Quantum Field Theory

Ling-Fong Li

National Center for Theoretical Science

Deep Inelastic Scattering

Introduction

Many important development comes from studies of proton $M_p = 938.3\ Mev/c^2$. But proton is a hadron with strong interaction which can not be handled by perturbation which was successful in QED. In earlier days we do not know what the right theory looks like. A series of experiments in late 60's and early 70'on elecron proton scatterings has led to the formulation of strong interaction in the form of QCD. Even though QCD works quite well in high energies, it is still hampered by large coupling constants at low erengies.

Structure of proton

Electron proton scattering

In *ep* scattereing, probe structure of *p* with *e* well described by QED. To reveal the structure of proton on some scale depends on the wavelength or energy of the probe. Here we list in the in the order of increasing enegy the description used to describe this reaction.

1. Rutherford formula

   Here electron energy is low and can be treated as non-relativistic particle and proton can be treated as a point particle and neglect the recoil of proton. The differential cross section is,

$$\left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}}$$

E : incident energy. θ : scattering angle. α: fine structure constant

   This is derived from classcial mechanics. In fact, Rutherford used this formula to infer that the atom is not a point particle but has a structure.

2. Mott fromula

Take into account of the spin of electron and relativistic nature of electron, we get Mott cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \left(\frac{d\sigma}{d\Omega}\right)_{Rutherford}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)$$

Here proton is still treated as a point particle with no structure.

3. Rosenbluth formula

As energy of electron is large enough, need to take consider strong interaction. In this simple case, we can parametrize the strong interaction effect of the proton in terms of form factors because em current of proton is local even when strong interaction is included. We will describe this as follows.

If proton were a pointed particle, interaction of proton with photon is,

$$\left\langle p'|J_\mu^{em}|p\right\rangle = \bar{u}(p')\gamma_\mu u(p)$$

Include strong interaction of proton, we can parametrize this interaction as

$$\langle p'|J_\mu^{em}|p\rangle = \bar{u}(p')\left[\gamma_\mu F_1(q^2) + i\frac{\sigma_{\mu\nu}q^\nu}{2m}F_2(q^2)\right]u(p)$$

where we have used the Lornetz covariance and current conservation to deduce thie simple form. Here $q = p - p'$, and $F_1(q^2), F_2(q^2)$ are functions which includes strong interaction are called *form factors*.As was described in Note 6 , the differential cross section is,

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\left[\frac{G_E^2(Q^2)+\tau G_M^2(Q^2)}{1+\tau} + 2\tau G_M^2(Q^2)\tan^2\frac{\theta}{2}\right],$$

This is Rosenbluth formula. Here $\tau = \frac{Q^2}{4M^2}$ and $Q^2 = -q^2$. The combinations

$$G_E(q^2) = F_1 + \tau F_2$$

$$G_M(q^2) = F_1 + F_2$$

are electric and magnetic form factors repectively and they satisfy

$$G_E(0) = F_1(0) = 1 \quad \text{total charge}$$

$$G_M(0) = F_1(0) + F_2(0) = 1 + F_2(0) \quad \text{magnetic moment}$$

Experimental measurements

$$G_M^p(0) = 2.79\mu_N \quad G_M^n(0) = -1.91\mu_N \quad \mu_N = \frac{e}{2M_p} \quad \textit{nuclear magneton}$$

are the anomalous magnetic moments of nucleons,

$$G_E^p(Q^2) = \frac{G_M^P(q^2)}{2.79} = \frac{G_n^M(Q^2)}{-1.91} \approx \frac{1}{(1 - q^2/0.7Gev^2)^2}$$

These are known as the dipole form factor. $F_1\left(q^2\right)$ can be related to Fourier transform of charge distribution,

$$F(q^2) = \int e^{i\vec{q}\cdot\vec{x}} \rho(x) d^3x \longrightarrow \rho(x) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} F(q^2)$$

The measurement of form factor will give information about the charge distribution. For spherical charge distribution, we can write

$$F(q^2) = 1 - \frac{1}{6}\vec{q}^2\langle r^2\rangle + \cdots$$

where

$$\langle r^2\rangle = 4\pi\int_0^\infty r^2 f(r) r^2 dr, \quad \text{charge radius,}$$

For proton

$$\langle r^2\rangle_p roton \simeq (0.86 fm)^2$$

Note that for $-q^2$ large, form factors decrease very fast $\sim \frac{1}{q^4}$

Summary:

1. Proton is not a point particle and has structure
2. The structure of proton can be described by two form factors $F_1\left(q^2\right)$, $F_2\left(q^2\right)$

The charge distribution of proton gives charge radius about 0.86 *fm*.

Deep Inelastic ep scattering

As energy of electron gets large, the inelastic channels become more important. As number of particles in the final state increase the form factors approach is no longer useful. It is remarkable that description becomes simple when we add up all the final states in the inelastic channels. To describ this, wrtie the inelastic scattering as

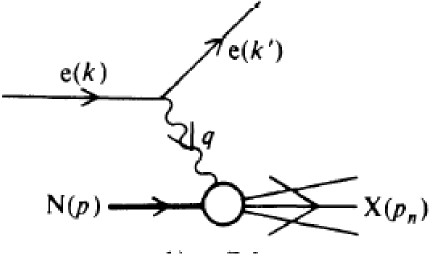
$$e + p \longrightarrow e + X$$

where X denotes generic final state. The cross section where the final state is summed over is called the **inclusive cross section**. For example the inclusive differential cross section is of the form,

$$\frac{d^2\sigma}{d\Omega dE'}\left(inclusive\right) = \sum_X \frac{d^2\sigma}{d\Omega dE'}\left(e + p \longrightarrow e + X\right)$$

E' energy of final state electron. Denote the momenta of this reaction as

$$e\left(k\right) + p\left(p\right) \longrightarrow e\left(k\right) + X\left(p_n\right)$$



Deifne kinematic variables by

$$q = k - k', \qquad \nu = \frac{p \cdot q}{M}, \qquad W^2 = p_n^2 = (p+q)^2$$

In the lab-frame we have

$$p_\mu = (M, 0, 0, 0)\,, \qquad k_\mu = (E, \vec{k}), \qquad k'_\mu = (E', \vec{k}')$$

