aligned} J_\mu^Z &= \sum_i (\bar{u}'_{Li} \gamma_\mu \left[\frac{1}{2} - \sin^2 \theta_W \left(\frac{2}{3} \right) \right] u'_{Li} + \bar{d}'_{Li} \gamma_\mu \left[-\frac{1}{2} + \sin^2 \theta_W \left(\frac{1}{3} \right) \right] d'_{Li} \\ & \quad + \bar{u}'_{Ri} \gamma_\mu \left[- \sin^2 \theta_W \left(\frac{2}{3} \right) \right] u'_{Ri} + \bar{d}'_{Ri} \gamma_\mu \left[\sin^2 \theta_W \left(\frac{1}{3} \right) \right] d'_{Ri} \end{aligned}$$

Since weak eigen states q_{iL} and mass eigen states q'_{iL} are related by unitary matries,

$$u'_{Li} = U \left(u_L \right)_{ij} u_{Lj}, \qquad \cdots$$

We see tha unitary matrices cancel out in the combination, $\bar{u}'_{Li} u'_{Li}$ so that the neutral current in terms of mass eigenstates has the same form as the one in terms of weak eigenstates. Thus it conserves all quark flavor. Note this feature is due to the fact that all quarks with same helicity and electric charge have the same quantum number with repect to $SU(2) \times U(1)$ gauge group.

Appendix

Unitarity

Scattering in Quantum Mechanics

For a free particle moving in $z-$direction, the plane wave can be expanded in terms of Legendre polynomials as,

$$e^{ikz} = \sum_l (2l+1)\, i^l j_l\,(kr)\, P_l\,(\cos\theta)$$

where $j_l\,(kr)$ is the spherical Bessel function and for large r it can be written as

$$j_l\,(kr) \sim \frac{1}{2ikr}\left[e^{i(kr-l\pi/2)} - e^{-i(kr-l\pi/2)}\right]$$

For the Schrodinger equation in central potential,

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V\,(r)\right]\psi\left(\vec{r}\right) = E\psi\left(\vec{r}\right)$$

the wave function $\psi\left(\vec{r}\right)$ can also be expanded in terms of Legendre polynomials in the form,

$$\psi\left(\vec{r}\right) = \sum_l (2l+1)\, i^l R_l\,(kr)\, P_l\,(\cos\theta)\,, \qquad \text{with} \qquad E = \frac{\hbar^2 k^2}{2m}$$

Here $R_l(kr)$ is the radial wave function. Assuming that $V(r)$ is a short range force, then for large r we can write

$$R_l(kr) \sim \frac{1}{2ikr}\left[\eta_l(k)\, e^{i(kr-l\pi/2)} - e^{-i(kr-l\pi/2)}\right]$$

Probability conservation (unitarity) requires

$$|\eta_l(k)| \leq 1$$

For the case of elastic scattering we have

$$|\eta_l(k)| = 1$$

and can be written as

$$\eta_l(k) = e^{2i\delta_l(k)}, \qquad \delta_l(k): \text{ phase shift}$$

Usually the asymptotic form of the wavefunction is written as

$$\psi\left(\vec{r}\right) \sim e^{ikz} + f(k,\theta)\,\frac{e^{ikr}}{r}, \qquad f(k,\theta): \text{ scattering amplitude}$$

Expansion in terms of Legendre polynomials gives,

$$f(k,\theta) = \sum_l (2l+1)\, f_l(k)\, P_l(\cos\theta) = \frac{1}{2ik}\sum_l (2l+1)\left[\eta_l(k) - 1\right] P_l(\cos\theta)$$

$f_l(k)$ is usually called the partial wave. Differential cross section is of the form,

$$\frac{d\sigma}{d\Omega} = |f(k,\theta)|^2$$

and the total cross section is

$$\sigma = 4\pi \sum_l (2l+1)|f_l(k)|^2 = \frac{\pi}{k^2}\sum_l (2l+1)|\eta_l(k)-1|^2$$

From unitarity, the maximum contribution of any partial wave to the total cross section is

$$\sigma_l \le \frac{4\pi(2l+1)}{k^2}$$

i. e. each partial wave cross section can not grow faster than $\frac{1}{k^2} \sim \frac{1}{E}$
For relativistic system, we have partial wave expansion for the helicity amplitude in the form,

$$T_{\lambda_3\lambda_4,\lambda_1\lambda_2}(E,\theta) = \sum_J (2J+1)\, T^J_{\lambda_3\lambda_4,\lambda_1\lambda_2}(E)\, d^J_{\mu\lambda}(\theta)$$

where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_3 - \lambda_4$ and $d^J_{\mu\lambda}(\theta)$ is the $d - function$. Then the unitarity requires

$$\left|T^J_{\lambda_3\lambda_4,\lambda_1\lambda_2}(E)\right| \le const$$