Then

$$\nu = E - E'$$

is the energy lost of lepton and,

$$q^2 = (k - k')^2 = -4EE' \sin^2 \frac{\theta}{2} \le 0, \qquad Q^2 = -q^2$$

θ is scattering angle. The scattering amplitude is,

$$T_n = e^2 \bar{u}(k', \lambda')\gamma^\mu u(k, \lambda) \frac{1}{q^2} \left\langle n \left| J_\mu^{em} \right| p, \sigma \right\rangle$$

where we have used J_μ^{em} to denote the interaction of photon with hadronic states. From the Feynman rule for QED, we get the unpolarized differential cross section,

$$\begin{aligned} d\sigma_n \quad &= \quad \frac{1}{\left|\vec{v}\right|} \frac{1}{2M} \frac{1}{2E} \frac{d^3k'}{(2\pi)^3 \, 2k_0'} \prod_{i=1}^{n} \left[\frac{d^3 p_i}{(2\pi)^3 \, 2p_{i0}} \right] \\ & \times \frac{1}{4} \sum_{\sigma\lambda\lambda'} |T_n|^2 \, (2\pi)^4 \, \delta^4 \left(p + k - k' - p_n \right) \end{aligned}$$

where $p_n = \sum_{i=1}^{n} p_i$. If we sum over all possible hadronic final states, we get inclusive cross section

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2}{q^4} \left(\frac{E'}{E} \right) l^{\mu\nu} W_{\mu\nu}$$

The leptonic tensor $l^{\mu\nu}$ is of the form,

$$l_{\mu\nu} = \frac{1}{2} tr \left(\not{k}' \gamma_\mu \not{k} \gamma_\nu \right) = 2 \left(k_\mu k_\nu' + k_\mu' k_\nu + \frac{q^2}{2} g_{\mu\nu} \right)$$

and the hadronic tensor $W^{\mu\nu}$ can be written as

$$\begin{aligned} W_{\mu\nu}(p,q) &= \frac{1}{4M}\sum_{\sigma}\sum_{n}\int\prod_{i=1}^{n}\left[\frac{d^3p_i}{(2\pi)^3\,2p_{i0}}\right]\left\langle p,\sigma\left|J_\mu^{em}\right|n\right\rangle\langle n\left|J_\nu^{em}\right|p,\sigma\rangle(2\pi)^3\,\delta^4(p_n-q-p) \\ &= \frac{1}{4M}\sum_{\sigma}\int\frac{d^4x}{2\pi}e^{iq\cdot x}\left\langle p,\sigma\left|J_\mu^{em}(x)\,J_\nu^{em}(0)\right|p,\sigma\right\rangle \end{aligned}$$

where we have used completeness in last step. It is more convenient to cast this in the form of matrix element of a commutator. To achieve this we note that the term with two current operators in reverse order can be written in the form,

$$\int\frac{d^4x}{2\pi}e^{iq\cdot x}\left\langle p,\sigma\left|J_\nu^{em}(0)\,J_\mu^{em}(x)\right|p,\sigma\right\rangle=\sum_n(2\pi)^3\,\delta^4(p_n+q-p)\,\langle p,\sigma\left|J_\nu^{em}\right|n\rangle\left\langle n\left|J_\mu^{em}\right|p,\sigma\right\rangle$$

The $\delta-function$ reqires the intermediate state $|n\rangle$ to have enegy with $E_n=M-q_0$ in order to have nonzero result. But since $q_0>0$ and the proton is stable, we can not satisfy the $\delta-function$ constraint and this matrix element is zero. We can therefore write

$$W_{\mu\nu}(p,q)=\frac{1}{4M}\sum_{\sigma}\int\frac{d^4x}{2\pi}e^{iq\cdot x}\left\langle p,\sigma\left|\left[J_\mu^{em}(x)\,,\;J_\nu^{em}(0)\right]\right|p,\sigma\right\rangle$$

From current conservation $\partial^\mu J_\mu^{em}=0$, we get

$$q^\mu\left\langle n\left|J_\mu^{em}\right|p,\sigma\right\rangle=0$$

which implies that

$$q^{\mu} W_{\mu\nu}(p,q) = q^{\nu} W_{\mu\nu}(p,q) = 0$$

From the fact that $W_{\mu\nu}$ is a second rank Lorentz tensor and depends on momenta p, q, one can deduce its covariant decomposition as,,

$$W_{\mu\nu}(p,q) = \left[-W_1 \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \frac{W_2}{M^2} \left(p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left(p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) \right]$$

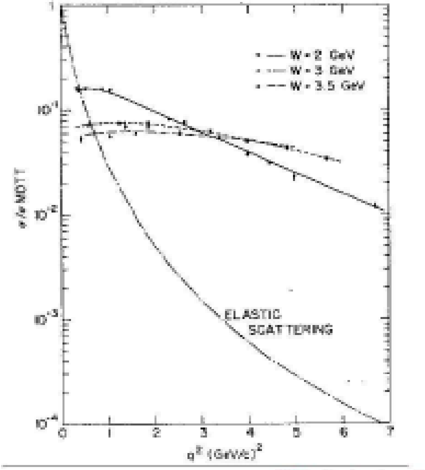
where $W_1(q^2, \nu)$, $W_2(q^2, \nu)$ are Lorentz invariant structure functions of the target proton. We can compute differential cross section in terms of the structure functions,

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2}{4E'^2 \sin^4 \frac{\theta}{2}} \left(2W_1 \sin^2 \frac{\theta}{2} + W_2 \cos^2 \frac{\theta}{2} \right)$$

By measuring differential cross section at different angles and energies, we can extract the structure functions, W_1 and W_2.

Bjorken scatting

Elastic *ep* scattering falls off very rapidly as $(-q^2)$ increases due to the compositness of proton. If this persists for other hadronic state, the total inelastic cross section will fall off rapidly as well. The surprise is that experimentally these cross section seems quite sizable instead of falling off rapidly for large q^2.



Define scaling variable

$$x = \frac{-q^2}{2Mv} = \frac{Q^2}{2Mv}\ , \quad Q^2 = -q^2$$

The range for x is

$$0 \leq x < 1$$

because invariant mass of final hadronic state is

$$W^2 = (p+q)^2 = q^2 + 2M\nu + M^2 \geq M^2$$

Also define

$$y = \frac{\nu}{E} = 1 - \frac{E'}{E}$$

the fraction of initial energy transfered to hadrons. Define

$$MW_1(Q^2, \nu) = F_1(x, q^2/M^2)$$

$$\nu W_2(Q^2, \nu) = F_2(x, q^2/M^2)$$

Write the inclusive cross section as

$$\frac{d^2\sigma}{dxdy} = \frac{8\pi\alpha^2}{MEx^2y^2}\left[xy^2F_1 + \left(1 - y - \frac{M}{2E}xy\right)F_2\right]$$

Bjorken scaling : in the large Q^2 limit the $F_i's$ are functions of x only, ,It turns out that all structure functions have the limiting behavior

$$\lim_{|q^2|\to\infty,\ x\ fixed} F_i(x, q^2/M^2) = F_i(x)$$

Experimentally for $Q^2 \geq 2GeV^2$ Bjorken scaling seems to be a good approximation. This seems to suggest that there are point-like constituents inside the proton.

Neutrino-nucleon scattering

Here we consider a very similar reaction,

$$\nu_l\left(k\right)+N\left(p\right)\longrightarrow l^-\left(k'\right)+X\left(p_n\right)$$

where we have weak interaction with

$$\mathcal{L}_{eff}=-\frac{G_F}{\sqrt{2}}J_\lambda J^\lambda+h.c.$$

where G_F is the Fermi constant. The charged weak current J^λ can be separated into leptonic and hadronic parts,

$$J^\lambda=J_l^\lambda+J_h^\lambda$$

The leptonic part is

$$J_l^\lambda=\bar{\nu}_e\gamma^\lambda\left(1-\gamma_5\right)e+\bar{\nu}_\mu\gamma^\lambda\left(1-\gamma_5\right)\mu$$

The differential cross sections are

$$\frac{d^2\sigma^{(\nu)}}{d\Omega dE'}=\frac{G_F^2}{2\pi}E'^2\left(2\sin^2\frac{\theta}{2}W_1^{(\nu)}+\cos^2\frac{\theta}{2}W_2^{(\nu)}-\frac{(E+E')}{M}\sin^2\frac{\theta}{2}W_3^{(\nu)}\right)$$

$$\frac{d^2\sigma^{(\bar{\nu})}}{d\Omega dE'}=\frac{G_F^2}{2\pi}E'^2\left(2\sin^2\frac{\theta}{2}W_1^{(\bar{\nu})}+\cos^2\frac{\theta}{2}W_2^{(\bar{\nu})}+\frac{(E+E')}{M}\sin^2\frac{\theta}{2}W_3^{(\bar{\nu})}\right)$$

where structure functions are defined as

$$
\begin{aligned}
W_{\mu\nu}^{(\nu)}(p,q) &= \frac{1}{4M}\sum_{\sigma}\int\frac{d^4x}{2\pi}e^{iq\cdot x}\left\langle p,\sigma\left|\left[J_{h\mu}(x),\ J_{h\nu}^{\dagger}(0)\right]\right|p,\sigma\right\rangle \\
&= -W_1^{(\nu)}g_{\mu\nu}+\frac{W_2^{(\nu)}p_{\mu}p_{\nu}}{M^2}-iW_3^{(\nu)}\frac{\varepsilon_{\alpha\beta\mu\nu}p^{\alpha}q^{\beta}}{M^2}+\cdots
\end{aligned}
$$

Bjorken Scaling for these structure functions are ,

$$MW_1^{(\nu)}(q^2,\nu)\longrightarrow F_1^{(\nu)}(x)$$

$$\nu W_2^{(\nu)}(q^2,\nu)\longrightarrow F_2^{(\nu)}(x)$$

$$\nu W_3^{(\nu)}(q^2,\nu)\longrightarrow F_3^{(\nu)}(x)$$

It is useful to use the structure functions with definite helicities. In the laboratory frame, choose the z-axis such that

$$p_{\mu}=(M,0,0,0),\qquad q_{\mu}=(q_0,0,0,q_3)$$

The longitudinal polarization of the virtual photon is

$$\varepsilon_{\mu}^{(s)}=\frac{1}{\sqrt{-q^2}}(q_3,0,0,q_0)$$

and the corresponding structure function is

$$W_s = \varepsilon_\mu^{(s)*} W^{\mu\nu} \varepsilon_\mu^{(s)} = -W_1 - \frac{q_3^2}{q^2 W_2} = \left(1 - \frac{\nu^2}{q^2}\right) W_2 - W_1$$

The right- and left-handed polarization vectors are

$$\varepsilon_\mu^{(R)} = \frac{1}{\sqrt{2}}(0,1,i,0)\,, \qquad \varepsilon_\mu^{(L)} = \frac{1}{\sqrt{2}}(0,1,-i,0)$$

and their structure functions are

$$W_R = W_1 + \frac{1}{2M}\sqrt{\nu^2 - q^2}\,W_3, \qquad W_L = W_1 - \frac{1}{2M}\sqrt{\nu^2 - q^2}\,W_3$$

In the scaling limit we get,

$$2MW_s \longrightarrow F_S = \frac{1}{x}F_2 - 2F_1$$

$$MW_L \longrightarrow F_2 - \frac{1}{2}F_3$$

$$MW_R \longrightarrow F_2 + \frac{1}{2}F_3$$

The differential cross sections can be written as

$$\frac{d^2\sigma^{(\nu)}}{dxdy} = G_F^2 \frac{MEx}{\pi}\left[(1-y)\,F_S^{(\nu)} + F_L^{(\nu)} + (1-y)^2\,F_R^{(\nu)}\right]$$

$$\frac{d^2\sigma^{(\bar{\nu})}}{dxdy} = G_F^2 \frac{MEx}{\pi} \left[(1-y)\, F_S^{(\bar{\nu})} + F_R^{(\bar{\nu})} + (1-y)^2\, F_L^{(\bar{\nu})} \right]$$

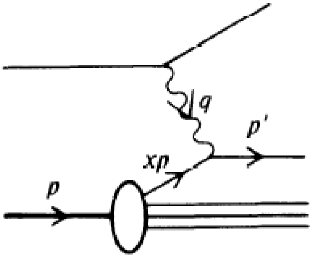
Note that the cross sections increase linearly with energy.

Parton model

Feynman (1969): deep inelastic scattering is due to incoherent elastic scattering from point-like constitents inside the neucleon : Parton.

Assuming parton has spin 1/2 and carries a fraction of proton momentum, ξ with $0 \leq \xi \leq 1$. Then contribution to hadronic tensor is

$$\begin{aligned} K_{\mu\nu}(\xi) &= W_{\mu\nu}(p,q) = \frac{1}{4\xi M}\sum_{\sigma\sigma'}\int\left[\frac{d^3p'}{(2\pi)^3\, 2p_0'}\right]\left\langle \xi p,\sigma\left|J_\mu^{em}\right|p',\sigma'\right\rangle\left\langle p',\sigma'\left|J_\nu^{em}\right|\xi p,\sigma\right\rangle \\ &\quad (2\pi)^3\,\delta^4\left(p'-q-\xi p\right) \\ &= \frac{1}{4\xi M}\sum_{\sigma\sigma'}\bar{u}(\xi p,\sigma)\,\gamma_\mu u\left(p',\sigma'\right)\bar{u}\left(p',\sigma'\right)\gamma_\nu u(\xi p,\sigma)\,\delta\left(p_0'-q_0-\xi p_0\right)/2p_0' \end{aligned}$$



The δ-function can be written as

$$\begin{aligned}\delta\left(p_0' - q_0 - \xi p_0\right)/2p_0' &= \theta\left(p_0'\right)\delta\left[p'^2 - \left(q-\xi p\right)^2\right] \\ &= \theta\left(q_0+\xi p_0\right)\delta\left(2M\nu\xi + q^2\right) = \theta\left(q_0+\xi p_0\right)\frac{\delta\left(\xi - x\right)}{2M\nu}\end{aligned}$$

For the spin sum,

$$\begin{aligned}&\frac{1}{2}\sum_{\sigma\sigma'}\bar{u}\left(\xi p,\sigma\right)\gamma_\mu u\left(p',\sigma'\right)\bar{u}\left(p',\sigma'\right)\gamma_\nu u\left(\xi p,\sigma\right) \\ =\;& \frac{\xi}{2} tr\left[\not{p}\gamma_\mu\left(\xi\not{p}+\not{q}\right)\gamma_\nu\right] \\ =\;& 2\xi\left[p_\mu\left(\xi p+q\right)_\nu + p_\nu\left(\xi p+q\right)_\mu - p\cdot\left(\xi p+q\right)g_{\mu\nu}\right] \\ =\;& 4M^2\xi^2\left(\frac{p_\mu p_\nu}{M^2}\right) - 2M\nu\xi g_{\mu\nu} + \cdots\end{aligned}$$

where we neglect parton mass. The parton tensor is,

$$K_{\mu\nu}\left(\xi\right) = \delta\left(\xi - x\right)\left(\frac{\xi p_\mu p_\nu}{M^2\nu} - \frac{1}{2M}g_{\mu\nu} + \cdots\right)$$

Let $f(\xi)\, d\xi$ be the number of partons with momentum between ξ and $\xi + d\xi$ (weighted by the squared charges). Then hadronic tensor is

$$\begin{aligned} W_{\mu\nu} &= \int_0^1 f(\xi)\, K_{\mu\nu}(\xi)\, d\xi \\ &= \frac{xf(x)}{\nu}\frac{p_\mu p_\nu}{M^2} - \frac{f(x)}{2M} g_{\mu\nu} + \cdots \end{aligned}$$

We can read out the structure functions,

$$MW_1 \to F_1(x) = \frac{1}{2} f(x) \tag{1}$$

$$\nu W_2 \to F_2(x) = xf(x) \tag{2}$$

Thus $F_{1,2}$ are the measures of momentum distribution of the partons insdie the target proton. Note that Eqs (1,2) implies that

$$2xF_1(x) = F_2(x)$$

which is known as **Callan-Gross** relation and is a consequence of partion has spin $\frac{1}{2}$.
Note that for spin 0 parton, we would have

$$\begin{aligned} K_{\mu\nu} &\propto \left\langle xp \left| J_\mu^{em} \right| xp + q \right\rangle \langle xp + q \left| J_\nu^{em} \right| xp \rangle \\ &\propto (2xp + q)_\mu (2xp + q)_\nu \end{aligned}$$

Since there is no $g_{\mu\nu}$ term,

$$F_1(x) = 0$$

In terms of helicity structure functions

$$\begin{aligned} F_S &= 0 \qquad \text{for a spin } 1/2 \text{ parton} \\ F_T &= 0 \qquad \text{for a spin } 0 \text{ parton} \end{aligned}$$

There is a simple explanation for this.

Sum rules and application of parton model

Identiy the parton with the quarks? The quarks are bounded togather by gluons. For a primitive model of 3 free quarks inside the proton, the structure function is $f(x)\,\tilde{}\,\delta\left(x-\frac{1}{3}\right)$. As we turn on interacton this distribution will be smeared and gluons can produce $q\bar{q}$ pairs and quarks can bremsstrahlung gluons. All these processes will produce a "$q\bar{q}$" at small x. Write em current as

$$J_\mu^{em}=\frac{2}{3}\bar{u}\gamma_\mu u-\frac{1}{3}\bar{d}\gamma_\mu d-\frac{1}{3}\bar{s}\gamma_\mu s$$

Structure function is

$$F_1^{ep}(x)=\frac{4}{9}(u+\bar{u})+\frac{1}{9}(d+\bar{d})+\frac{1}{9}(s+\bar{s})$$

$q_i(x)$ probability of finding a parton with longitudinal momentum fraction x with quantum member of quark q in the proton. (Parton distribution function). From isospin symmetry, we get *en* structure functions by $u\leftrightarrow d$

$$F_1^{en}(x)=\frac{4}{9}(d+\bar{d})+\frac{1}{9}(u+\bar{u})+\frac{1}{9}(s+\bar{s})$$

These parton distribution functions are contrained by the quantum numbers of proton. For example,

$$\text{Isospin:}\quad \frac{1}{2}\int_0^1\left\{[u(x)-\bar{u}(x)]-\left[d(x)-\bar{d}(x)\right]\right\}dx=\frac{1}{2}$$

$$\text{Strangeness:}\quad \int_0^1\left[s(x)-\bar{s}(x)\right]=0$$

Charge: $$\int_0^1 \frac{2}{3}\left[u(x) - \bar{u}(x)\right] - \frac{1}{3}\int_0^1 \left[d(x) - \bar{d}(x)\right] - \frac{1}{3}\int_0^1 \left[s(x) - \bar{s}(x)\right] dx = 1$$

Neutrino deep inelastic scattering

$$\nu_\mu + N \to \mu + X$$

$$\nu_e + N \to e + X$$

$$J_\mu^W \approx \cos\theta_c \bar{u}\gamma^\mu(1-\gamma_5)d + \sin\theta_c \bar{u}\gamma^\mu(1-\gamma_5)s + \cdots$$

Here structure functions can also be expressed in terms of parton distribution functions $q_i(x)$
All data are consistent with partons carrying quark quantum numbers.

Light-cone Singularity and Bjorke Scaling

Bjorken scaling is connected with the light-cone behavior in field theory. Recall hadronic tensors can be written as

$$W_{\mu\nu}(p,q) = \frac{1}{4M}\sum_{\sigma}\int \frac{d^4x}{2\pi} e^{iq\cdot x} \left\langle p,\sigma \middle| \left[J_\mu^{em}(x),\ J_\nu^{em}(0) \right] \middle| p,\sigma \right\rangle \tag{3}$$

The exponential can be written as

$$q\cdot x = \frac{(q_0+q_3)}{\sqrt{2}}\frac{(x_0-x_3)}{\sqrt{2}} + \frac{(q_0-q_3)}{\sqrt{2}}\frac{(x_0+x_3)}{\sqrt{2}} - \vec{q}_T\cdot\vec{x}_T$$

where $\vec{q}_T = (q_1, q_2)$, $\vec{x}_T = (x_1, x_2)$. In rest frame of nucleon,

$$p_\mu = (M,0,0,0), \qquad q_\mu = (\nu,0,0,\sqrt{\nu^2-q^2})$$

In scaling limit $-q^2$, $\nu\to\infty$ with $-q^2/2M\nu$ fixed

$$q_0+q_3 \sim 2\nu, \qquad q_0-q_3 \sim \frac{q^2}{2\nu}$$

We expect dominant contribution to the integral in Eq(3) comes from regions with less rapid oscillations i.e. $q\cdot x = O(1)$, which implies that

$$x_0-x_3 \sim O\left(\frac{1}{\nu}\right), \qquad \text{and} \qquad x_0+x_3 \sim O\left(\frac{1}{xM}\right)$$

Or

$$x_0^2 - x_3^2 \sim O\left(\frac{1}{-q^2}\right)$$

Thus $x^2 = x_0^2 - x_3^2 - \vec{x}_T^{\,2} \leq x_0^2 - x_3^2 \sim O\left(\frac{1}{-q^2}\right)$ which vanishes as $-q^2 \to \infty$. So in scaling limit we are probing the current product near the light cone.

Free Field Light-cone Singularity

1) Product of fields

In free field theory, product of fields are singular on the light-cone $\left(x^2 \approx 0\right)$ and the leading singularities are indepenet of masses. Consider

$$\langle 0 | T(\phi(x)\phi(0)| 0\rangle = i\Delta_F(x) = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2 - m^2 + i\varepsilon}$$

Carry out the integration

$$\Delta_F(x) = \frac{-1}{4\pi}\delta\left(x^2\right) + \frac{m}{8\pi\sqrt{x^2}}\theta\left(x^2\right)\left[J_1\left(m\sqrt{x^2}\right) - iN_1\left(m\sqrt{x^2}\right)\right] - \frac{im}{4\pi^2\sqrt{x^2}}\theta\left(-x^2\right)K_1\left(m\sqrt{-x^2}\right)$$

where J_n, N_n and K_n are Bessel functions. For $x^2 \approx 0$, we have

$$\Delta_F(x) = \frac{i}{4\pi^2}\frac{1}{\left(x^2 - i\varepsilon\right)} + O\left(m^2x^2\right)$$

One can also show

$$\begin{aligned}
[\phi(x), \phi(0)] &= i\Delta(x) = \frac{1}{(2\pi)^3}\int d^4k e^{-ik\cdot x}\varepsilon(k_0)\,\delta\left(k^2 - m^2\right) \\
&= \frac{-i}{2\pi}\varepsilon(x_0)\,\delta\left(x^2\right) \qquad \text{for} \qquad x^2 \approx 0
\end{aligned}$$

Setting $m^2 \to 0$, we get,

$$i \int d^4 k e^{-ik\cdot x} \varepsilon(k_0)\, \delta(k^2) = (2\pi)^2 \varepsilon(x_0)\, \delta(x^2)$$

Thus light-cone singularties of the commutator $\Delta(x)$ and propagator function $\Delta_F(x)$ are directly related,

$$\Delta(x) = 2\varepsilon(x_0)\, \mathrm{Im}\,(i\Delta_F(x))$$

This reflects the singular function identity

$$\frac{1}{-x^2 + i\varepsilon} - \frac{1}{-x^2 - i\varepsilon} = -2\pi i \varepsilon(x_0)\, \delta(x^2)$$

which is a special case of general identity

$$\left(\frac{1}{-x^2 + i\varepsilon}\right)^n - \left(\frac{1}{-x^2 - i\varepsilon}\right)^n = -\frac{2\pi i}{(n-1)!} \varepsilon(x_0)\, \delta^{(n-1)}(x^2)$$

In the following we shall obtain the commutator singularities from those of propagators by the replacement,

$$\left(\frac{1}{-x^2 + i\varepsilon}\right)^n \longrightarrow \frac{2\pi i}{(n-1)!} \varepsilon(x_0)\, \delta^{(n-1)}(x^2)$$

For the fermions the results are

$$\left\{\psi_\alpha(x), \bar{\psi}_\beta(y)\right\} = iS_{\alpha\beta}(x-y), \qquad S_{\alpha\beta}(x) = (i\gamma\cdot\partial + m)_{\alpha\beta}\, \Delta(x)$$

$$\left\langle 0 \left| T\left(\psi_\alpha(x), \bar{\psi}_\beta(y)\right) \right| 0 \right\rangle = i S^F_{\alpha\beta}(x-y), \qquad S^F_{\alpha\beta}(x) = (i\gamma\cdot\partial + m)_{\alpha\beta}\Delta^F(x)$$

For $x^2 \approx 0$, we have

$$S_{\alpha\beta}(x) \approx (i\gamma\cdot\partial)_{\alpha\beta}\left[\frac{1}{2\pi}\varepsilon(x_0)\,\delta\left(x^2\right)\right]$$

$$S^F_{\alpha\beta}(x) \approx (i\gamma\cdot\partial)_{\alpha\beta}\left[\frac{1}{2\pi}\frac{1}{x^2 - i\varepsilon}\right]$$

2) Product of scalar currents

Consider composite operators like scalar current ,

$$J(x) =: \phi^2(x) :$$

Note normal ordering remove the singularities in the product $\phi(x+\zeta)\,\phi(x-\zeta)$ as $\zeta^\mu \to 0$. The singularities in product of the currents can be worked out by using Wick's therem,

$$\begin{aligned} T(J(x)\,J(0)) &= T(:\phi^2(x)::\phi^2(0):) = 2\left[\langle 0\,|\,T(\phi(x)\,\phi(0))|\,0\rangle\right]^2 \\ +4\,\langle 0\,|\,T(\phi(x)\,\phi(0))|\,0\rangle &: \phi(x)\,\phi(0) : + : \phi^2(x)\,\phi^2(0) : \\ &= -2\left[\Delta_F(x,m)\right]^2 + 4i\Delta_F(x,m) : \phi(x)\,\phi(0) : + : \phi^2(x)\,\phi^2(0) : \end{aligned}$$

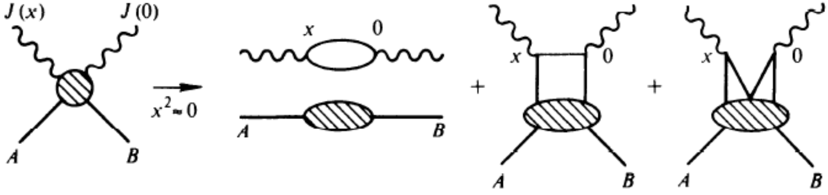
Hence for $x^2 \approx 0$, we get

$$T(J(x)\,J(0)) \approx \frac{1}{8\pi^4\left(x^2 - i\varepsilon\right)^2} - \frac{:\phi(x)\,\phi(0):}{\pi^2\left(x^2 - i\varepsilon\right)} + :\phi^2(x)\,\phi^2(0):$$

In this expansion the singularitis as $x^2 \approx 0$ are all contained in the c-number functions which are independent of the initial or final states. If we take this between 2 arbitrary states,

$$\langle A\,|\,T(J(x)\,J(0))|\,B\rangle \approx \frac{\langle A|B\rangle}{8\pi^4\left(x^2 - i\varepsilon\right)^2} - \frac{\langle A\,|:\phi(x)\,\phi(0):|\,B\rangle}{\pi^2\left(x^2 - i\varepsilon\right)} + \left\langle A\,\middle|:\phi^2(x)\,\phi^2(0):\middle|\,B\right\rangle$$

which corresponds to diagrams below.



(a) (b) (c)

Free Field Singularities and Scaling

Consider the electromagnetic current given by

$$J_\mu(x) =: \bar{\psi}(x)\gamma_\mu Q\psi(x):$$

where Q is the electric charge operator. We will first calculate the time-ordered product by Wick's theorem,

$$\begin{aligned} T\left(J_\mu(x)J_\nu(0)\right) &= T\left(:\bar{\psi}(x)\gamma_\mu Q\psi(x)::\bar{\psi}(0)\gamma_\nu Q\psi(0):\right) \qquad (4) \\ &= Tr\left[iS_F(-x)\gamma_\mu iS_F(x)\gamma_\nu Q^2\right] + :\bar{\psi}(x)\gamma_\mu Q S_F(x)\gamma_\nu Q\psi(0): \\ + &: \bar{\psi}(0)\gamma_\nu Q S_F(-x)\gamma_\mu Q\psi(x): + :\bar{\psi}(x)\gamma_\mu Q\psi(x)\bar{\psi}(0)\gamma_\nu Q\psi(0): \end{aligned}$$

Using the identity

$$\gamma_\mu\gamma_\nu\gamma = \left(S_{\mu\nu\lambda\rho} + i\varepsilon_{\mu\nu\lambda\rho}\right)\gamma^\rho, \qquad \text{where} \qquad S_{\mu\nu\lambda\rho} = g_{\mu\nu}g_{\lambda\rho} + g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}$$

we can write Eq(4) in the limit $x^2 \approx 0$ as

$$\begin{aligned} T\left(J_\mu(x)J_\nu(0)\right) &\approx \left(trQ^2\right)\frac{\left(x^2 g_{\mu\nu} - x_\mu x_\nu\right)}{\pi^4\left(x^2 - i\varepsilon\right)^4} + \frac{ix^\alpha}{2\pi^2\left(x^2 - i\varepsilon\right)^2} \\ &\left\{S_{\mu\alpha\nu\beta}\left[V^\beta(x,0) - V^\beta(0,x)\right] + i\varepsilon_{\mu\alpha\nu\beta}\left[A^\beta(x,0) - A^\beta(0,x)\right]\right\} \\ + &: \bar{\psi}(x)\gamma_\mu Q\psi(x)\bar{\psi}(0)\gamma_\nu Q\psi(0): \end{aligned}$$

where

$$V^{\beta}(x,y) =: \bar{\psi}(x)\gamma^{\beta}Q^{2}\psi(y):$$

$$A^{\beta}(x,y) =: \bar{\psi}(x)\gamma^{\beta}\gamma_{5}Q^{2}\psi(y):$$

If we write

$$\frac{x^{2}g_{\mu\nu}-2x_{\mu}x_{\nu}}{(x^{2}-i\varepsilon)^{4}} = \frac{2}{3}\frac{g_{\mu\nu}}{(x^{2}-i\varepsilon)^{3}} - \frac{1}{12}\partial_{\mu}\partial_{\nu}\frac{1}{(x^{2}-i\varepsilon)^{2}}$$

and

$$\frac{x^{\alpha}}{(x^{2}-i\varepsilon)^{2}} = -\frac{1}{2}\partial^{\alpha}\left(\frac{1}{x^{2}-i\varepsilon}\right)$$

we get for the commutator,

$$\left[J_{\mu}(x), J_{\nu}(0)\right] \approx \frac{itrQ^{2}}{\pi^{3}}\left\{\frac{2}{3}g_{\mu\nu}\delta''\left(x^{2}\right)\varepsilon(x_{0}) + \frac{1}{6}\partial_{\mu}\partial_{\nu}\left[\delta'\left(x^{2}\right)\varepsilon(x_{0})\right]\right\} \quad (5)$$

$$+\left\{S_{\mu\alpha\nu\beta}\left[V^{\beta}(x,0)-V^{\beta}(0,x)\right]+i\varepsilon_{\mu\alpha\nu\beta}\left[A^{\beta}(x,0)-A^{\beta}(0,x)\right]\right\}$$

$$\partial^{\alpha}\frac{\left[\delta\left(x^{2}\right)\varepsilon(x_{0})\right]}{2\pi} + \quad : \quad \bar{\psi}(x)\gamma_{\mu}Q\psi(x)\bar{\psi}(0)\gamma_{\nu}Q\psi(0): \quad (6)$$

We can then apply these to the cross sections of $e^{+}e^{-}$ annihilation and inelastic eN scattering.

1. **$e^+e^- \to$ hadrons**

Following the same procedure as in the discussion of inelastic eN scattering, it is straightforward to show that the total hadronic cross section for $\mathbf{e}^+\mathbf{e}^-$ annihilation can be written as a current commutator,

$$\sigma\left(e^+e^- \to hadrons\right) = \frac{8\pi^2\alpha^2}{3\left(q^2\right)^2}\int d^4xe^{iq\cdot x}\left\langle 0\left|\left[J_\mu\left(x\right), J^\mu\left(0\right)\right]\right|0\right\rangle$$

The most singular light-cone term comes from the first term on the right-handed side of Eq (5) and we get from this term

$$\sigma\left(e^+e^- \to hadrons\right) \approx \frac{8\pi^2\alpha^2 i\left(trQ^2\right)}{3\pi^3\left(q^2\right)^2}\int d^4xe^{iq\cdot x}\left\{\frac{8}{3}\delta''\left(x^2\right)\varepsilon\left(x_0\right) + \frac{1}{6}\partial^2\left[\delta'\left(x^2\right)\varepsilon\left(x_0\right)\right]\right\}$$

in the large q^2 limit. Using the identity

$$i\int d^4xe^{-iq\cdot x}\varepsilon\left(q^0\right)\delta\left(q^2\right) = \left(2\pi\right)^2\varepsilon\left(x^0\right)\delta\left(x^2\right)$$

we get

$$\begin{aligned}\sigma\left(e^+e^- \to hadrons\right) &\approx \frac{8\pi^2\alpha^2 i\left(trQ^2\right)}{3\pi^3\left(q^2\right)^2}\left[\frac{8}{3}\frac{q^2}{4} - \frac{q^2}{6}\right]\varepsilon\left(q^0\right)\delta\left(q^2\right) \\ &= \frac{4\pi\alpha^2}{3q^2}tr\left(Q^2\right)\end{aligned}$$

Recall that

$$\sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{4\pi\alpha^2}{3q^2}$$

Thus we get the simple result

$$\frac{\sigma\left(e^+e^- \to hadrons\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = tr\left(Q^2\right)$$

This justify the simple naive picture that in the deep inelastic limit, $q^2 \to \infty$, the virtual photon will first produce quarks where is coupling is point like and then quarks trun into hadrons through some strong interaction which is difficult to compute.

2. **Lepton-hadron scattering**

For deep inelastic *lN* scattering the first term on the right-handed side of Eq (5) will not contribute since it is a *c*-number and the non-trivial leading singular term will be the second term which for convenience is written in the form,

$$\left[J_\mu(\frac{x}{2}), J_\nu(-\frac{x}{2})\right] \approx \left\{ \begin{array}{c} S_{\mu\alpha\nu\beta}\left[:\bar{\psi}\left(\frac{x}{2}\right)\gamma^\beta Q^2\psi\left(-\frac{x}{2}\right): - :\bar{\psi}\left(-\frac{x}{2}\right)\gamma^\beta Q^2\psi\left(\frac{x}{2}\right):\right] \\ +i\varepsilon_{\mu\alpha\nu\beta}\left[:\bar{\psi}\left(\frac{x}{2}\right)\gamma^\beta\gamma_5 Q^2\psi\left(-\frac{x}{2}\right): - :\bar{\psi}\left(-\frac{x}{2}\right)\gamma^\beta\gamma_5 Q^2\psi\left(\frac{x}{2}\right):\right] \end{array}\right.$$

$$\partial^\alpha \frac{\left[\delta\left(x^2\right)\varepsilon\left(x_0\right)\right]}{2\pi}$$

We can expand the bilocal operator

$$\begin{aligned}\bar{\psi}\left(\frac{x}{2}\right)\psi\left(-\frac{x}{2}\right) &= \bar{\psi}(0)\left[1+\overleftarrow{\partial}_{\mu_1}\frac{x^{\mu_1}}{2}+\frac{1}{2!}\overleftarrow{\partial}_{\mu_1}\overleftarrow{\partial}_{\mu_2}\frac{x^{\mu_1}}{2}\frac{x^{\mu_2}}{2}+\cdots\right]\times \\ &\quad \left[1-\frac{x^{\nu_1}}{2}\overrightarrow{\partial}_{\nu_1}+\frac{1}{2!}\frac{x^{\nu_1}}{2}\frac{x^{\nu_2}}{2}\overrightarrow{\partial}_{\nu_1}\overrightarrow{\partial}_{\nu_2}+\cdots\right]\psi(0) \\ &= \sum_n \frac{1}{n!}\frac{x^{\mu_1}}{2}\frac{x^{\mu_2}}{2}\cdots\frac{x^{\mu_n}}{2}\bar{\psi}(0)\overleftrightarrow{\partial}_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\cdots\overleftrightarrow{\partial}_{\mu_n}\psi(0)\end{aligned}$$

to get

$$\begin{aligned}\left[J_\mu\left(\frac{x}{2}\right),J_\nu\left(-\frac{x}{2}\right)\right] &= \sum_{n=odd}\frac{1}{n!}\frac{x^{\mu_1}}{2}\frac{x^{\mu_2}}{2}\cdots\frac{x^{\mu_n}}{2}O^{(n+1)}_{\beta\mu_1\mu_2\cdots\mu_n}(0)\,S_{\mu\alpha\nu\beta}\partial^\alpha\frac{\left[\delta\left(x^2\right)\varepsilon\left(x_0\right)\right]}{2\pi} \\ &\quad +\sum_{n=even}\frac{1}{n!}\frac{x^{\mu_1}}{2}\frac{x^{\mu_2}}{2}\cdots\frac{x^{\mu_n}}{2}O'^{(n+1)}_{\beta\mu_1\mu_2\cdots\mu_n}(0)\,i\varepsilon_{\mu\alpha\nu\beta}\partial^\alpha\frac{\left[\delta\left(x^2\right)\varepsilon\left(x_0\right)\right]}{2\pi}\end{aligned}$$

where

$$O^{(n+1)}_{\beta\mu_1\mu_2\cdots\mu_n}(0)=\bar{\psi}(0)\overleftrightarrow{\partial}_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\cdots\overleftrightarrow{\partial}_{\mu_n}\gamma_\beta Q^2\psi(0)$$

$$O'^{(n+1)}_{\beta\mu_1\mu_2\cdots\mu_n}(0)=\bar{\psi}(0)\overleftrightarrow{\partial}_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\cdots\overleftrightarrow{\partial}_{\mu_n}\gamma_\beta\gamma_5 Q^2\psi(0)$$

To calculate the structure function we write,

$$\frac{1}{2}\sum_{\sigma}\left\langle p\sigma\left|O^{(n+1)}_{\beta\mu_1\mu_2\cdots\mu_n}(0)\right|p\sigma\right\rangle = A^{(n+1)}p^{\beta}p_{\mu_1}p_{\mu_2}\cdots p_{\mu_n} + \textit{trace terms}$$

where $A^{(n+1)}$ is some constant and trace terms contain one or more $g_{\mu_i\mu_j}$ factors. Also $O'^{(n+1)}$ tetm will not contribute to the spin average structure functions due to the antisymmetric property of $\varepsilon_{\mu\alpha\nu\beta}$. We then have for the structure function,

$$W_{\mu\nu}(p,q) \approx \frac{1}{2M}\int\frac{d^4x}{2\pi}e^{iq\cdot x}\sum_{odd\ n}^{\infty}\left(\frac{x\cdot p}{2}\right)^n\frac{p^{\beta}}{n!}A^{(n+1)}S_{\mu\alpha\nu\beta}\partial^{\alpha}\frac{\left[\delta\left(x^2\right)\varepsilon\left(x_0\right)\right]}{2\pi}$$

Define

$$\sum_{odd\ n}^{\infty}\left(\frac{x\cdot p}{2}\right)^n\frac{A^{(n+1)}}{n!} = \int d\xi e^{ix\cdot\xi p}f(\xi)$$

then

$$W_{\mu\nu}(p,q) \approx \frac{1}{2M}\int\frac{d^4x}{2\pi}e^{iq\cdot x}\int d\xi e^{ix\cdot\xi p}f(\xi)\,S_{\mu\alpha\nu\beta}(q+\xi p)^{\alpha}p^{\beta}\frac{\left[\delta\left(x^2\right)\varepsilon\left(x_0\right)\right]}{2\pi}$$

Using the identity,

$$i\int\frac{d^4x}{2\pi}e^{ix\cdot(q+\xi p)}\delta\left(x^2\right)\varepsilon\left(x_0\right) = \delta\left(\left(q+\xi p\right)^2\right)\varepsilon\left(q_0+\xi p_0\right)$$

we have

$$\begin{aligned} W_{\mu\nu}(p,q) &\approx \frac{1}{M}\int d\xi f(\xi)\,\delta\left(q^2+2M\nu\xi\right)\left(g_{\mu\alpha}g_{\beta\nu}+g_{\mu\beta}g_{\alpha\nu}-g_{\mu\nu}g_{\alpha\beta}\right)(q+\xi p)^{\alpha}\,p^{\beta} \\ &= \frac{1}{2M^2\nu}\int d\xi f(\xi)\,\delta\left(\xi+\frac{q^2}{2M\nu}\right)\left(-M\nu g_{\mu\nu}+2\xi p_\mu p_\nu+\cdots\right) \\ &= f(x)\left[-\frac{g_{\mu\nu}}{2M}+\frac{x}{\nu}\frac{p_\mu p_\nu}{M^2}+\cdots\right] \end{aligned}$$

for $x=-\dfrac{q^2}{2M\nu}$. Thus we recover the parton modle results

$$MW_1 \longrightarrow F_1(x)=\frac{1}{2}f(x)$$

$$\nu W_2 \longrightarrow F_2(x)=xf(x)$$

This implies that the assumption of canonical free-field light-cone structure is equivalent to that of parton model.